A DIOPHANTINE DIVISOR PROBLEM AND HECKE ZETA FUNCTIONS

OLGA BALKANOVA, WILLIAM DUKE, AND DMITRY FROLENKOV

Abstract. Various results about a divisor function with Diophantine properties are obtained, including a simple asymptotic formula for its sum and a Voronoï-type formula. The proofs rely on analytic properties of certain Dirichlet series that are expressed in terms of Hecke's zeta functions with Grössencharaktere associated to a real quadratic number field. Also used are new estimates and asymptotics for the standard hypergeometric function that are uniform in parameters, which are of independent interest.

1. Introduction and statement of main results

Let $d(n)$ be the usual divisor function. A method of Dirichlet [12] from 1849 leads to the identity

$$\Delta(x) := \sum_{n\le x} d(n) - x\log x - (2\gamma-1)x = -2\sum_{1\le d\le x^{\frac12}} \rho\left(\frac{x}{d}\right) + O(1), \tag{1.1}$$

where γ is Euler's constant and ρ is the sawtooth function defined for $x\in\mathbb{R}$ by

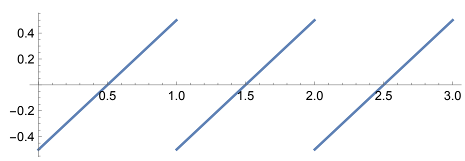

Figure 1. Sawtooth function $\rho(x)$

$$\rho(x) = \{x\} - \frac12 = x - \lfloor x\rfloor - \frac12. \tag{1.2}$$

This identity immediately implies the first result in the divisor problem, which is the bound

$$\Delta(x) = O(x^{\frac12}).$$

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In 1903 Voronoï [50] improved this estimate by showing

$$\sum_{1\le d\le x^{\frac12}} \rho\left(\frac{x}{d}\right) = O(x^{\frac13}\log x),$$

still using an elementary method. Essentially the same bound was obtained using a different and more general elementary method in 1917 by Vinogradov in his first paper [49]. See [10, p.165 and p. 213] for expositions of both methods.

In 1904 Voronoï [51] took an analytic approach and established a formula for $\Delta(x)$ that better displays its oscillatory behavior with respect to $x^{\frac14}$. It can be expressed in the form

$$\sum_{n\le x}\big(d(n)-\log n-2\gamma\big) = \frac{x^{\frac14}}{\pi\sqrt{2}}\sum_{n\ge1}\frac{d(n)}{n^{\frac34}}\cos(4\pi\sqrt{nx}-\frac14\pi)+O(\log x). \tag{1.3}$$

This formula reflects the functional equation of the Dirichlet series

$$\zeta^2(s) = \big(\sum_{n\ge1} n^{-s}\big)^2 = \sum_{n\ge1} d(n)n^{-s}.$$

It also implies $\Delta(x) = O_\epsilon(x^{\frac13+\epsilon})$, by truncating the right hand side of (1.3) at $n = x^{\frac13}$ and estimating the remainder. A number of relatively small, but quite difficult, successive improvements of the one third exponent have been given using estimates of exponential sums. The latest results are 131/416 due to Huxley [31] and 0.3144.. due to Li–Yang [36], while the conjecture is

$$\Delta(x) = O_\epsilon(x^{\frac14+\epsilon}),$$

which would be essentially best possible. See [32] for a survey and references on the Dirichlet divisor problem. A standard general reference covering the divisor problem is [48].

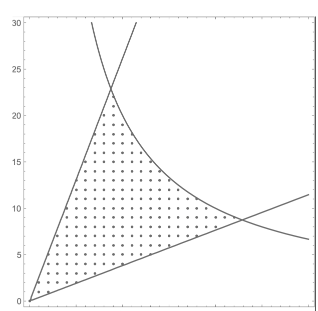

Figure 2. Diophantine divisor function

In this paper we will consider a divisor function that counts divisors that are restricted in a certain way. For a fixed $\alpha \ge 1$ define

$$d(n,\alpha) = \#\{d \mid n;\ \alpha^{-1}n \le d^2 \le \alpha n\}. \tag{1.4}$$

Thus $d(n,\alpha)$ counts divisors of n that are quite close together; it can be written as

$$d(n,\alpha) = \#\{n = d_1 d_2;\ \alpha^{-1} d_1 \le d_2 \le \alpha d_1\}. \tag{1.5}$$

As with the usual divisor function, let

$$\Delta(x,\alpha) = \sum_{n \le x} d(n,\alpha) - x\log\alpha. \tag{1.6}$$

This sum counts the difference between the number of lattice points in a sector under the hyperbola of the type shown in Figure 2, and its area. A standard argument yields the trivial estimate

$$\Delta(x,\alpha) = O_\alpha(x^{\frac{1}{2}}). \tag{1.7}$$

As Figure 2 suggests, this problem is closely related to both the standard Dirichlet divisor problem (the limit case $\alpha \to \infty$, informally speaking) and the Hardy-Littlewood (H-L) problem of counting lattice points in a right triangle. The H-L problem reduces to understanding the sum[1]

$$S(x,\alpha) = \sum_{n \le x} \rho(n\alpha).$$

Our first result makes precise this connection.

Theorem 1.1. *Let $\Delta(x,\alpha)$ be given in (1.6). For $x,\alpha \ge 1$ we have*

$$\Delta(x,\alpha) = -2S\left((x/\alpha)^{\frac{1}{2}}, \alpha\right) - 2 \sum_{(\frac{x}{\alpha})^{\frac{1}{2}} < d \le x^{\frac{1}{2}}} \rho\left(\frac{x}{d}\right) + O(\alpha) \tag{1.8}$$

with an absolute implied constant.

While this result clearly implies the trivial bound (1.7), Vinogradov's method already mentioned applies directly to give the following.

Corollary 1.2. *For any $\epsilon > 0$ we have that*

$$\Delta(x,\alpha) = -2S\left((x/\alpha)^{\frac{1}{2}}, \alpha\right) + O_{\alpha,\epsilon}(x^{\frac{1}{3}+\epsilon}). \tag{1.9}$$

Remark. Further improvements are possible using the methods of Huxley [31] and Li–Yang [36], which give the same estimates for the remainder term in (1.9) as in the classical Dirichlet divisor problem.

The size of the sum $S(x,\alpha)$ depends essentially on the arithmetic (Diophantine) nature of α. The occurrence of $S((\frac{x}{\alpha})^{\frac{1}{2}}, \alpha)$ in (1.8) indicates that the behavior of $\Delta(x,\alpha)$ should also depend on the arithmetic nature of α and not merely its size. This is the reason for calling our divisor problem "Diophantine". Also, we emphasize that α is fixed.

[1] Useful general references for the sum $S(x,\alpha)$ are [34, §2 p.102] and the recent book [4].

