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ESTIMATING HECKE EIGENVALUES OF SIEGEL MODULAR FORMS

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0. Introduction. A central problem in the theory of automorphic forms is to estimate the Hecke eigenvalues of cusp forms. Historically, such eigenvalues first arose as multiplicative remainder terms in formulas for the number of representations of integers by certain quadratic forms [Ra]. In this case the eigenfunctions are holomorphic cusp forms for congruence subgroups of the modular group, and the best possible estimate for their eigenvalues was obtained by Deligne when he proved the Ramanujan-Petersson conjecture [De].

In this paper we shall consider the case of holomorphic Siegel cusp forms of degree $n \geqslant 2$. As when $n = 1$, the Hecke eigenvalues here are closely connected to representations by a positive quadratic form of scalar multiples of a fixed form in n variables. In contrast to the case $n = 1$, however, only relatively weak bounds are known for these eigenvalues. Such bounds have previously been obtained by generalizing two classical approaches: the Rankin-Selberg method and the method of Poincaré series and Kloosterman sums. In the case $n \geqslant 2$, we shall go well beyond these methods by estimating the matrix coefficients of certain representations. For precise statements of our results, see formulas (1.7), (1.8), and (1.9) below.

For the sake of exposition, we shall restrict our attention to modular forms for the full Siegel modular group (with trivial multiplier system when $n = 2$), although the method employed generalizes considerably. We remark that Shahidi [Sh] has obtained good estimates for the Hecke eigenvalues of generic cusp forms on quite general groups. However, when $n \geqslant 2$, holomorphic Siegel modular forms are not generic; so his estimates do not apply here.

In the final section further results are given concerning the absolute convergence of certain L-functions and the Hecke eigenvalues of singular modular forms (see [A]).

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1. Classical approach for cusp forms. Let $G = GSp_{2n}$ be the group of symplectic similitudes with respect to

$$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

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where $n \in \mathbf{Z}^+$. The connected component

$$G^+(\mathbf{R}) = \{g \in G(\mathbf{R}) \mid {}^t g J g = rJ, r > 0\}$$

acts on $\mathscr{H}$, the Siegel upper half-plane of degree n, by $Z \mapsto g(Z) = (AZ + B) \times (CZ + D)^{-1}$, where $g = \begin{pmatrix} AB \\ CD \end{pmatrix} \in G^+(\mathbf{R})$. Define the automorphy factor

$$j(g, Z) = (det\ g)^{-1/2} \det(CZ + D).$$

For $k \in \mathbf{Z}^+$ and f a function on $\mathscr{H}$, define the weight-k slash operator

$$f|_g(Z) = j(g, Z)^{-k} f(g(Z)). \tag{1.1}$$

For $m \in \mathbf{Z}^+$ set

$$\Gamma_m = \{\gamma \in M_{2n}(\mathbf{Z});\ {}^t\gamma J \gamma = mJ\} \tag{1.2}$$

and $\Gamma = \Gamma_1 = Sp_{2n}(\mathbf{Z})$. The weight-$k$ Hecke operator is defined by

$$T_m f = m^{nk/2 - n(n+1)/2} \sum_{\gamma \in \Gamma \backslash \Gamma_m} f|_\gamma. \tag{1.3}$$

This makes sense provided we have

$$f|_\gamma = f$$

for all $\gamma \in \Gamma$. Also, the sum is finite since Γ_m is a finite union of (left or right) cosets of Γ.

Let S_n^k be the (finite-dimensional) space of Siegel cusp forms of weight k for Γ. We refer to [Kl] for many basic properties of Siegel modular forms. The theory of Hecke operators can be found in [A] and [Fr]. The latter sources may be consulted for proofs of the facts used below. The space S_n^k possesses a natural inner product, making it a Hilbert space. The operators $\{T_m\}_{m=1}^\infty$ comprise a family of commuting Hermitian linear operators on S_n^k, and S_n^k possesses an orthonormal basis of simultaneous eigenforms [Fr, pp. 272–274]. Let f be such an eigenform. Then for each m

$$T_m f = \lambda_m f$$

where $\lambda_m \in \mathbf{R}$. The λ_m are the *Hecke eigenvalues*; estimating them is our main concern. We shall restrict our attention to square-free m; since λ_m is multiplicative, we are reduced to considering λ_p for p prime. Now $\Gamma \backslash \Gamma_p$ has $\prod_{j=1}^n (1 + p^j)$ elements, and since for a cusp form $f|_\gamma(Z)$ is uniformly bounded on $\mathscr{H}$, the trivial bound for λ_p is

$$\lambda_p \ll p^{nk/2}.$$

In fact, it is not much harder to prove (see [Ko1], [W])

$$|\lambda_p| < 2^n p^{nk/2}.$$

Classical approaches to improving this bound first connect λ_p with the Fourier coefficients $a(N)$, where

$$f(Z) = \sum_{N>0} a(N)e(tr\, NZ) \qquad \text{for } Z \in \mathscr{H}$$

with N running over positive semi-integral $n \times n$ matrices. If $n = 1$, we have the simple relation

$$\lambda_p a(N) = a(pN)$$

for $p \nmid N$ a prime. For $n \geqslant 2$ we only have this asymptotically.

PROPOSITION 1.1. *Let $T_p f = \lambda_p f$ for f a Siegel cusp form of degree $n \geqslant 2$. Then for M a positive semi-integral $n \times n$ matrix*

$$(1.4) \qquad \lambda_p a(M) - a(pM) \ll (det\, M)^{k/2} p^{nk/2-1}$$

as $p \to \infty$ through primes. Here, the implied constant depends only on f.

Proof. It is convenient to use the language of matrix residue classes. For $D \in Mat_n(\mathbf{Z})$ the symmetric residue classes $(mod\, D)$ are elements of the quotient group $\mathscr{R}_D = \mathscr{A}_D/\mathscr{B}$, where

$$\mathscr{A}_D = \{B \in Mat_n(\mathbf{Z});\, {}^tBD = {}^tDB\}$$

and $\mathscr{B} = \{B \in \mathscr{A}_D;\, B = SD \text{ for some } S = {}^tS \in Mat_n(\mathbf{Z})\}$. Thus, we write $B_1 \equiv B_2 (mod\, D)$ if $B_1 - B_2 = SD$ for some $S = {}^tS$.

LEMMA 1.2. *A set of representatives for $\Gamma \backslash \Gamma_m$ is given by $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, where D runs over a fixed set of left nonassociated (with respect to $GL_n(\mathbf{Z})$) right divisors of $m = m1_n$, $A = m{}^tD^{-1}$, and B runs over representatives of $\mathscr{R}_D$.*

Proof. For completeness we will provide a proof of this standard result. There is a $\sigma = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Gamma$ such that $\sigma\rho = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ for $\rho \in \Gamma_m$. If

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix},$$

then $A_1 A = A'$, $C_1 A = 0$, $A_1 B + B_1 D = B'$, and $C_1 B + D_1 D = D'$. This implies that $C_1 = 0$ since $det\ A \neq 0$; so

$$A' = A_1 A \qquad \text{and} \qquad D' = D_1 D$$

with ${}^tA_1 D_1 = 1_n$, and hence $A_1, D_1 \in GL_n(\mathbf{Z})$. Also ${}^tD_1 B_1 = {}^tB_1 D_1$; so $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is uniquely determined up to $GL_n(\mathbf{Z})$ association and symmetric residue classes.

