ESTIMATES FOR COEFFICIENTS OF L–FUNCTIONS. I

W. DUKE† AND H. IWANIEC†

Department of Mathematics, Rutgers University
New Brunswick, NJ 08903, USA

1. INTRODUCTION

Of the various methods for estimating Fourier coefficients of Maass cusp forms the most successful one so far has been based on properties of symmetric power L–functions associated to the form together with an analytic lemma of Landau. This gives

$$|\lambda_n| \leq \tau(n) n^{\frac{1}{5}}$$

for the n^{th} Hecke eigenvalue λ_n (see [3],[4],[5] and [6]). Landau's lemma requires positivity of the coefficients; this limits the complete exploitation of deeper knowledge of analytic properties of these L–functions.

In this work we present a method which does not require the coefficients to be positive but which instead appeals to functional equations of the L–functions twisted by Dirichlet characters. The signs of these functional equations play a role in this method; in many cases they give rise to multi-dimensional Kloosterman sums for which Deligne's estimate is available. In a sense Deligne's estimate yields an effect similar to that of the method of stationary phase for Landau's lemma (J. Hafner has informed us that S. Friedberg and J. Hoffstein have also noticed such an effect). When applied to the Rankin-Selberg convolution of the symmetric square L–function our Theorem 1 also gives the above estimate for λ_n.

It seems likely that advances in the theory of symmetric power L–functions will allow the application of Theorem 1 in some new cases. For example, if the twisted fourth symmetric power L–functions are shown to satisfy the conditions of our Theorem then we would get the bound

$$|\lambda_n| \leq \tau(n) n^{\frac{1}{6}} \quad .$$

If the twisted fourth symmetric power L–function multiplied by the Dirichlet L–function satisfies these conditions then we would have

$$|\lambda_n| \leq \tau(n) n^{\frac{5}{28}} \quad .$$

In the second paper [2] of this series we shall present a different method for L–functions having Euler products of degree three and compatible functional equations. No estimate for Kloosterman sums is used in [2]. When specified to the Shimura symmetric square L–function this method yields the bound

$$|\lambda_n| \leq \tau(n) n^{\frac{7}{32}} \quad ,$$

which is slightly weaker than $\frac{1}{5}$ but better than $\frac{1}{4}$ and is obtained within GL_2 theory only.

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2. STATEMENT OF THE RESULT

Let $\mathcal{A} = (a_n)$ be a sequence of complex numbers. Suppose that the series

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

converges absolutely in $\operatorname{Re} s > 1$. Let us set

$$\mathcal{A}(s,\chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s} \quad , \tag{2}$$

where χ is a Dirichlet character modulo p. For technical simplification we confine ourselves to prime moduli, so that any non-principal character is primitive. Then we assume that for all but a small number of exceptional characters $\chi \pmod p$, the series $\mathcal{A}(s,\chi)$ may be analytically continued to an entire function which satisfies a functional equation of the type

$$\mathcal{A}(1-s,\chi) = \epsilon_\chi \Phi(s) \mathcal{A}(s,\overline{\chi}) \tag{3}$$

where $|\epsilon_\chi| = 1$ and $\Phi(s)$ is holomorphic in $\operatorname{Re} s > 1$. $\Phi(s)$ depends on p and it may depend on the parity of χ, but not on χ otherwise. Moreover, we assume there exists a constant $c \geq 1$ such that

$$\Phi(s) \ll \left(p|s|\right)^{(2\sigma-1)c} \tag{4}$$

on $\operatorname{Re} s = \sigma > 1$, the implied constant depending on σ.

The sign ϵ_χ of the functional equation features in our argument. We require that $\epsilon_\chi \overline{\chi}(a)$ be randomly distributed on the unit circle. More precisely, we assume that for any a we have

$$K_p(a) := \sum_{\chi \notin \mathcal{H}_p} \epsilon_\chi \overline{\chi}(a) \ll p^{\frac{1}{2}} \quad , \tag{5}$$

where $\mathcal{H}_p$ is the set of exceptional characters $\chi \pmod p$, including the trivial character, for which we do not claim the functional equation (3). We assume that the number of exceptional characters is bounded:

$$|\mathcal{H}_p| \leq H \quad , \tag{6}$$

say, where H does not depend on p.

THEOREM 1. *If the above conditions hold for a set of primes of positive density then for any $n \geq 1$ we have*

$$a_n \ll n^{\frac{2c-1}{2c+1}+\epsilon}$$

with any $\epsilon > 0$, the implied constant depending on ϵ.

REMARKS: The hypotheses of our theorem are quite realistic. Indeed, in practice we have the functional equations (3) with $\Phi(s) = \Theta(s)/\Theta(1-s)$, where

$$\Theta(s) = \left(\frac{p}{\pi}\right)^{\frac{ks}{2}} \prod_{j=1}^{k} \Gamma\left(\frac{s}{2} + \kappa_j\right) ,$$

with $\mathrm{Re}\,\kappa_j \geq -\frac{1}{2}$, and $\epsilon_\chi = \left(\tau(\chi)/\sqrt{p}\right)^k$, where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{x \,(\mathrm{mod}\, p)} \chi(x) e_p(x) .$$

In this situation we have $2c = k$ and

$$\sum_{\chi \,(\mathrm{mod}\, p)} \epsilon_\chi \overline{\chi}(a) = (p-1)p^{-\frac{k}{2}} \sum_{\substack{x_1 \dots x_k \equiv a \\ (\mathrm{mod}\ p)}} e_p(x_1 + \cdots + x_k) , \tag{7}$$

so (5) follows from the Deligne estimate for generalised Kloosterman sums (see [1]).