Known properties of $S(x, \alpha)$ imply corresponding results for $\Delta(x, \alpha)$. For any $\epsilon > 0$ we deduce from [21, §3] (see also [33]) that irrational α exist so that

$$S(x, \alpha) = O(x^{1-\epsilon})$$

does not hold. Thus by the Corollary of Theorem 1.1, for any $\epsilon > 0$ there exist irrational α for which

$$\Delta(x, \alpha) = O(x^{\frac{1}{2}-\epsilon}) \tag{1.10}$$

does not hold. That is, for general irrational α we cannot essentially improve the trivial bound (1.7).

If the simple continued fraction expansion of irrational α has bounded partial quotients, which includes quadratic irrational α, then we have the essentially optimal results

$$S(x, \alpha) = O(\log x) \quad \text{and} \quad S(x, \alpha) = \Omega_{\pm}(\log x), \tag{1.11}$$

which were proven independently by Hardy-Littlewood [21], [22] and Ostrowski [41]. Thus for such α, by the above Corollary

$$\Delta(x, \alpha) = O_{\epsilon}(x^{\frac{1}{3}+\epsilon}). \tag{1.12}$$

Just like analytic properties of

$$\zeta^2(s) + \zeta'(s) - 2\gamma\zeta(s)$$

influence the growth of $\Delta(x)$ through Voronoï's formula, analytic properties of the Dirichlet series

$$\phi(s, \alpha) = \sum_{n \geq 1} (d(n, \alpha) - \log \alpha) n^{-s} \tag{1.13}$$

affect the behavior of $\Delta(x, \alpha)$. Hardy-Littlewood [24] and Behnke [5] studied the corresponding relationship between $S(x, \alpha)$ and the Dirichlet series

$$\psi(s, \alpha) = \sum_{n \geq 1} \rho(n\alpha) n^{-s}. \tag{1.14}$$

For "most" α, the function $\psi(s, \alpha)$ has a natural boundary, whose location is determined by the simple continued fraction of α.

The analysis of sums like $S(x, \alpha)$ and $\Delta(x, \alpha)$ using analytic properties of ψ and ϕ is greatly simplified through smoothing. It is natural to use the standard Riesz means

$$\Delta_r(x, \alpha) = \sum_{n \leq x} (d(n, \alpha) - \log \alpha) \left(1 - \frac{n}{x}\right)^r \tag{1.15}$$

and

$$S_r(x, \alpha) = \sum_{n \leq x} \rho(n\alpha) \left(1 - \left(\frac{n}{x}\right)^2\right)^r. \tag{1.16}$$

In general, one strives to work with such means with $r \geq 0$ as small as possible, thereby approximating the case $r = 0$ as closely as possible. For any

$r \geq 0$, the behavior of both $\Delta_r(x,\alpha)$ and $S_r(x,\alpha)$ still depend significantly on Diophantine properties of α.

One can say much more about the sums if α is real quadratic. Here we assume that α has the form

$$\alpha = \frac{a+\sqrt{a^2-4}}{2}, \tag{1.17}$$

where $a \in \mathcal{A}$ and

$$\mathcal{A} = \{a |\, a > 3,\ D = a^2-4 \ \text{ is the discriminant of a real quadratic field}\}. \tag{1.18}$$

The first few elements of the set $\mathcal{A}$ are

$$\mathcal{A} = \{4, 5, 8, 9, 12, 13, 15, 17, 19, 21, 24, 28, 31, 32, \ldots\}. \tag{1.19}$$

The simple continued fraction of α is

$$\alpha = a-1+\frac{1}{1+}\ \frac{1}{(a-2)+}\ \frac{1}{1+}\ \frac{1}{(a-2)+}\cdots. \tag{1.20}$$

Note that α is the fundamental unit in $\mathbb{F} = \mathbb{Q}(\sqrt{D})$ and that α is totally positive. Now $\psi(s,\alpha)$ from (1.14) has a meromorphic continuation to $\mathbb{C}$ with infinitely many poles. For even D their positions, degrees and residues were determined by Hecke using his Zeta functions with Grössencharakteren (see [28, p.63, p. 323 in Werke]) and his proof immediately extends to cover all α with $a \in \mathcal{A}$. For these α we will see that $\phi(s,\alpha)$ from (1.13) also has a meromorphic continuation with infinitely many poles whose positions, degrees, and residues we determine.

Although Theorem 1.1 provides the relation between the sums $\Delta_0(x_1,\alpha)$ and $S_0(x_2,\alpha)$, it is not easy to extract simple asymptotic information from it as a result of the behavior of the second sum on the RHS. However, due to a remarkable coincidence between the poles on the imaginary axis of $\phi(s,\alpha)$ and those of $\psi(2s,\alpha)$, it is possible to construct a combination of $\Delta_r(x,\alpha)$ and $S_r(\sqrt{\alpha x},\alpha)$ so that the resulting asymptotic formula is simple. We give two such asymptotic formulas under different assumptions on r.

Theorem 1.3. *For $x \to +\infty$, $a \in \mathcal{A}$, α as in (1.17), $r > 1$ and any sufficiently small $\epsilon > 0$ (say $0 < \epsilon < (r-1)/100$) we have*

$$\Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x},\alpha) = C(a) + O_a(x^{-\epsilon}), \tag{1.21}$$

where

$$C(a) = \frac{a-3}{12} + \frac{1}{2}\log\alpha - \frac{\sqrt{D}}{12}. \tag{1.22}$$

Theorem 1.4. *For $x \to +\infty$, $a \in \mathcal{A}$, α as in (1.17), $r > 3$ and any sufficiently small $\epsilon > 0$ we have*

$$\Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x},\alpha) = C(a) + O_a(x^{-1+\epsilon}). \tag{1.23}$$

In the table below we provide some numerical computations to illustrate the results of Theorems 1.3 and 1.4. To this end, let us define for $a \in \mathcal{A}$ and α as in (1.17) the error term function

$$E(r,a,x) = \Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x},\alpha) - C(a). \tag{1.24}$$

r	$E(r,8,10^5)$	$E(r,8,10^6)$	$E(r,9,10^5)$	$E(r,9,10^6)$
3	-1.93245e-05	-2.64042e-06	-2.80304e-05	-2.74769e-06
2	-9.07321e-06	-1.58076e-06	-3.22359e-05	-7.08911e-07
1	0.00185194	-0.000413826	0.00521695	-0.000775616
0.6	0.0357489	0.0328198	0.0758085	0.0195105
0.5	0.0679295	0.121604	0.135241	0.0879656
0.4	0.120632	0.412836	0.227332	0.327092
0.25	0.223755	2.38438	0.396916	2.03341
0.1	0.0787089	13.2703	0.166314	11.8602
0.01	-0.710027	37.1763	-1.04555	34.029
0	5.06076	48.639	4.51732	45.086

Table 1

A heuristic argument given in the remark at the end of section 8 suggests that Theorem 1.3 might hold with the condition $r > 1$ replaced by $r > 1/2$. This possible improvement also appears to be consistent with the numerics in Table 1. Furthermore, it seems unlikely that the statement of Theorem 1.3 holds for each $r > r_0$ if $r_0 < 1/2$.