In view of this lemma, (1.3), and (1.1), we may write, for $\alpha = nk - \frac{1}{2}n(n+1)$ and $m \in \mathbf{Z}^+$,

$$\begin{aligned} T_m f(Z) &= m^\alpha \sum_{D|m} (det\ D)^{-k} \sum_{B(mod\,D)} f((m{}^tD^{-1}Z + B)D^{-1}) \\ &= m^\alpha \sum_{N>0} a(N) \sum_{D|m} (det\ D)^{-k} e(tr\ mD^{-1}N{}^tD^{-1}Z) \sum_{B(mod\,D)} e(tr\ NBD^{-1}) \end{aligned}$$

where the sums are over appropriate D and B. Thus, the coefficient of $e(tr\ MZ)$ in $T_m f(Z)$ is

$$a(mM) + O\left(m^\alpha \sum_{\substack{D|m \\ D \neq m}} (det\ D)^{-k} a(m^{-1}DM{}^tD) \# \mathcal{R}_D \right)$$

since $\# \mathcal{R}_m = m^{(1/2)n(n+1)}$. Using the Hecke-type bound (see [Rag, p. 450])

$$a(N) \ll_f (det\ N)^{k/2}$$

we get, since $\lambda_m a(M)$ is also the coefficient of $e(tr\ MZ)$ in $T_m f(z)$, that

$$\lambda_m a(M) - a(mM) \ll_f (det\ M)^{k/2} m^{nk/2 - n(n+1)/2} \sum_{\substack{D|m \\ D \neq m}} \# \mathcal{R}_D$$

for any $m \in \mathbf{Z}^+$.

Now for $m = p$ we have from Lemma 1.2 that

$$\sum_{D|p} \# \mathcal{R}_D = \# \Gamma \backslash \Gamma_p = \prod_{j=1}^{n} (1 + p^j)$$

as before, so that in fact

$$\sum_{\substack{D|p \\ D \neq p}} \# \mathcal{R}_D \ll p^{(1/2)n(n+1)-1}.$$

Thus, the estimate (1.4) follows.

By means of Proposition 1.1 and estimates for $a(pM)$, we can thus derive estimates for λ_p, provided $a(M) \neq 0$. As we shall see, even the error term $p^{nk/2-1}$ in (1.4), which is the limitation of the Fourier coefficient method as outlined here, is superceded by the techniques of this paper when $n > 3$. (It should be noticed, however, that estimates for general Fourier coefficients cannot be obtained from those for λ_p, which only control the growth of these coefficients along scalar multiples of a fixed positive matrix.)

The Rankin-Selberg method gives the best known bound for $a(N)$ when $n > 2$ in terms of *det N* (see [BR] and [Fo]):

$$a(N) \underset{\varepsilon}{\ll} (det\ N)^{k/2-\delta_n+\varepsilon} \tag{1.5}$$

where $\delta_n^{-1} = 2n + 4[n/2] + 2 + 2/(n+1)$ with $[\alpha]$ = integral part of α. From (1.4) and (1.5) we deduce by choosing M with $a(M) \neq 0$ that

$$\lambda_p \underset{\varepsilon}{\ll} p^{nk/2-n\delta_n+\varepsilon} \tag{1.6}$$

where $n\delta_n \to \frac{1}{4}$ as $n \to \infty$. In Corollary 4.3(a) below, it is proved that, if $n > 1$, then

$$|\lambda_p| \leqslant 2^n p^{nk/2-n(n+1)/12}, \tag{1.7}$$

which is clearly much stronger than (1.6) for all $n > 1$. When $n = 2$, (1.7) is essentially the same as that which follows from Kitaoka's [Ki] improvement of (1.5) gotten by the method of Poincaré series and Kloosterman sums, namely,

$$a(N) \underset{\varepsilon}{\ll} (det\ N)^{k/2-1/4+\varepsilon}.$$

However, for $n = 2^r$, $r \geqslant 1$, (1.7) is improved in Corollary 4.3(b) below to

$$|\lambda_p| \leqslant 2^n p^{nk/2-n(n+1)/8}, \tag{1.8}$$

and when $n = 2$, in Corollary 4.5 to

$$|\lambda_p| \leqslant 4p^{k-1}. \tag{1.9}$$

Without further assumptions, the exponent $k - 1$ in (1.9) cannot be improved since for f in the Maass space we have by the Saito-Kurokawa correspondence (see [EZ, p. 79] and reference there) that

$$\lambda_p = c_p + p^{k-1} + p^{k-2}$$

where $\sum_{n \geqslant 1} c_n e(nz)$ is a normalized Hecke eigenform of weight $2k - 2$ for $SL_2(\mathbf{Z})$. Thus, $\lambda_p = p^{k-1} + 0(p^{k-3/2})$. It has been conjectured (see [Kur]) that for f not in the Maass space

$$|\lambda_p| \leqslant 4p^{k-3/2}$$

holds, this being the Ramanujan conjecture for cusp forms of degree two.

2. From classical to adelic. In this section we reinterpret the Hecke operator T_p in terms of representation theory of the symplectic group with coefficients in the p-adic numbers $\mathbf{Q}_p$. In §3 we use this reinterpretation to get a bound on λ_p in terms of the asymptotics of matrix coefficients of representations. Most of this material, especially the first few lemmas, is known to experts. But since it is essential for our main results, we have given a fairly detailed account.

Let $G = GSp_{2n}$, $G^+(\mathbf{R})$ the connected component of $G(\mathbf{R})$, $\Gamma = Sp_{2n}(\mathbf{Z})$ as in §1. Let $\mathbf{A}$ be the ring of adeles [W] of $\mathbf{Q}$. For a finite prime p let $\mathbf{Q}_p$ be the p-adic numbers, $\mathbf{Z}_p$ the ring of p-adic integers, and $K_p = G(\mathbf{Z}_p)$. Let K_f be the product of all K_p, p finite. Using strong approximation [Kn] for Sp_{2n} and the fact that $\mathbf{Z}$ is a principal ideal domain, one obtains

$$G(\mathbf{A}) = G(\mathbf{Q})G^+(\mathbf{R})K_f. \tag{2.1}$$

We observe that $G^+(\mathbf{R})K_f$ is an open subgroup of $G(\mathbf{A})$ and that $G(\mathbf{Q}) \cap G^+(\mathbf{R})K_f = \Gamma$.

Suppose f is a Siegel cusp form of weight k with respect to Γ. We define a function ϕ_f on $G(\mathbf{A})$ as follows. Write $g \in G(\mathbf{A})$ as $g = \gamma g_\infty \kappa$ according to (2.1). Then set

$$\phi_f(g) = f(g_\infty(i)) \cdot j(g_\infty, i)^{-k}. \tag{2.2}$$

Here, i means $\sqrt{-1}$ times the identity matrix. Note that ϕ_f is a function on $G(\mathbf{Q})\backslash G(\mathbf{A})$ which is right invariant under K_f and the center of $G(\mathbf{A})$.

LEMMA 2.1. *Let* $a(p) = \begin{pmatrix} 1_n & 0 \\ 0 & p1_n \end{pmatrix}$ *which we consider as an element of* $G(\mathbf{Q})$ *or* $G(\mathbf{Q}_p)$. *If* $\{\gamma\}$ *is a set of coset representatives for* $\Gamma\backslash\Gamma_p$ *(see §1), then* $\{p \cdot \gamma^{-1}\}$ *is a set of coset representatives for* $K_p a(p) K_p / K_p$.