3. THE DISTRIBUTION IN ARITHMETIC PROGRESSIONS

Let f be a smooth, compactly supported function in $\mathbb{R}^+$. We shall study the distribution of $\mathcal{A}_f = (a_n f(n))$ in residue classes $l \,(\mathrm{mod}\, p)$ with $(l,p) = 1$. We set

$$\mathcal{A}_f(p;l) = \sum_{n \equiv l \,(\mathrm{mod}\, p)} a_n f(n) ,$$

and

$$\mathcal{A}_f(\chi) = \sum_n a_n \chi(n) f(n) ,$$

thus

$$\mathcal{A}_f(p;l) = \frac{1}{p-1} \sum_{\chi \,(\mathrm{mod}\, p)} \overline{\chi}(l) \mathcal{A}_f(\chi) . \tag{8}$$

If χ is not exceptional we apply contour integration and the functional equation (3), getting

$$\mathcal{A}_f(\chi) = \epsilon_\chi \mathcal{A}_g(\overline{\chi}) ,$$

where

$$g(y) = \frac{1}{2\pi i} \int_{(1+\epsilon)} F(s)\Phi(s) y^{-s} ds \tag{9}$$

and $F(s)$ is given by the Mellin integral

$$F(s) = \int_0^\infty f(x) x^{-s} dx . \tag{10}$$

Note that g depends on the parity of χ, but not on χ otherwise. We display this dependence by writing g_+ and g_- respectively in place of g. We also split the sum $K_p(a)$ into even and odd characters getting

$$S_\pm(a) = \frac{1}{2}K_p(a)\pm\frac{1}{2}K_p(-a) \quad .$$

Let us denote the contribution of exceptional characters to (8) by

$$\mathcal{M}_f(p;l) = \frac{1}{p-1}\sum_{\chi\in\mathcal{H}_p}\overline{\chi}(l)\mathcal{A}_f(\chi)$$

and call it the main term. Then we have the error term equal to

$$\begin{aligned}\Delta\mathcal{A}_f(p;l) &= \mathcal{A}_f(p;l) - \mathcal{M}_f(p;l)\\ &= \frac{1}{p-1}\sum_{\chi\notin\mathcal{H}_p}\overline{\chi}(l)\epsilon_\chi\mathcal{A}_g(\overline{\chi})\\ &= \frac{1}{p-1}\sum_{m,\pm}a_m g_\pm(m)S_\pm(lm) \quad .\end{aligned}$$

By (5) we deduce that By (5) we deduce that

$$\Delta\mathcal{A}_f(p;l) \ll p^{-\frac{1}{2}}\sum_m\left|a_m g(m)\right| \quad . \tag{11}$$

It remains to estimate $g(m)$. To this end let us assume that the Mellin integral (10) is bounded by

$$F(s) \ll |s|^{-r} \tag{12}$$

for some $r > c+1$. Then by (4) and (12) we infer from (9) that

$$g(m) \ll m^{-1-\epsilon}p^{c(1+2\epsilon)} \tag{13}$$

for arbitrarily small $\epsilon > 0$. By (11), (13) and (1) we conclude the following:

THEOREM 2. *If (12) holds then*

$$\Delta\mathcal{A}_f(p;l) \ll p^{c-\frac{1}{2}+\epsilon} \tag{14}$$

for any $\epsilon > 0$, the implied constant depending on ϵ but not on l and p.

REMARK: The condition (12) is easily satisfied with any $r \geq 0$ for functions of the type

$$f(x) = \omega(xl^{-1}) \quad , \tag{15}$$

where $\omega(t)$ is a smooth function supported in the interval $[\frac{1}{2}, 2]$ and $l \geq 1$ is a parameter.

4. PROOF OF THEOREM 1

We apply Theorem 2 for a test function f of type (15) giving

$$\sum_{n\equiv l\,(\mathrm{mod}\,p)} a_n\omega(nl^{-1}) \ll p^{-1} \sum_{\frac{l}{2}<n<2l} |a_n| + p^{c-\frac{1}{2}+\epsilon} \ .$$

This holds for primes p in a set of positive density. On the left-hand side the term $a_l\omega(1)$ occurs with high multiplicity since every p divides $l-l$. For $n \neq l$ there are at most $O(\log l)$ prime divisors of $n-l$. Therefore, summing the above inequality over admissible primes $p \sim P$, we get

$$P|a_l| \ll \sum_{\frac{l}{2}<n<2l} |a_n|(\log l)^2 + P^{c+\frac{1}{2}+\epsilon} \ .$$

Hence

$$|a_l| \ll \left(lP^{-1} + P^{c-\frac{1}{2}}\right)l^{\epsilon} \ll l^{\frac{2c-1}{2c+1}+\epsilon}$$

on choosing the optimal P.

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