Still for α from (1.17) with D even, Hecke was able bound $S(x,\alpha)$ using the analytic properties of $\psi(s,\alpha)$ together with a clever and intricate application of (a slight extension of) the Schnee-Landau theorem. He showed [28, (2) p.55, p.314 in Werke]

$$S(x,\alpha) = O_\epsilon(x^\epsilon) \quad \text{for all } \epsilon > 0. \tag{1.25}$$

This is weaker than (1.11), but the ingredients used in the proof can also be applied to obtain results about the averages like $S_r(x,\alpha)$ that are inaccessible for general α [28, p. 71, p. 330 in Werke]. It is worth noting that in his proof of (1.25), Hecke made strong use of the fact that $\psi(s,\alpha)$ has relatively mild ("degree one") growth in vertical strips away from poles. We will see that $\phi(s,\alpha)$ from (1.13) has "degree two". The meaning of this will become clear. Roughly speaking, it is the difference between $\zeta^2(s)$ and $\zeta(s)$. This causes significant new difficulties in the study of our divisor problem over the H-L problem. It is for this reason we work mostly with Riesz means. Another new difficulty that appears has to do with the appearance of certain hypergeometric functions in the representation of $\phi(s,\alpha)$. The exact nature of the difficulty will be explained later.

Our next result is a weighted Voronoï-type formula for $\Delta_r(x,\alpha)$ with α from (1.17). This formula comes out readily from the proof of Theorem 1.3.

Theorem 1.5. *For $x \to +\infty$, $a \in \mathcal{A}$ with $a > 3$ and α as in* (1.17)*, $r > 1$ and any sufficiently small $\epsilon > 0$ (say $0 < \epsilon < (r-1)/100$) we have*

$$\begin{aligned}(1.26)\quad \Delta_r(x,\alpha) = \frac{3-a}{12\log\alpha}\log x + C_r(a) - \frac{\Gamma(1+r)}{4\log\alpha}\times \\ \times \sum_{n\neq 0}\frac{(-1)^n\Gamma(in\kappa)}{\Gamma(1+r+in\kappa)}(xD)^{in\kappa}\left(\zeta(in\kappa,\nu_1\lambda^n)+\zeta(in\kappa,\nu_1\lambda^{-n})\right) + O_a(x^{-\epsilon}),\end{aligned}$$

where $\zeta(s,\nu_j\lambda^n)$ is the Hecke zeta function defined by (4.4)*, $\kappa = \pi/\log\alpha$, $\psi(z)$ - digamma function and*

$$\begin{aligned}(1.27)\quad C_r(a) = \frac{3-a}{12\log\alpha}\Big(-\psi(1+r) - \psi(1/2) - \gamma + \log(\pi\sqrt{D})\Big) + \\ + \log\sqrt{\alpha} - \frac{1}{4} - \frac{\zeta'(0,\nu_1)}{2\pi\log\alpha}.\end{aligned}$$

For $r > 3$ one can replace the error term $O_a(x^{-\epsilon})$ in (1.26) *by $O_a(x^{-1+\epsilon})$.*

By trivially estimating the infinite sum on the right hand side of the formula we get for $r > 1$ that

$$(1.28)\qquad \sum_{n\le x}\left(1-\frac{n}{x}\right)^r\left(d(n,\alpha) - \log\,\alpha\right) = \frac{3-a}{12\log\alpha}\log x + O(1).$$

It follows, in particular, that $|\Delta(x,\alpha)|$ is unbounded. More precisely, there is a $C > 0$ so that

$$\Delta(x,\alpha) < -C\log x$$

for arbitrarily large x. See e.g. [25].

Other kinds of restricted divisor functions have been introduced and studied. For some examples see [52], [11] (also [47, p.207]), [45], [18], [20], [2], listed chronologically. A number of different generalizations and applications of Hecke's original method are given in the papers [5], [6], [14], [1].

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2. Outline and Discussion of methods

In this section we outline the structure of the paper and then discuss a few of the main ideas and methods used in the proofs of our theorems, in particular Theorem 1.3.

One of these methods, a new precise hypergeometric asymptotic formula, has independent interest and for convenience we will state it at the end of this section.

Theorem 1.1 and its corollary are proven in the next section. The proof of Theorem 1.1 is elementary and may be viewed as a development of Dirichlet's proof of (1.1). The proof of the corollary is a standard application of a method of Vinogradov.

The groundwork for the proofs of Theorems 1.3, 1.4 and 1.5 is begun in section 4, where the definitions and basic properties of Hecke's zeta functions are explained. For the most part we refer to Hecke's original papers but also to some useful papers of Rademacher.

In section 5 we present the core results on the function $\phi(s,\alpha)$. We show how to represent $\phi(s,\alpha)$ as an infinite sum involving Hecke zeta functions, gamma functions and hypergeometric functions. This yields some information about the poles of $\phi(s,\alpha)$.

Section 6 is devoted to the properties of $\psi(s,\alpha)$ and is based on Hecke's paper [28]. In section 7, we prove the required estimates on the hypergeometric function. In section 8, we establish Theorems 1.3, 1.4 and 1.5.

The first step in the proof of Theorem 1.3 is to construct a combination of the sums $\Delta_r(x_1,\alpha)$ and $S_r(x_2,\alpha)$ such that the corresponding combination of the functions $\phi(s_1,\alpha)$ and $\psi(s_2,\alpha)$ does not have poles at the points $in\kappa$. Computations performed in [28] and section 5 suggested us to use slightly different smooth factors in (1.15) and (1.16). The reason for this is that the functions $\phi(s,\alpha)$ and $\psi(2s,\alpha)$ have poles at the same points. Although we have managed to construct such a combination, it turns out that it still has infinitely many poles at the points $-j+in\kappa$, $j\geq 1$. This is why we are not able to improve the error term $O(x^{-1+\epsilon})$ in Theorem 1.4 even for sufficiently large r. An interesting question is whether it is possible to construct a more complex combination of $\Delta_r(x_1,\alpha)$ and $S_r(x_2,\alpha)$ that will not have poles at $-j+in\kappa$ for $0\leq j\leq J$.

The next step is to study the left hand side of (1.21) using the results of [28] and section 5. To do this, it is required to estimate as well as we can the growth of

$$\phi(\sigma+it,\alpha)\ll(1+|t|)^{\mathfrak{k}(\sigma)}$$

for $\sigma>-C$ and $t\to\infty$. It turns out that not only the value of $\mathfrak{k}(\sigma)$ is important, but also the fact that this estimate holds for a large negative σ. This fact allows us to better optimize some parameters in the proof and thus establish Theorem 1.3 for $r>1$.