Proof. If $\gamma = \alpha \cdot a(p) \cdot \beta$ with $\alpha, \beta \in \Gamma$, then $p\gamma^{-1} = \beta^{-1}\begin{pmatrix} p1_n & 0 \\ 0 & 1_n \end{pmatrix}\alpha^{-1}$, which is clearly in $K_p a(p) K_p$. If $p\gamma_1^{-1} \equiv p \cdot \gamma_2^{-1} (mod\ K_p)$, then $\gamma_2\gamma_1^{-1} \in K_p$. We have $\gamma_2\gamma_1^{-1} \in Sp_{2n}(\mathbf{Q})$, and all the entries of $\gamma_2\gamma_1^{-1}$ are of the form t/p^c with $t \in \mathbf{Z}$, $c = 0$, or 1. The condition $\gamma_2\gamma_1^{-1} \in K_p$ insures that $t/p^c \in \mathbf{Z}$. Hence, $\gamma_2\gamma_1^{-1} \in \Gamma$, $\Gamma\gamma_1 = \Gamma\gamma_2$. This shows that the natural map $\gamma \mapsto p \cdot \gamma^{-1}$ from $\Gamma\backslash\Gamma_p$ to $K_p a(p) K_p / K_p$ is injective.

Let $k_1 a(p) k_2$ represent a right K_p-coset. We may assume $k_1 \in Sp_{2n}(\mathbf{Z}_p)$ and $k_2 = 1$. Let $\bar{k}_1$ be the image of k_1 in $Sp_{2n}(\mathbf{Z}_p/p\mathbf{Z}_p) = Sp_{2n}(\mathbf{Z}/p\mathbf{Z})$. Standard approximation results let us choose $\gamma_1 \in \Gamma$ with $\bar{\gamma}_1 = \bar{k}_1$. Then $\gamma_1^{-1}k_1 \equiv 1_{2n} (mod\ p)$ and $a(p)^{-1}\gamma_1^{-1}k_1 a(p) \in K_p$. It follows that the above map from $\Gamma\backslash\Gamma_p$ to $K_p a(p) K_p / K_p$ is also surjective. Q.E.D.

COROLLARY 2.2. *Let $\tilde{T}(p)$ be the operator on functions on $G(\mathbf{Q})\backslash G(\mathbf{A})$ defined by convolution on the right by the characteristic function of the double coset $X_p = K_p a(p) K_p$. Then, $\tilde{T}(p)\phi = \phi * ch_{X_p}$, where ch_X is the characteristic function of a set X and Haar measure is normalized to have $Vol(K_p) = 1$.* We have

$$p^{nk/2-n(n+1)/2} \cdot \tilde{T}(p)\phi_f = \phi_{T_p f}. \tag{2.3}$$

Proof. We calculate $\tilde{T}(p)\phi_f(g)$ for $g \in G(\mathbf{A})$.

By formula (2.1) we may assume $g \in G^+(\mathbf{R})$.

Let $\{\gamma\}$ be a set of coset representatives for $\Gamma\backslash\Gamma_p$. By the above lemma the set $\{p\gamma^{-1}\}$ represents the cosets $K_p a(p) K_p/K_p$. Let $(p\gamma^{-1})_p$ be the element of $G(\mathbf{A})$ which is equal to $p\gamma^{-1}$ at the pth component and 1 elsewhere. Since $g \in G^+(\mathbf{R})$, we have $g(p\gamma^{-1})_p = (p\gamma^{-1})_p g$. Hence,

$$\tilde{T}(p)\phi_f(g) = \sum_\gamma \phi_f((p\gamma^{-1})_p \cdot g).$$

But $(p\gamma^{-1})_p = p\gamma^{-1} \cdot (p\gamma^{-1})_\infty \cdot k$ with $k \in K_f$ (see (2.1)). Hence,

$$\begin{aligned}\tilde{T}(p)\phi_f(g) &= \sum_\gamma \phi_f((p^{-1}\gamma)_\infty g)\\ &= \sum_\gamma f((p^{-1}\gamma)_\infty g(i)) j((p^{-1}\gamma)_\infty g, i)^{-k}.\end{aligned}$$

Write $g(i) = z$. Since j is a factor of automorphy, we have

$$\begin{aligned}j((p^{-1}\gamma)_\infty g, i) &= j((p^{-1}\gamma)_\infty, z) \cdot j(g, i)\\ &= j(p^{-1}, \gamma(z)) j(\gamma, z) j(g, i).\end{aligned}$$

Of course, $j(p^{-1}, \gamma(z)) = 1$. Hence,

$$\begin{aligned}\tilde{T}(p)\phi_f(g) &= \sum_\gamma f(\gamma(z)) j(\gamma, z)^{-k} j(g, i)^{-k}\\ &= \sum_{\gamma \in \Gamma\backslash\Gamma_p} f|_\gamma(z) \cdot j(g, i)^{-k}.\end{aligned}$$

Comparing this with (2.2) and the definition of T_p (1.3), we obtain (2.3). Q.E.D.

Now assume the function ϕ_f defined by (2.2) generates an irreducible automorphic cuspidal representation $\pi = \otimes \pi_p$ of $G(\mathbf{A})$. (We refer to [B] for basic terminology concerning automorphic representations.) For each finite p the representation π_p is spherical with respect to K_p; that is, it admits a nonzero vector invariant under K_p. In fact, one checks using (2.1) and the Γ-invariance of f that ϕ_f is K_p-invariant for all p. Let A be the group of diagonal elements in G. Then π_p is determined by (the Weyl group orbit of) an unramified character χ of $A(\mathbf{Q}_p)$ (see [Sa]). We will call χ, or rather its orbit under the Weyl group, the *Satake parameter* of π_p. Let $\Phi_p = ch_{X_p}$ where $X_p = K_p a(p) K_p$, as in Corollary 2.2.

LEMMA 2.3. *The operator $\pi_p(\Phi_p)$ acts in the space of K_p-fixed vectors of π_p by the scalar*

$$\tilde{\lambda}_p = p^{n(n+1)/4} \cdot \sum_{a \in S} \chi(a) \tag{2.4}$$

where

$$S = \{a = diag(a_1, \ldots, a_n, pa_1^{-1}, \ldots, pa_n^{-1}) \,|\, a_i = 1 \text{ or } p\}.$$

Proof. Let P be the Borel subgroup of G consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

where a is an $n \times n$ upper triangular matrix, and b, d are whatever they must be. Thus, $A \subseteq P$. Let B be the Iwahori subgroup (see [I]) compatible with the choice of P, so that $P(\mathbf{Z}_p) \subseteq B$. Let W be the Weyl group for (A, G). Then

$$K_p = \bigcup_{w \in W} P(\mathbf{Z}_p) w B. \tag{2.5}$$

(We choose representatives of $w \in W$ in K_p.) One checks by an easy calculation that $a(p)^{-1} B a(p) \subseteq K_p$. Hence, (2.5) implies

$$K_p a(p) K_p = \bigcup_{w \in W} P(\mathbf{Z}_p) w(a(p)) K_p \tag{2.6}$$

where $w(a(p)) = wa(p)w^{-1}$. (Identify w with an element of K_p.)