Note that this is the result of a deep study of the asymptotic properties of the hypergeometric function that arises in $\phi(s,\alpha)$. More precisely, the function $\phi(s,\alpha)$ can be expressed as a series over n whose terms include

$$ {}_2F_1\left(s-in\kappa,s+in\kappa,1/2+s;\frac{2-a}{4}\right). \tag{2.1}$$

To estimate $\phi(\sigma+it,\alpha)$, one needs to obtain uniform estimates in n and t on (2.1). The main difficulty is that real parts of all three parameters of the hypergeometric function could be negative and this does not allow us to apply most of its integral representations. Luckily, we can transform it into a combination of hypergeometric functions that has σ only in the first parameter. As a consequence, it is possible to use a standard integral representation for the hypergeometric function. To obtain an asymptotic formula, we apply a version of the saddle point method due to McKee-Sun-Ye [39], which allows the saddle point to be close to the end point of the integration interval. As a result, the following proposition is proven, which might also be of use in other contexts.

Proposition 2.1. *Let $s = \sigma + it$ with $-\infty < \sigma_1 < \sigma < \sigma_2 < \infty$ and $t \to +\infty$. Let $\beta = n\kappa/t$. Then for $t^{-1+\epsilon_0} < \beta < 1 - t^{-1+\epsilon}$ and $0 < z_1 < z < z_2 < \infty$ we have*

$$\begin{aligned}(2.2)\quad & \frac{\Gamma(s-in\kappa)\Gamma(s+in\kappa)}{\Gamma(2s)}{}_2F_1\left(s-in\kappa, s+in\kappa, 1/2+s; -z\right) = \\ &= e^{-\frac{\pi i}{4}} 2^{1-2\sigma-2it}\sqrt{\pi}(1-\beta)^{\sigma-1/2+it(1-\beta)}\beta^{-1/2+it\beta}\frac{y_+^{1/2-it\beta}(1-y_+)^{1/2-it\beta}}{\sqrt{|t|}(y_+ + z)^{\sigma+it(1-\beta)}} \\ &\times\left(1+\frac{(2z+1)\beta}{\sqrt{4\beta^2z^2+4\beta^2z+1}}\right)^{1/2}\left(1+\sum_{j=1}^{n}\frac{c_j}{(t\beta)^j}+\sum_{j=1}^{n}\frac{d_j}{(t(1-\beta))^j}\right)+ \\ &\qquad + O_{z,\sigma}\left(\frac{(1-\beta)^{\sigma-1/2}\sqrt{\beta}}{(t\beta)^{n+1}}+\frac{(1-\beta)^{\sigma-1/2}}{(t(1-\beta))^{n+3/2}\sqrt{t}}\right),\end{aligned}$$

where $c_j, d_j \ll_{z,\sigma} 1$ and

$$y_+ = \frac{1-2\beta z+\sqrt{4\beta^2z^2+4\beta^2z+1}}{2(1+\beta)}.$$

3. Dirichlet's method developed and Vinogradov's applied

The following lemma is standard, but for convenience a proof follows.

Lemma 3.1. *For $x \geq 1$ we have*

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma - \frac{\rho(x)}{x} + O(x^{-2}).$$

Proof. Let

$$B_1(x) = x - \frac{1}{2} \quad \text{and} \quad B_2(x) = x^2 - x + \frac{1}{6}$$

be the Bernoulli polynomials. By partial summation,

$$\begin{aligned}
(3.1)\quad \sum_{n\le x}\frac{1}{n} &= \int_{1-}^{x}\frac{d\lfloor u\rfloor}{u} = \int_{1}^{x}\frac{du}{u} - \int_{1-}^{x}\frac{dB_1(\{u\})}{u} \\
&= \log x - \Big[\frac{B_1(\{u\})}{u}\Big]_{u=1-}^{x} - \int_{1}^{x}\frac{B_1(\{u\})}{u^2}du \\
&= \log x + \frac{1}{2} - \frac{B_1(\{x\})}{x} - \int_{1}^{x}\frac{B_1(\{u\})}{u^2}du \\
&= \log x + \gamma - \frac{B_1(\{x\})}{x} + \int_{x}^{\infty}\frac{B_1(\{u\})}{u^2}du \\
&= \log x + \gamma - \frac{B_1(\{x\})}{x} - \frac{B_2(\{x\})}{2x^2} + \int_{x}^{\infty}\frac{B_1(\{u\})}{u^3}du.
\end{aligned}$$

This finishes the proof since $\rho(x) = B_1(\{x\})$. $\square$

Proof of Theorem 1.1. By decomposing the sum according to if $d \le m$ or $d > m$ we have

$$\sum_{n\le x} d(n,\alpha) = \sum_{\substack{dm\le x\\ \alpha^{-1}d\le m\le \alpha d}} 1 = \sum_{\substack{dm\le x\\ \alpha^{-1}d\le m\le \alpha d\\ d\le m}} 1 + \sum_{\substack{dm\le x\\ \alpha^{-1}d\le m\le \alpha d\\ d>m}} 1.$$

Since $d = m$ always satisfies $\alpha^{-1}d \le m \le \alpha d$, by using symmetry, we have

$$\sum_{n\le x} d(n,\alpha) = 2\sum_{\substack{dm\le x\\ \alpha^{-1}d\le m\le \alpha d\\ d\le m}} 1 - \sum_{d^2\le x} 1.$$

Thus

$$\begin{aligned}
\sum_{n\le x} d(n,\alpha) &= 2\sum_{d\le x^{\frac12}}\ \sum_{d\le m\le \min(\alpha d, \frac{x}{d})} 1 - \sum_{d\le x^{\frac12}} 1 \\
&= 2\sum_{d\le x^{\frac12}}\ \sum_{m\le \min(\alpha d, \frac{x}{d})} 1 - 2\sum_{d\le x^{\frac12}}(d-1) - \sum_{d\le x^{\frac12}} 1 \\
&= 2\sum_{d\le x^{\frac12}}\ \sum_{m\le \min(\alpha d, \frac{x}{d})} 1 - \lfloor x^{\frac12}\rfloor(\lfloor x^{\frac12}\rfloor - 1) - \lfloor x^{\frac12}\rfloor \\
&= 2\sum_{d\le x^{\frac12}}\ \sum_{m\le \min(\alpha d, \frac{x}{d})} 1 - \lfloor x^{\frac12}\rfloor^2.
\end{aligned}$$

Since $\alpha d \leq \frac{x}{d}$ if and only if $d \leq (\frac{x}{\alpha})^{\frac{1}{2}}$, we have

$$\begin{aligned}\sum_{n\leq x} d(n,\alpha) &= 2\sum_{d\leq(\frac{x}{\alpha})^{\frac{1}{2}}} \lfloor \alpha d\rfloor + 2\sum_{(\frac{x}{\alpha})^{\frac{1}{2}}<d\leq x^{\frac{1}{2}}} \left\lfloor \frac{x}{d}\right\rfloor - \lfloor x^{\frac{1}{2}}\rfloor^2 \\ &= 2\sum_{d\leq(\frac{x}{\alpha})^{\frac{1}{2}}} \alpha d + 2\sum_{(\frac{x}{\alpha})^{\frac{1}{2}}<d\leq x^{\frac{1}{2}}} \frac{x}{d} - 2\sum_{d\leq(\frac{x}{\alpha})^{\frac{1}{2}}} \rho(\alpha d) - \\ &\qquad - 2\sum_{(\frac{x}{\alpha})^{\frac{1}{2}}<d\leq x^{\frac{1}{2}}} \rho\left(\frac{x}{d}\right) - \lfloor x^{\frac{1}{2}}\rfloor - \lfloor x^{\frac{1}{2}}\rfloor^2.\end{aligned}$$

Note that

$$\begin{aligned}\lfloor x^{\frac{1}{2}}\rfloor + \lfloor x^{\frac{1}{2}}\rfloor^2 =& \lfloor x^{\frac{1}{2}}\rfloor(\lfloor x^{\frac{1}{2}}\rfloor+1) = \left(x^{\frac{1}{2}} - \rho(x^{\frac{1}{2}}) - \frac{1}{2}\right)\left(x^{\frac{1}{2}} - \rho(x^{\frac{1}{2}}) + \frac{1}{2}\right) \\ &= x - 2\rho(x^{\frac{1}{2}})x^{\frac{1}{2}} + O(1).\end{aligned}$$

Similarly, we have

$$2\sum_{d\leq(\frac{x}{\alpha})^{\frac{1}{2}}} \alpha d = \alpha\left\lfloor\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}\right\rfloor\left(\left\lfloor\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}\right\rfloor+1\right) = x - 2\rho\left(\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}\right)(x\alpha)^{\frac{1}{2}} + O(\alpha).$$

By Lemma 3.1

$$2\sum_{(\frac{x}{\alpha})^{\frac{1}{2}}<d\leq x^{\frac{1}{2}}} \frac{x}{d} = x\log\alpha + 2\rho\left(\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}\right)(x\alpha)^{\frac{1}{2}} - 2\rho(x^{\frac{1}{2}})x^{\frac{1}{2}} + O(\alpha).$$

Combining the above we arrive at

$$\sum_{n\leq x} d(n,\alpha) = x\log\alpha - 2\sum_{d\leq(\frac{x}{\alpha})^{\frac{1}{2}}} \rho(\alpha d) - 2\sum_{(\frac{x}{\alpha})^{\frac{1}{2}}<d\leq x^{\frac{1}{2}}} \rho\left(\frac{x}{d}\right) + O(\alpha),$$

which gives Theorem 1.1. □

The Corollary of Theorem 1.1 follows easily from the next lemma, due to Vinogradov, given in the form presented in [30, Thm 11.3]. Here we choose $f(y) = \frac{x}{y}$.

Lemma 3.2. *For $k \geq 1$ and $f \in \mathcal{C}^2[M, M+M']$ with*

$$\frac{1}{C} \leq |f''(y)| \leq \frac{k}{C}$$

we have

$$\sum_{n=M}^{M+M'-1} \{f(n)\} = \frac{M'}{2} + O(k^2 M' \log C + kC)C^{-\frac{1}{3}}).$$

4. Hecke zeta functions

In this section we define and record some important properties of Hecke's zeta functions for the real quadratic field $\mathbb{F} = \mathbb{Q}(\sqrt{D})$.

For α as in (1.17) with $a > 3$ let $\mathcal{O} = \mathbb{Z} + \alpha\mathbb{Z}$ be the ring of integers of $\mathbb{F}$ and set

$$\kappa = \frac{\pi}{\log \alpha}, \quad A = \frac{\sqrt{D}}{\pi}. \tag{4.1}$$

Furthermore, we define the Hecke sign characters

$$\nu_0(\mu) = 1,\ \nu_1(\mu) = sgn(\mu\mu'), \tag{4.2}$$

where μ' stands for the Galois conjugate of μ, and let

$$\lambda(\mu) = e^{i\kappa \log |\frac{\mu}{\mu'}|}. \tag{4.3}$$

For $n \in \mathbb{Z}$ the Grössencharaktere $\nu_j(\mu)\lambda^n(\mu)$ satisfies $\nu_j(\alpha)\lambda^n(\alpha) = 1$ and so is well-defined on principal ideals of $(\mu) \subset \mathcal{O}$. The associated (partial) zeta function is given for $\Re s > 1$ by

$$\zeta(s, \nu_j\lambda^n) = \sum_{(\mu)} \frac{\nu_j(\mu)\lambda^n(\mu)}{|N(\mu)|^s}, \tag{4.4}$$

where the sum is over distinct nonzero principal ideals of $\mathcal{O}$.

Lemma 4.1. *For $j \in \{0, 1\}$ and $n \in \mathbb{Z}$, the completed zeta function given by*

$$\xi(s, \nu_j\lambda^n) = A^s \Gamma\left(\frac{s+j}{2} + \frac{in\kappa}{2}\right) \Gamma\left(\frac{s+j}{2} - \frac{in\kappa}{2}\right) \zeta(s, \nu_j\lambda^n) \tag{4.5}$$

is entire and satisfies the functional equation

$$\xi(1 - s, \nu_j\lambda^n) = \xi(s, \nu_j\lambda^{-n}), \tag{4.6}$$

except that, when $n = j = 0$, it has simple poles at $s = 1, 0$ with residues

$$\operatorname{res}_{s=0} \xi(s, \nu_0) = -2\log\alpha, \quad \operatorname{res}_{s=1} \xi(s, \nu_0) = 2\log\alpha. \tag{4.7}$$

In any fixed vertical strip and away from these poles, the function $\xi(s, \nu_j\lambda^n)$ is bounded uniformly in the variables s and n.

We have the evaluation

$$\xi(0, \nu_1) = \pi\zeta(0, \nu_1) = \frac{a-3}{6}\pi. \tag{4.8}$$

The following estimate holds:

$$\zeta(\sigma + it, \nu_j\lambda^n) \ll (1 + |t - n\kappa|)^{k(\sigma)} (1 + |t + n\kappa|)^{k(\sigma)}, \tag{4.9}$$

where

$$k(\sigma) = \begin{cases} 0, & \text{if} \quad \sigma > 1, \\ (1-\sigma)/2 + \epsilon, & \text{if} \quad 0 \le \sigma \le 1, \\ 1/2 - \sigma, & \text{if} \quad \sigma < 0. \end{cases} \tag{4.10}$$

Proof. The first statement follows from [26, p.35, p.73 in Werke] (see also [28, section 3]). The second statement, implicit in [26], is proven in detail in [43, Hilfssatz 7, p.126, p. 323 in Collected Papers I]. The proof is a standard application of the integral representation of ξ in terms of a theta function given by Hecke.