Let us describe the Weyl group W in detail. Write a typical element of A as

$$\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & b_1 & & \\ & & & & \ddots & \\ & & & & & b_n \end{bmatrix}$$

where $a_1 b_1 = a_2 b_2 = \cdots = a_n b_n$. Let S_n be the symmetric group on n letters. Then S_n acts on A by permuting the coordinates $a_1, \ldots, a_n$. Note that since $a_1 b_1 =$

$a_2b_2 = \cdots = a_nb_n$, each $s \in S_n$ will also permute the coordinates $b_1, \ldots, b_n$ via the same permutation. Now S_n, acting on A via the above manner, is a subgroup of W. For $i = 1, 2, \ldots, n$, let σ_i be the automorphism of A which interchanges a_i and b_i, while leaving the other coordinates fixed. Let R_n be the group generated by σ_i, $1 \leqslant i \leqslant n$. Then R_n is of order 2^n and is another subgroup of W. In fact, W itself is the semidirect product of R_n and S_n.

The stabilizer of $a(p)$ in W (or more accurately, the stabilizer in W of the class of $a(p)$ in $A(\mathbf{Q}_p)/A(\mathbf{Z}_p)$) is nothing but S_n. Since $W = R_nS_n$, (2.6) gives

$$K_pa(p)K_p = \bigcup_{w \in R_n} P(\mathbf{Z}_p)w(a(p))K_p. \tag{2.7}$$

This is now a *disjoint* union. From our description of R_n, it is clear that the $w(a(p))$'s here are precisely the elements of S in (2.4).

We will show below that

$$Vol\, P(\mathbf{Z}_p)w(a(p))K_p = p^{n(n+1)/4} \cdot \delta(w(a(p)))^{-1/2} \tag{2.8}$$

where δ is the Jacobian for the adjoint action of A on the unipotent radical of P.

The representation π_p is the unique irreducible spherical subquotient of the induced representation $Ind_{P(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}\chi$ (see [C]). Here, *Ind* refers to unitary, or normalized, induction. Let $\phi_{K,\chi}$ be the K_p-invariant function in the space of this induced representation, normalized so that $\phi_{K,\chi}(1) = 1$. Then

$$\begin{aligned} \tilde{\lambda}_p &= \int_{K_pa(p)K_p} \phi_{k,\chi}(x)\, dx \\ &= \sum_{w \in R_n} p^{n(n+1)/4}\chi(w(a(p))) \end{aligned} \tag{2.9}$$

by (2.7) and (2.8). This is of course equivalent to (2.4). Q.E.D.

Proof of (2.8) Let N be the unipotent radical of P and write $a = w(a(p))$. We claim that the map $n \mapsto na$ gives rise to a bijection between $N(\mathbf{Z}_p)/N(\mathbf{Z}_p) \cap aN(\mathbf{Z}_p)a^{-1}$ and $P(\mathbf{Z}_p)aK_p/K_p$. It is clear that the map is well defined and injective. To see that it is also surjective, observe that $P(\mathbf{Z}_p) = N(\mathbf{Z}_p)A(\mathbf{Z}_p)$ and hence $P(\mathbf{Z}_p)aK_p = N(\mathbf{Z}_p)aK_p$. The claim follows. Thus, the volume of $P(\mathbf{Z}_p)aK_p$ is equal to the index $[N(\mathbf{Z}_p)\colon N(\mathbf{Z}_p) \cap aN(\mathbf{Z}_p)a^{-1}]$. Let Δ_n be the set of roots of A in (the Lie algebra of) N. For each root $\alpha \in \Delta_n$ there is a one-dimensional subgroup $N_\alpha \subseteq N$, and we have $N = \prod_{\alpha \in \Delta_n} N_\alpha$, the product being taken in any fixed order. See Steinberg [St] for these facts.

Thus, we have $N(\mathbf{Z}_p) = \prod_{\alpha \in \Delta_n} N_\alpha(\mathbf{Z}_p)$ and

$$[N(\mathbf{Z}_p)\colon N(\mathbf{Z}_p) \cap aN(\mathbf{Z}_p)a^{-1}] = \prod_{\alpha \in \Delta_n} [N_\alpha(\mathbf{Z}_p)\colon N_\alpha(\mathbf{Z}_p) \cap aN_\alpha(\mathbf{Z}_p)a^{-1}].$$

Let $|\cdot|$ denote the normalized p-adic norm [W]. One checks easily that

$$[N_\alpha(\mathbf{Z}_p)\colon N_\alpha(\mathbf{Z}_p) \cap aN_\alpha(\mathbf{Z}_p)a^{-1}] = \begin{cases} |\alpha(a)|^{-1}, & \text{if } |\alpha(a)| < 1 \\ 1, & \text{otherwise}. \end{cases}$$

It follows that the volume of $P(\mathbf{Z}_p)aK_p$ is equal to $(\prod_{\alpha \in \Delta_n(a)} |\alpha(a)|)^{-1}$, where $\Delta_n(a) \subseteq \Delta_n$ is the set of $\alpha \in \Delta_n$ with $|\alpha(a)| < 1$.

If a typical element of A is written as

$$\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & b_1 & & \\ & & & & \ddots & \\ & & & & & b_n \end{bmatrix}$$

as before, then the roots in Δ_n are given by $a_i a_j^{-1}$, $1 \leqslant i < j \leqslant n$, and $a_i b_j^{-1}$, $1 \leqslant i \leqslant j \leqslant n$. Now write $a = a' \cdot a''$ with

$$a' = \begin{bmatrix} a_1 & & & & & & & \\ & \ddots & & & & & & \\ & & a_{n-1} & & & & & \\ & & & 1 & & & & \\ & & & & b_1 & & & \\ & & & & & \ddots & & \\ & & & & & & b_{n-1} & \\ & & & & & & & 1 \end{bmatrix},$$

$$a'' = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & a_n & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & b_n \end{bmatrix}.$$

We will also identify a' with the element α

$$\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_{n-1} & & & \\ & & & b_1 & & \\ & & & & \ddots & \\ & & & & & b_{n-1} \end{bmatrix}$$

of $GSp_{2(n-1)}$. We have

$$\prod_{\alpha \in \Delta_n(a)} |\alpha(a)|^{-1} = \prod_{\alpha \in \Delta_{n-1}(a')} |\alpha(a)|^{-1} \cdot \Bigg(\prod_{\substack{i<n \\ |a_i|<|a_n|}} |a_i a_n^{-1}|^{-1} \Bigg) \cdot \Bigg(\prod_{\substack{i \leqslant n \\ |a_i|<|b_n|}} |a_i b_n^{-1}|^{-1} \Bigg).$$

Since $a = w(a(p)) \in S$, we have $a_i = 1$ or p, $a_i b_i = p$ for all i. From this one finds easily that

$$\Bigg(\prod_{\substack{i<n \\ |a_i|<|a_n|}} |a_i a_n^{-1}|^{-1} \Bigg) \cdot \Bigg(\prod_{\substack{i \leqslant n \\ |a_i|<|b_n|}} |a_i b_n^{-1}|^{-1} \Bigg) = |a_1 \dots a_n|^{-1}.$$

Hence,

$$\prod_{\alpha \in \Delta_n(a)} |\alpha(a)|^{-1} = \Bigg(\prod_{\alpha \in \Delta_{n-1}(a')} |\alpha(a)|^{-1} \Bigg) \cdot |a_1 \dots a_n|^{-1}.$$

Using induction on n, we obtain

$$\begin{aligned} \prod_{\alpha \in \Delta_n(a)} |\alpha(a)|^{-1} &= (|a_1| \cdot |a_1 a_2| \dots |a_1 \dots a_n|)^{-1} \\ &= (|a_1|^n \cdot |a_2|^{n-1} \dots |a_n|)^{-1}. \end{aligned}$$

On the other hand, using $a_i b_i = p$, we have

$$\begin{aligned} p^{n(n+1)/4} \cdot \delta(a)^{-1/2} &= p^{n(n+1)/4} \cdot \Bigg(\prod_{i<j} |a_i a_j^{-1}| \Bigg)^{-1/2} \cdot \Bigg(\prod_{i \leqslant j} |a_i b_j^{-1}| \Bigg)^{-1/2} \\ &= (|a_1|^n \cdot |a_2|^{n-1} \dots |a_n|^1)^{-1}. \end{aligned}$$

This proves (2.8).