The fact that $\zeta(0, \nu_1)$ can be evaluated in elementary terms and is rational is a special case of a phenomenon discovered by Hecke [27]. The specific evaluation in (4.8) can readily be derived from [46, Thm. 13, p.143.].

Finally, (4.9) follows from Rademacher's uniform version of the Phragmén-Lindelöf theorem [44, Thm. 2], which he applied in his Theorem 5 to Hecke zeta functions.

□

Remark. Note that $\zeta(s, \nu_j \lambda^n)$ need not have an Euler product unless $\mathbb{Q}(\alpha)$ has (wide) class number one. It is known (see [8], also [7]) that this holds exactly for

$$D = 12, 21, 77, 437,$$

with corresponding values of a given by $4, 5, 9, 21$. In these cases we have

$$\zeta(s, \nu_0) = \zeta(s)L(s, \chi_D) \quad \text{and} \quad \zeta(s, \nu_1) = L(s, \chi_{D_1})L(s, \chi_{D_2}),$$

where $L(s, \chi_D)$ is the Dirichlet L-functions with Kronecker symbol χ_D and $D = D_1 D_2$ with $12 = (-3)(-4), 21 = (-3)(-7), 77 = (-7)(-11)$ and $437 = (-19)(-23)$. It can be checked that (4.8) yields well-known evaluations of Dirichlet L-functions for these examples.

5. Properties of $\phi(s, \alpha)$

In order to use the Hecke zeta functions to prove our theorems, we will express the Dirichlet series

$$\phi^*(s, \alpha) = \sum_n d(n, \alpha) n^{-s}, \tag{5.1}$$

with $d(n, \alpha)$ defined in (1.4), in terms of them. We use the following lemma, which is easily shown by direct calculation.

Lemma 5.1. *Let $\mathcal{O} = \mathbb{Z} + \alpha\mathbb{Z}$ where $\alpha = \frac{1}{2}(a + \sqrt{a^2 - 4})$, with $D = a^2 - 4$ the discriminant of a real quadratic field. The map*

$$\beta \mapsto (d_1, d_2) = \left(\left| \frac{\beta}{\sqrt{D}} \right| + \left| \frac{\beta'}{\sqrt{D}} \right|, \left| \frac{\alpha'\beta}{\sqrt{D}} \right| + \left| \frac{\alpha\beta'}{\sqrt{D}} \right| \right)$$

gives a bijection from

$$\{\beta \in \mathcal{O};\ \beta > 0 \text{ and } \beta' < 0\} \ \text{ to } \ \{(d_1, d_2) \in (\mathbb{Z}^+)^2;\ \alpha^{-1} d_1 < d_2 < \alpha d_1\}.$$

Here, as usual, $\mathbb{Z}^+$ denotes the set of positive integers.

This lemma can be adapted to apply to more general units α, but we must restrict to (d_1, d_2) that satisfy a certain congruence (see [13, Lemma 2]).

For $\operatorname{Re} s > 1$ let

$$\Phi_j(s) = \Phi_j(s,\alpha) = {\sum_{\beta\in\mathcal{O}}}' \nu_j(\beta)(|\beta| + |\beta'|)^{-s}(|\alpha'\beta| + |\alpha\beta'|)^{-s},$$

where as usual the prime in the sum means to leave out $\beta = 0$. Convergence follows easily since the sum is over a two dimensional lattice. The next identity follows straight from Lemma 5.1.

Lemma 5.2. *For $\phi^*(s,\alpha)$ defined in (5.1) we have the identity*

$$\phi^*(s,\alpha) = \frac{D^s}{4}\big(\Phi_0(s) - \Phi_1(s)\big),$$

when $\operatorname{Re} s > 1$.

Thus to study $\phi^*(s,\alpha)$, hence $\phi(s,\alpha)$ from (1.13), we are reduced to considering the Dirichlet series $\Phi_j(s)$ for $j = 1, 2$.

Proposition 5.3. *Fix $j \in \{0, 1\}$ and a as above and recall κ from (4.1). Suppose that s is in a compact subset of $\mathbb{C}$ that does not contain any of the points $-2k - 1 + j \pm in\kappa$, for $n, k \in \{0, 1, 2, \dots\}$ and also does not contain $s = 1$ if $j = 0$. Then we have the uniformly convergent expansion*

$$\Phi_j(s) = B\sum_{n\in\mathbb{Z}}(-1)^n \frac{\Gamma(\frac{s+1-j+in\kappa}{2})\Gamma(\frac{s+1-j-in\kappa}{2})}{(\log\alpha)\Gamma(2s)}\times \tag{5.2}$$

$$\times\, {}_2F_1(s+in\kappa, s-in\kappa; s+\frac{1}{2}; \frac{1}{2}-\frac{a}{4})\xi(s,\nu_j\lambda^n), \tag{5.3}$$

where $B = B(s) = (2\sqrt{\pi})^{2s-2}D^{-\frac{s}{2}}$.

Together with Lemma 5.2, Proposition 5.3 gives the meromorphic continuation of

$$\phi(s,\alpha) = \phi^*(s,\alpha) - \zeta(s)\log\alpha \tag{5.4}$$

with the location of its possible poles.

We will first restrict s so that $\operatorname{Re} s > 1$ and prove the following variant identity, which will also be used later upon analytic continuation.