3. Hecke eigenvalues and matrix coefficients. In this section we write G for $G(\mathbf{Q}_p)$; similar notations apply to subgroups of G.

Consider a number r, $1 \leqslant r < \infty$. We say that an admissible representation π of G with unitary central character has *L^r matrix coefficients* if, for every pair of smooth vectors u, v in the space of π, one has

$$\int_{G/Z} |(\pi(g)u, v)|^r \, dg < \infty .$$

(Here, Z denotes the center of G.)

Let π be an irreducible spherical representation of G with Satake parameter χ as above (see [Sa]). We write $\tilde{a}$ for $\begin{pmatrix} a & 0 \\ 0 & pa^{-1} \end{pmatrix}$, where a is the $n \times n$ matrix $a = diag(a_1, \ldots, a_n)$ with $a_i \in \mathbf{Q}_p^\times$.

LEMMA 3.1. *Let $1 \leqslant r < \infty$ and suppose π has L^r matrix coefficients. Then for any $\tilde{a} \in S$ as in* (2.4) *we have*

$$|\chi(\tilde{a})| < p^{(1-2/r)[(1/4)n(n+1)]}. \tag{3.1}$$

Proof. To prepare for the proof, we introduce some notations. Let Δ be the set of simple roots determined by the choice of P. Set

$$A^- = \{a \in A \,|\, |\alpha(a)| \leqslant 1 \text{ for all } \alpha \in \Delta\}.$$

Let π be an admissible representation of G with unitary central character. Let V be the space of π. Let V_N be its Jacquet module [C] with respect to N. Following Harish-Chandra, we call a character of A a *P-exponent* for π if it occurs as a subquotient of V_N. From Casselman [C, §4] we may deduce the following (see [C, Corollary 4.4.5] and the proof of [C, Theorem 4.4.5]).

LEMMA 3.2. *If π has L^r matrix coefficients, then for every P-exponent μ one has*

$$|\mu\delta^{-1/r}(a)| < 1 \qquad \textit{for all } a \in A^- \backslash ZA(\mathbf{Z}_p).$$

Let π and its Satake parameter be as before. Replacing χ by $\omega(\chi)$, $w \in W$, if necessary, we may assume that χ is in the negative Weyl chamber. That is, $|\chi(a)| \geqslant 1$ for all $a \in A^-$. The relation between π and its Satake parameter is that π is the subquotient of $Ind_P^G \chi$ containing the K_p-fixed vector. We claim that, with χ as specified, π is a subrepresentation of $Ind_P^G \chi$. To see this, we appeal to the Langlands classification [Si]. This implies that we may realize π as a subrepresentation of $Ind_{P_1}^G \pi_1$ where $P_1 \supseteq P$ is a suitable parabolic subgroup, with Levi decomposition $P_1 = M_1 N_1$, and $\pi_1 \simeq \sigma \otimes \nu$ where σ is a tempered representation of M_1 and ν is a positive-valued quasi character of M_1 whose restriction to A is in the negative Weyl chamber. Frobenius reciprocity [C] says that for π to be a subrepresentation of $Ind_{P_1}^G \pi_1$, the representation π_1 must also be spherical. Hence, σ is also spherical, and since it is tempered, it is the unique irreducible spherical subrepresentation of $Ind_{P\cap M_1}^{M_1}\chi_1$ for a suitable unramified unitary character χ_1 of A. Thus, π is a subrepresentation of $Ind_P^G\chi_1 \otimes \nu$. Hence, $\chi_1 \otimes \nu$ is also a Satake parameter for π; that is, χ and $\chi_1 \otimes \nu$ are in the same W-orbit. But $|\chi|$ and $|\chi_1 \otimes \nu| = \nu$ are both in the negative Weyl chamber; so they must be equal. It follows easily (see [Bk], §3.3, Prop. 1) that χ and $\chi_1 \otimes \nu$ are conjugate by an element of $W \cap M_1$. Therefore, $Ind_{P\cap M_1}^{M_1}\chi$ is equivalent to $Ind_{P\cap M_1}^{M_1}\chi_1 \otimes \nu$ (both being unitary), so that π is also a subrepresentation of $Ind_P^G\chi$.

Let $(\pi)_N$ denote the Jacquet module of π (see [C]). By Frobenius reciprocity (see [C, Theorem 2.4.1]) we obtain

$$Hom_A((\pi)_N, \chi\delta^{1/2}) \simeq Hom_G(\pi, Ind_P^G\chi) \neq 0.$$

Thus, $\chi\delta^{1/2}$ is a P-exponent of π, and we have by Lemma 3.2

$$|\chi\delta^{1/2-1/r}(a)| < 1, \qquad (a \in A^-\backslash ZA(\mathbf{Z}_p)). \tag{3.2}$$

Write a typical element of A as $\tilde{a} = \begin{pmatrix} a & 0 \\ 0 & ba^{-1} \end{pmatrix}$ with $a = diag(a_1, \ldots, a_n)$, $b \in \mathbf{Q}_p^\times$. Write $\chi(\tilde{a}) = |a_1|^{\nu_1}\ldots|a_n|^{\nu_n}\cdot|b|^s$, with $\nu_j \in \mathbf{C}$, $s \in \mathbf{C}$. The ν_j and s are defined modulo $2\pi i/log(p)$; in particular, the real parts $Re\ \nu_j$ and $Re\ s$ are well defined. Since the restriction of χ to Z is unitary, we have

$$Re(s) = -\frac{1}{2}Re(\nu_1 + \cdots + \nu_n). \tag{3.3}$$

Having χ in the negative Weyl chamber means

$$Re(\nu_1) \leqslant Re(\nu_2) \leqslant \cdots \leqslant Re(\nu_n) \leqslant 0. \tag{3.4}$$

Taking $\tilde{a} = \begin{pmatrix} pI_n & 0 \\ 0 & p^{-1}I_n \end{pmatrix} \in A^-$ in (3.2), we obtain

$$-Re(\nu_1 + \cdots + \nu_n) < \left(1 - \frac{2}{r}\right)\cdot\frac{n(n+1)}{2}. \tag{3.5}$$

Now let $\tilde{a} \in S$ as in (2.4). Then $a_i = 1$ or p, and we get

$$|\chi(a)| \leqslant p^{-Re(\nu_1+\cdots+\nu_n)}\cdot p^{-Re(s)}.$$

By (3.3)–(3.5) we obtain

$$|\chi(a)| < p^{(1-2/r)[(1/4)n(n+1)]}$$

as required. This concludes the proof of Lemma 3.1.

Now let f be a Siegel cusp form of weight k with respect to $\Gamma = Sp_{2n}(\mathbf{Z})$ and assume that the function ϕ_f generates the irreducible representation $\pi = \otimes \pi_p$. Let λ_p be the eigenvalue of T_p on f. From (2.3), (2.4), and Lemma 3.1, we obtain the following corollary.

COROLLARY 3.3. *If π_p has L^r matrix coefficients, then*

$$|\lambda_p| < 2^n \cdot p^{nk/2 - n(n+1)/2r}. \tag{3.6}$$

4. Estimates of matrix coefficients. In this section we let F be a local field of characteristic other than 2. Let $W_n = F^{2n}$ be endowed with a nondegenerate symplectic form and let $Sp(W_n)$ be the corresponding group of isometries. For each integer l with $1 \leqslant l \leqslant n$, we fix once for all a nondegenerate subspace $W_l \subseteq W_n$ of dimension $2l$ and let $Sp(W_l)$ be the corresponding symplectic group. Then $Sp(W_l) \subseteq Sp(W_n)$. In order to use some inductive arguments from [H1], we must also consider $\widetilde{Sp}(W_n)$, the metaplectic two-fold cover of $Sp(W_n)$. Henceforth, we will consider representations of $Sp(W_n)$ as being representations of $\widetilde{Sp}(W_n)$ trivial on the two-element kernel of the covering map $\widetilde{Sp}(W_n) \to Sp(W_n)$. For $1 \leqslant l \leqslant n$ let $\widetilde{Sp}(W_l)$ be the inverse image of $Sp(W_l)$ in $\widetilde{Sp}(W_n)$.

Let ρ be a unitary representation of $\widetilde{Sp}(W_n)$. We recall from [H1] the notion of *rank* of ρ. Consider the unipotent subgroup N of $Sp(W_n)$ consisting of matrices of the form

$$n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \tag{4.1}$$

where b is a symmetric $n \times n$ matrix with entries in F. It is well known that N can be uniquely lifted to a subgroup of $\widetilde{Sp}(W)$, which we again denote by N. The correspondence $b \to n(b)$ identifies N with the additive group of $n \times n$ symmetric matrices. Fix a nontrivial character ψ of F. The Pontrjagin dual $\hat{N}$ of N is also identified with the group of $n \times n$ symmetric matrices: to each such matrix β we associate the character ψ_β defined by

$$\psi_\beta(n(b)) = \psi(tr(\beta \cdot b)). \tag{4.2}$$

Here, tr denotes the trace of a matrix. Now the restriction of ρ to N is decomposed into a direct integral over $\hat{N}$; such a decomposition is determined by a projection-valued measure on $\hat{N}$. We call this measure the spectral measure of $\rho | N$. Following [H1], we will say that ρ has *N-rank* r if the spectral measure of $\rho | N$ is supported on the set of matrices of rank $\geqslant r$ but not (entirely) supported on the set of matrices of rank $> r$. We say that ρ has *pure rank* r if the spectral measure of $\rho | N$ is supported on a subset of the set of matrices of rank r.

Recall also that ρ is said to be *strongly* L^p if, for a dense set of vectors in the (Hilbert) space of ρ, the associated matrix coefficients are in $L^p(\widetilde{Sp}(W_n))$. The representation ρ is *strongly* $L^{p+\varepsilon}$ if it is strongly $L^{p'}$ for any $p' > p$. If ρ is admissible and strongly L^p, then one checks easily that matrix coefficients associated to smooth vectors are all in $L^p(\widetilde{Sp}(W_n))$.

LEMMA 4.1. *Let $l < n$. If $\rho|_{\widetilde{Sp}(W_l)}$ is strongly $L^{p+\varepsilon}$, where $p \leqslant 2k$, k a positive integer, then ρ itself is strongly $L^{q+\varepsilon}$, where $q = 2k([n/l] + 1)$. Here, $[x]$ denotes the largest integer not greater than x.*

Proof. This follows immediately from Proposition 8.3 of [H1] and the corollary on page 108 of [CHH].

LEMMA 4.2. *Suppose l divides n. Then we can take $q = 2kn/l$ in Lemma 4.1.*

Proof. This follows from the remark after the proof of Proposition 8.3 in [H1].

THEOREM 4.3. *Let ρ be a unitary representation of $\widetilde{Sp}(W_n)(n > 1)$ of pure rank n. Then*

(a) *if $n \neq 1, 2, 4$, then ρ is strongly $L^{6+\varepsilon}$;*
(b) *if $n = 2, 4$, then ρ is strongly $L^{8+\varepsilon}$.*

Proof. We use induction. Suppose it has been proven for $\widetilde{Sp}(W_m)$ with $2 \leqslant m < n$. Choose an integer l with $1 \leqslant l < n$. By Corollary 2.13 of [H1], $\rho|_{\widetilde{Sp}(W_l)}$ is a finite sum of representations of the form $\sigma \otimes \tau$, where σ is of pure rank l and τ is an $(n - l)$-fold tensor power of the oscillator representation. By induction, σ is strongly $L^{p(l)+\varepsilon}$ where

$$p(l) = \begin{cases} \infty, & l = 1 \\ 8, & l = 2, 4 \\ 6, & \text{otherwise}. \end{cases}$$

One can directly estimate matrix coefficients of the oscillator representation, to conclude that τ is strongly $L^{4l/(n-l)+\varepsilon}$ (see [H1, Proposition 8.1]). Thus, $\sigma \otimes \tau$, and hence $\rho|_{\widetilde{Sp}(W_l)}$ is strongly $L^{r+\varepsilon}$ with

$$r = \left(\frac{1}{p(l)} + \frac{n-l}{4l}\right)^{-1} \equiv r(l).$$

(We use the Jordan-Hölder inequality.)

Assume first $n \geqslant 12$. We choose $l \geqslant 5$. Then $r(l) = 12l/(3n - l)$. To get $q = 2k([n/l] + 1) \leqslant 6$ in Lemma 4.1, we need $n/l < 3$ and $k = 1$. Thus, $12l/(3n - l) \leqslant 2$. Hence, the conditions on l are

$$\frac{n}{3} < l \leqslant \frac{3}{7}n, \qquad l \geqslant 5.$$

Such an integer l always exists since $n \geqslant 12$. Thus, the theorem will follow once we prove it for $2 \leqslant n \leqslant 11$. But this can be proved in the same way as for $n \geqslant 12$, by choosing appropriate l. In the case l divides n we use Lemma 4.2 instead. The proof is indicated by the following table.

n	2	3	4	5	6	7	8	9	11
l	1	1	2	2	2	3	3	3	4
Lemma Used	4.2	4.2	4.2	4.1	4.2	4.1	4.1	4.2	4.1

PROPOSITION 4.4. *Assume $F \neq \mathbf{C}$. Suppose $n = 2^r (r \geqslant 1)$ and ρ is a representation of $Sp(W_n)$ (not $\widetilde{Sp}(W_n)$) of pure rank n. Then ρ is strongly $L^{4+\varepsilon}$.*

Proof. Throughout, we consider ρ as a representation of $\widetilde{Sp}(W_n)$. First, consider the case $n = 2$. By Corollary 2.12 of [H1], $\rho|_{\widetilde{Sp}(W_1)}$ is a finite sum of representations of the form $\sigma \otimes \omega$, where ω is an oscillator representation of $\widetilde{Sp}(W_1) = \widetilde{SL}(2)$ and σ is a representation of $\widetilde{SL}(2)$ of pure rank 1. Since ρ factors through $Sp(W_2)$, $\rho|_{\widetilde{Sp}(W_1)}$ must factor through $Sp(W_1)$. But it is well known that under ω the kernel of the covering $\widetilde{Sp}(W) \to Sp(W)$ acts by its unique nontrivial character. The same therefore must be true for σ; i.e., σ is a genuine representation [G] of $\widetilde{Sp}(W_1)$. It is well known, and easy to see, that any irreducible, genuine, unitary representation of $\widetilde{Sp}(W_1)$ is strongly $L^{4+\varepsilon}$. In fact, the genuine, irreducible unitary representation of $\widetilde{Sp}(W_1)$ with slowest decay of matrix coefficients is a piece of the oscillator representation. Thus σ, which is a direct integral of irreducibles, must also be strongly $L^{4+\varepsilon}$. It follows then that $\rho|_{\widetilde{Sp}(W_1)}$ is strongly $L^{2+\varepsilon}$; i.e., it is tempered. Applying Lemma 4.2, we see that ρ itself is strongly $L^{4+\varepsilon}$.