Lemma 5.4. *For $j \in \{0, 1\}$ and* $\operatorname{Re} s > 1$

$$\Phi_j(s) = \sum_{n\in\mathbb{Z}}(-1)^n \frac{\Gamma(s+in\kappa)\Gamma(s-in\kappa)}{(\log\alpha)\Gamma(2s)}\left(\frac{a}{2}\right)^{-s+in\kappa}\times \tag{5.5}$$

$$\times\, {}_2F_1\left(\frac{s+1}{2} - \frac{in\kappa}{2}, \frac{s}{2} - \frac{in\kappa}{2}; s+\frac{1}{2}; \frac{D}{a^2}\right)\zeta(s,\nu_j\lambda^n). \tag{5.6}$$

Proof. Define $\Phi_j(s,x)$ for $x \in \mathbb{R}$ and $\operatorname{Re} s > 1$ by

$$\Phi_j(s,x) = \sum_{\beta\in\mathcal{O}}{}' \nu_j(\beta)(|\beta|e^x + |\beta'|e^{-x})^{-s}(|\alpha'\beta|e^x + |\alpha\beta'|e^{-x})^{-s}.$$

To prove (5.5) we apply the basic principle of Fourier analysis, which was used in brilliant and unexpected ways by Hecke to study algebraic numbers [29, p. 104, p.338 in Werke]:

Wenn eine Funktion bei einer Substitution (von unendlich hoher Ordnung) invariant bleibt, so entwickle man die Funktion in eine Fouriersche Reihe nach einer geeignet gewählten Variablen, welche diese Invarianz in Evidenz setzt.[2]

Clearly $\Phi_j(s,x)$ is a C^1 function in x for fixed s with $\operatorname{Re} s > 1$ and

$$\Phi_j(s, x + \log\alpha) = \Phi_j(s,x).$$

Thus $\Phi_j(s,x)$ has an absolutely convergent Fourier expansion

$$\Phi_j(s,x) = \sum_{n\in\mathbb{Z}} A_j(n,s)e\left(\frac{nx}{\log\alpha}\right), \qquad \text{where} \quad e(z) = e^{2\pi i z}.$$

We apply Hecke's well-known unfolding trick to compute $A_j(n,s)$:

$$A_j(n,s) = \frac{1}{\log\alpha}\int_0^{\log\alpha} e\left(-\frac{nx}{\log\alpha}\right)\sum_{\beta\in\mathcal{O}}{}' \nu_j(\beta)(|\beta|e^x + |\beta'|e^{-x})^{-s}\times$$
$$\times(|\alpha'\beta|e^x + |\alpha\beta'|e^{-x})^{-s}dx = \frac{2}{\log\alpha}K_n(s)\zeta(s;\nu_j\lambda^n),$$

where

$$\begin{aligned} K_n(s) &= \int_{-\infty}^{\infty}\big((e^x + e^{-x})(\alpha' e^x + \alpha e^{-x})\big)^{-s} e\left(-\frac{nx}{\log\alpha}\right)dx \\ &= \alpha^s\int_{-\infty}^{\infty}\big((1+e^{-2x})(1+\alpha^2 e^{-2x})\big)^{-s}(e^{-2x})^{s+\frac{\pi i n}{\log\alpha}}dx \\ &= \frac{\alpha^s}{2}\int_0^1 t^{s+in\kappa}(1-t)^{s-in\kappa}(1-(1-\alpha^2)t)^{-s}dt \end{aligned}$$

after making the change of variables $t = (e^{2x}+1)^{-1}$ and recalling κ from (4.1). Now the Euler integral formula [37, p.57] gives

$$K_n(s) = \frac{\alpha^s}{2}\frac{\Gamma(s+in\kappa)\Gamma(s-in\kappa)}{\Gamma(2s)}{}_2F_1(s, s+in\kappa; 2s; 1-\alpha^2). \tag{5.7}$$

By the quadratic transformation [37, (8) p. 93] we get

$$K_n(s) = (-1)^n\frac{\Gamma(s+in\kappa)\Gamma(s-in\kappa)}{2\Gamma(2s)}\left(\frac{a}{2}\right)^{-s+in\kappa}\times$$
$$\times\, {}_2F_1\left(\frac{s+1}{2} - \frac{in\kappa}{2}, \frac{s}{2} - \frac{in\kappa}{2}; s+\frac{1}{2}; \frac{D}{a^2}\right), \tag{5.8}$$

from which (5.5) follows. □

[2]If a function remains invariant under a substitution (of infinitely high order), then expand the function in a Fourier series with respect to a suitably chosen variable that makes this invariance evident.

In order to analytically continue the sum in (5.2) in s to the left, we need to estimate the hypergeometric function uniformly in the large parameter n, which occurs in both the first and second parameter of the hypergeometric function. Note that a more detailed analysis of the hypergeometric function is given in section 7.

Lemma 5.5. *For $|n| \to \infty$ one has*

$$ {}_2F_1\left(s+in\kappa, s-in\kappa; s+\frac{1}{2}; \frac{1}{2}-\frac{a}{4}\right) \ll |n|^{1/2-\sigma}, \tag{5.9}$$

where the implied constant depends on a and s.

Proof. To prove (5.9) we apply [40, 15.8.1] followed by [40, 15.6.6], getting

$$ {}_2F_1\left(s+in\kappa, s-in\kappa, 1/2+s; -z\right) = \frac{(z+1)^{1/2-s}\Gamma(1/2+s)}{\Gamma(1/2in\kappa)\Gamma(1/2+in\kappa)} \\ \times \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(1/2-in\kappa+w)\Gamma(1/2+in\kappa+w)}{\Gamma(1/2+s+w)}\Gamma(-w)z^w dw, \tag{5.10}$$

where $-1/2 < c < 0$. Let $s = \sigma + it$ and $w = u + iv$. Applying the Stirling formula [40, 5.11.9] we obtain

$$ \frac{\Gamma(1/2+s)}{\Gamma(1/2-in\kappa)\Gamma(1/2+in\kappa)} \frac{\Gamma(1/2-in\kappa+w)\Gamma(1/2+in\kappa+w)}{\Gamma(1/2+s+w)}\Gamma(-w) \ll \\ \ll |t|^{\sigma} \frac{(1+|v+n\kappa|)^u(1+|v-n\kappa|)^u}{(1+|v+t|)^{u+\sigma}(1+|v|)^{1/2+u}} e^{-\pi g(t,n\kappa,v)/2}, \tag{5.11}$$

where

$$ g(t,y,v) = |v+y| + |v-y| + |v| - |v+t| + t - 2y. \tag{5.12}$$

Using (5.12) one can easily deduce that the part of the integral (5.10) with $|v| \gg n\kappa + \log^2 n$ is negligible. Estimating the remaining integral trivially, we obtain (5.9). □

Proof of Proposition 5.3. To derive (5.2) from Lemma 5.4, first observe that by applying another quadratic transformation [15, (4) p.111] (or [40, 15.8.17]) to the hypergeometric function in (5.8) we get for $a > 2$:

$$ \left(\frac{a}{2}\right)^{-s+i\kappa n} {}_2F_1\left(\frac{s+1}{2} - \frac{in\kappa}{2}, \frac{s}{2} - \frac{in\kappa}{2}; s+\frac{1}{2}; \frac{D}{a^2}\right) = \\ = \left(\frac{a}{2}\right)^{-s-i\kappa n} {}_2F_1\left(\frac{s+1}{2} + \frac{in\kappa}{2}, \frac{s}{2} + \frac{in\kappa}{2}; s+\frac{1}{2}; \frac{D}{a^2}\right) = \\ = {}_2F_1\left(s+in\kappa, s-in\kappa; s+\frac{1}{2}; \frac{1}{2}-\frac{a}{4}\right). \tag{5.13}$$

Note that (5.2) can also be derived from (5.7) using a single quadratic transformation [15, (30) p.113]. By using the duplication formula

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s),$$

we see that the gamma factors on the right hand side of (5.5) combine with those in (4.5) to allow us to deduce (5.2) from (5.5).