For general $n = 2^r (r > 1)$ we apply Lemma 4.2 with $l = 2^{r-1}$. From Corollary 2.12 of [H1], we see $\rho|_{\widetilde{Sp}(W_l)}$ is a finite sum of representations $\sigma \otimes \tau$ with σ a unitary representation of $\widetilde{Sp}(W_l)$ of pure rank l and τ an l-fold tensor product of oscillator representations. Since $l = 2^{r-1}$ is even, τ factors through $Sp(W_l)$. Since ρ is a representation of $Sp(W_n)$, σ must also factor through $Sp(W_l)$. Hence, we may assume σ to be strongly $L^{4+\varepsilon}$ by induction. But then Lemma 4.2 shows that ρ is strongly $L^{4+\varepsilon}$.

Remark. It is expected that every unitary representation of $\widetilde{Sp}(W_2)$ of pure rank 2 is strongly $L^{4+\varepsilon}$. Given this, the arguments above show that the representation ρ in Theorem 4.3 is $L^{6+\varepsilon}$ even for $n = 2, 4$; and in Proposition 4.4, ρ can be any unitary representation of $\widetilde{Sp}(W_n)$ of pure rank $n = 2^r$ (not necessarily factoring through $Sp(W_n)$).

COROLLARY 4.5. *Let ρ be any unitary representation of $Sp(W_n)$ $(n > 1)$ of pure rank n. Then*

(a) *ρ is strongly $L^{6+\varepsilon}$;*

(b) *if $n = 2^r (r \geqslant 1)$, then ρ is strongly $L^{4+\varepsilon}$.*

Proof. This follows trivially from Theorem 4.3 and Proposition 4.4.

Remark. There is no reason to think the cases $n = 2^r$ singled out in part (b) of Corollary 4.5 are in fact different from general n; the discrepancy between parts (a) and (b) seems to be purely an artifact of our method of proof. Thus, one expects that a representation ρ of $Sp(W_n)$ of pure rank n should be $L^{4+\varepsilon}$ for any n. Knowing this would give rise to corresponding improvements in Proposition 5.2, Corollary 5.3, and Proposition 5.6.

5. Conclusions and complements. We now come back to the setting of §2. Because of the close relationship between $G = GSp_{2n}$ and Sp_{2n}, Corollary 4.5 obviously implies the following proposition.

PROPOSITION 5.1. *Let p be a place of $\mathbf{Q}$. Let ρ be a unitary representation of $G(\mathbf{Q}_p)$ of pure rank n. We allow $p = \infty$ with $\mathbf{Q}_\infty = \mathbf{R}$. Then*

(a) *if $n > 1$, then ρ is strongly $L^{6+\varepsilon}$;*

(b) *if $n = 2^r$, $r \geqslant 1$ then ρ is strongly $L^{4+\varepsilon}$.*

PROPOSITION 5.2. *Let $\pi = \otimes \pi_p$ be any irreducible automorphic cuspidal representation of $G(\mathbf{A})$ (not necessarily coming from a classical Siegel cusp form).*

(a) *If $n > 1$, then each local component π_p has $L^{6+\varepsilon}$ matrix coefficients.*

(b) *If $n = 2^r$, $r \geqslant 1$, then each local component π_p has $L^{4+\varepsilon}$ matrix coefficients.*

Proof. Let N be the unipotent subgroup of G consisting of elements of the form (4.1) but now considered to be defined over $\mathbf{Q}$. The Pointrjagin dual of $N(\mathbf{Q})\backslash N(\mathbf{A})$ can be identified with the set of $n \times n$ symmetric matrices β with entries in $\mathbf{Q}$ (see (4.2)). Let f be a nonzero smooth function in the space of π. For each fixed $g \in G(\mathbf{A})$ the function $n \mapsto f(ng)$ is defined on $N(\mathbf{Q})\backslash N(\mathbf{A})$, and is of course smooth. Hence, it has a Fourier expansion of the form

$$f(ng) = \sum_{\beta} \psi_\beta(n) f_\beta(g)$$

where β runs through the set of $n \times n$ symmetric matrices with entries in $\mathbf{Q}$. Since π is cuspidal, the main result of [L2] says that there is at least one β of full rank n, for which the corresponding Fourier coefficient $f_\beta(g)$ is nonzero. By Lemma 2.4 of [H3], this implies that each component π_p is of pure rank n. Now use Proposition 5.1.

Remark. This proposition is valid for any number field (instead of $\mathbf{Q}$).

Combining Proposition 5.2 with Corollary 3.3, we obtain the following corollary.

COROLLARY 5.3. *Let f be a Siegel cusp form of weight k as in §2. Assume that it is an eigenform under the Hecke operator T_p, with eigenvalue λ_p.*

(a) *If $n > 1$, then*

$$|\lambda_p| \leqslant 2^n \cdot p^{nk/2 - n(n+1)/12}.$$

(b) *If $n = 2^r$, $r \geqslant 1$, then*

$$|\lambda_p| \leqslant 2^n \cdot p^{nk/2 - n(n+1)/8}.$$

For $n = 2^r$ our estimate of matrix coefficients for cusp forms is essentially sharp (see Example 5.8 below), but the estimate for the Hecke eigenvalue λ_p can be improved somewhat, at least for $n = 2$.

Let A be the group of diagonal elements as in §2. Let Σ be the set of roots of A in G. We use other notations as in §2. Let p be a finite prime. For each $\alpha \in \Sigma$ we let $\check{\alpha}$ be the corresponding coroot and set $a_\alpha = \check{\alpha}(p) \in A(\mathbf{Q}_p)$. Of course, a_α ought to be considered a class in $A(\mathbf{Q}_p)/A(\mathbf{Z}_p)$.

PROPOSITION 5.4. *Let $n = 2$. Let ρ be an irreducible, spherical unitary representation of $G(\mathbf{Q}_p)$ which is not one-dimensional. Let χ be the Satake parameter for ρ (see §2). Then*

$$|\chi(a_\alpha)| \leqslant p \qquad \text{for any } \alpha \in \Sigma. \tag{5.1}$$

Proof. This may be seen by inspection of the classification of the unramified unitary dual of $G(\mathbf{Q}_p)$ given by Rodier; see [R, §7.2].

Note that the representation ρ in the above Proposition is necessarily of rank 2 since by [H1], any (irreducible unitary) representation of $G(\mathbf{Q}_p)$ of rank 0 must be one-dimensional, and any representation of rank 1 must be a component of the oscillator representation which can only live on the metaplectic cover of $G(\mathbf{Q}_p)$.

COROLLARY 5.5. *Let $n = 2$ and f a Siegel cusp form of weight k and eigenvalues λ_p. Then*

$$|\lambda_p| \leqslant 4p^{k-1}.$$

Proof. As in the proof of Lemma 3.1, we write a typical element of A as $\tilde{a} = diag(a_1, a_2, a_1^{-1}b, a_2^{-1}b)$. Write $\chi(\tilde{a}) = |a_1|^{\nu_1}|a_2|^{\nu_2}|b|^s$ and assume χ is in the negative Weyl chamber. For any $\tilde{a} \in S$ as in Lemma 2.3, we have by (3.4) and (3.5)

$$|\chi(\tilde{a})| \leqslant p^{-(1/2)Re(\nu_1+\nu_2)}.$$

For α the root given by $\alpha(\tilde{a}) = a_1 a_2 b^{-1}$, we have $a_\alpha = diag(p, p, p^{-1}, p^{-1})$, and (5.1) gives $p^{-(\nu_1+\nu_2)} \leqslant p$. Hence, $|\chi(\tilde{a})| \leqslant p^{1/2}$ for each $\tilde{a} \in S$. This together with (2.3) and (2.4) gives the result as claimed.

PROPOSITION 5.6. *Let $\pi = \otimes \pi_p$ be any irreducible automorphic cuspidal representation of $Sp_{2n}(\mathbf{A})$. Let S be a finite set of places outside of which the representations π_p are unramified. Let*

$$L^S(s, \pi) = \prod_{p \notin S} L(s, \pi_p) \tag{5.2}$$

be the partial Langlands L-function attached to the $(2n+1)$-dimensional representation of ${}^LG$ (see [B]).

(a) *The Euler product (5.2) is absolutely convergent for $Re(s) > \frac{2}{3}n + 1$.*
(b) *If $n = 2^r$, $r \geqslant 1$, then (5.2) is absolutely convergent for $Re(s) > n/2 + 1$.*

Proof. Assume $p \notin S$. Let us review the definition of $L(s, \pi_p)$. Let A, P be as in §2–4, and A_1, P_1 their intersections with Sp_{2n}. The representation π_p is determined by an unramified character χ of A_1. Write χ as

$$\chi\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & a_n^{-1} & & \\ & & & & \ddots & \\ & & & & & a_n^{-1} \end{bmatrix} = |a_1|^{\nu_1}\dots|a_n|^{\nu_n};$$

then

$$L(s, \pi_p) = (1 - p^{-s})^{-1} \prod_{i=1}^{n} [(1 - p^{-s-\nu_i})(1 - p^{-s+\nu_i})]^{-1}. \tag{5.3}$$

Now suppose π_p has L^r matrix coefficients. We may assume that χ is in the negative Weyl chamber, so that

$$Re(\nu_1) \leqslant \cdots \leqslant Re(\nu_n) \leqslant 0.$$

By the obvious analogue of formula (3.2), we obtain

$$|\chi\delta^{1/2-1/r}(a)| < 1 \qquad (a \in A_1^- - A_1(\mathbf{Z}_p)) \tag{5.4}$$

where $A_1^- = A_1 \cap A^-$. Taking $a = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}$ with $\delta = diag(p, 1, \dots, 1)$ in (5.4), we find

$$|Re(\nu_i)| < \left(1 - \frac{2}{r}\right)n, \qquad i = 1, 2, \dots, n. \tag{5.5}$$

Hence, we have the following lemma.

LEMMA 5.7. *If all π_p, $p \notin S$, have L^r matrix coefficients, then $L^S(\pi, s)$ converges for $Re(s) > (1 - 2/r)n + 1$.*

Now the proposition follows from Proposition 5.2.

Example 5.8. Assume that n is even. Let O_n be an orthogonal group associated to a nondegenerate quadratic form in n variables. Let $\sigma = \otimes\sigma_p$ be a one-dimensional automorphic representation of $O_n(\mathbf{A})$ which is trivial on $SO_n(\mathbf{A})$. Assume that $O_n(\mathbf{R})$ is compact and σ_∞ is nontrivial. Then σ can be theta lifted to a nontrivial cuspidal automorphic representation $\pi = \theta(a)$ of $Sp_{2n}(\mathbf{A})$ (see [HPS] for the case $n = 2$). Let S be the finite set of places of $\mathbf{Q}$ such that at $p \notin S$, O_n is unramified and σ_p is trivial.

Then $\pi_p = \theta(a)_p$ is spherical. Let χ_p be the Satake parameter for π_p. From work of Rallis [R], one has

$$(5.6) \qquad \chi_p \begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & a_n^{-1} & & \\ & & & & \ddots & \\ & & & & & a_n^{-1} \end{bmatrix}$$

$$= |a_1| \cdot |a_2|^2 \ldots |a_{n/2}|^{n/2} \cdot |a_{n/2+1}|^{n/2-1} \ldots |a_{n-1}|^1 \cdot \nu(a_1 \ldots a_n)$$

where ν is the unique unramified quadratic character of $\mathbf{Q}_p^\times$ if O_n is nonsplit over $\mathbf{Q}_p$ and is trivial otherwise.

From this it follows that, for $p \notin S$, π_p is $L^{4+\varepsilon}$ but not L^4, and $L^S(\pi, s)$ converges absolutely for $Re(s) > n/2 + 1$ but not for $Re(s) = n/2 + 1$. Thus, for $n = 2^r$ the estimates of Proposition 5.2 and 5.6 are sharp. From (5.6) we can also see that the estimate of Corollary 5.5 is sharp.

Finally, let $\pi = \otimes \pi_p$ be a unitary automorphic representation of $Sp_{2n}(\mathbf{A})$ (or $GSp_{2n}(\mathbf{A})$) which is singular in the sense of Maass [M]. Then there is an even integer $l < n$ so that each component π_p is of rank l in the sense of [H1] (see [H3]). By Howe [H2] and Li [L1], there must be an orthogonal group O_l in l variables, and an irreducible unitary representation σ_p of $O_l(\mathbf{Q}_p)$ for each p, such that π_p is the local theta lift of σ_p. If π_p is spherical (p finite), then the Satake parameter of π_p is related to that of σ_p given by a formula of Rallis [R]. From that we obtain (we omit the routine details of proof) the following proposition.

PROPOSITION 5.9. *Suppose π is singular of rank $l < n$. Then each π_p has $L^{4n/l+\varepsilon}$ matrix coefficients.*

PROPOSITION 5.10. *Suppose f is a Siegel modular form of weight $k < n/2$ (hence singular of rank $2k < n$ by [H3]) which is an eigenvector for the Hecke operators T_p with eigenvalues λ_p. Then*

$$|\lambda_p| \leqslant 2^n \cdot p^{k(n-1)/4}.$$

Remark. In Proposition 5.9, if $l > 2n/3$, then $4n/l < 6$, so that we can establish stronger decay of matrix coefficients for singular representations of rank l in the range $2n/3 < l < n$ than for representations of rank n. This anomalous situation is further evidence that Corollary 4.5 is not optimal and that we should expect matrix coefficients in $L^{4+\varepsilon}$ for any representation of $Sp(W_n)$ of pure rank n. This would give us corresponding better estimates on the λ_p for cusp forms.

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