Proposition 5.3 now follows by the properties of the completed Hecke zeta functions given in Lemma 4.1, including the uniform boundedness of $\xi(s,\nu_j\lambda^n)$ in n and s, together with (5.9) and the standard fact that

$$\lim_{|t|\to\infty} \frac{\Gamma(\sigma+it)}{e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}} = \sqrt{2\pi}. \tag{5.14}$$

□

Remark. To study $\psi(s,\alpha)$ from (1.14) for certain real quadratic α, Hecke used the simpler "degree one" functions

$$\Psi_j(s) = \sum_{\mu}{}' \nu_j(\mu)(|\mu|+|\mu'|)^{-s},$$

with the appropriate summation over μ. The fact that $\Phi_j(s)$ has degree two accounts for one of the new difficulties in treating the restricted divisor problem using Hecke's method, due to its increased growth in vertical strips. Also, for $\Psi_j(s)$ no hypergeometric function occurs in the corresponding Fourier coefficient.

One can group together n and $-n$ terms in (5.2) getting

$$\begin{aligned}\Phi_j(s) = {}& \frac{(2\sqrt{\pi})^{2s-2}}{D^{s/2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(2s)\log\alpha} \times \\ & \times \Gamma\left(\frac{s+1-j+in\kappa}{2}\right)\Gamma\left(\frac{s+1-j-in\kappa}{2}\right) \times \\ & \times (\xi(s,\nu_j\lambda^n)+\xi(s,\nu_j\lambda^{-n}))\,{}_2F_1\left(s-in\kappa, s+in\kappa, 1/2+s; \frac{2-a}{4}\right) + \\ & + \frac{(2\sqrt{\pi})^{2s-2}\xi(s,\nu_j)}{D^{s/2}\Gamma(2s)\log\alpha}\Gamma^2\left(\frac{s+1-j}{2}\right){}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right).\end{aligned} \tag{5.15}$$

The function $\Phi_0(s)$ has poles at the points:

$$1, \quad 0, \quad -1-2m \pm in\kappa, \quad \text{for } m,n \in \mathbb{Z}_{\geq 0}. \tag{5.16}$$

The first two poles come from the poles of $\xi(s,\nu_0)$, and the series of poles comes from the poles of Gamma factors. The function $\Phi_1(s)$ has poles at the points:

$$-2m \pm in\kappa, \quad \text{for } m,n \in \mathbb{Z}_{\geq 0}. \tag{5.17}$$

Note that though the hypergeometric function in (5.2) has poles at $-\frac{1}{2}-m$, $m \in \mathbb{Z}_{\geq 0}$, the functions $\Phi_j(s)$ are holomorphic at these points due to the presence of $\Gamma^{-1}(2s)$. According to (1.13), (5.4) and Lemma 5.2

$$\phi(s,\alpha) = \frac{D^s}{4}\Phi_0(s) - \frac{D^s}{4}\Phi_1(s) - \zeta(s)\log\alpha. \tag{5.18}$$

It turns out that $\phi(s,\alpha)$ is holomorphic at $s=1$. From (5.15) and (4.7) we have

$$\operatorname{res}_{s=1}\frac{D^s}{4}\Phi_0(s) = \frac{\sqrt{D}}{2}{}_2F_1\left(1,1,3/2;\frac{2-a}{4}\right). \tag{5.19}$$

This hypergeometric function [42, Sec. 7.3.4., eq. (17)] reduces to

$${}_2F_1\left(1,1,3/2;-x\right) = \frac{\log(\sqrt{x}+\sqrt{x+1})}{\sqrt{x(x+1)}}, \tag{5.20}$$

and using the fact that $D = a^2-4$ we obtain

$$\operatorname{res}_{s=1}\frac{D^s}{4}\Phi_0(s) = 2\log\left(\frac{\sqrt{a-2}}{2}+\frac{\sqrt{a+2}}{2}\right) \tag{5.21}$$

$$= \log\left(\frac{a+\sqrt{a^2-4}}{2}\right) = \log\alpha. \tag{5.22}$$

Therefore, this residue cancels out with the one of $-\zeta(s)\log\alpha$ so that $\phi(s,\alpha)$ is holomorphic at $s=1$.

For our further computations we also need to know residues of $\phi(s,\alpha)$ at $s=in\kappa$ with $n\neq 0$. Applying (5.15) and (5.18) we prove the following result.

Lemma 5.6. *We have*

$$\operatorname{res}_{s=in\kappa}\phi(s,\alpha) = \operatorname{res}_{s=in\kappa}\frac{-D^s}{4}\Phi_1(s) = \tag{5.23}$$

$$= \frac{-(2\sqrt{\pi})^{2in\kappa-2}D^{in\kappa/2}}{2\log\alpha}\frac{(-1)^n\Gamma(in\kappa)}{\Gamma(2in\kappa)}\left(\xi(in\kappa,\nu_1\lambda^n)+\xi(in\kappa,\nu_1\lambda^{-n})\right).$$

6. Properties of $\psi(2s,\alpha)$

In this section, we state some results of [28] needed for our calculations.

First, we remark that η, defined on [28, p.55], coincides with α in (1.17). Furthermore, $e(1)$, introduced in [28, p.60], is equal to 2, since for $a>3$ the number α is a totally positive fundamental unit in $\mathbb{Q}(\sqrt{D})$. For $j=0,1$ let (see [28, (3)])

$$\Psi_j(s) = \sum_{\mu\in\mathcal{O}}{}' \frac{\nu_j(\mu)}{(|\mu|+|\mu'|)^s}, \tag{6.1}$$

where ν_j are the Hecke sign characters defined by (4.2), the prime sign above the sum means that $\mu \neq 0$, and $\mathcal{O}$ is a ring of integers of $\mathbb{Q}(\sqrt{D})$. For $a \equiv 0 \pmod 4$ and $D = a^2 - 4$ Hecke proved that (see [28, (5)])

$$\Psi_j(2s) = \frac{2^{2s-2}A^{-s}}{\pi\Gamma(2s)\log\alpha} \sum_{n=-\infty}^{\infty} \xi(s, \nu_j\lambda^n) \times \times \Gamma\left(\frac{s+1-j+in\kappa}{2}\right)\Gamma\left(\frac{s+1-j-in\kappa}{2}\right). \tag{6.2}$$

However, an analysis of Hecke's proof shows that (6.2) is also true for odd values of a.

Lemma 6.1. *The following identity holds:*

$$\psi(2s,\alpha) = \frac{D^s}{8}\Psi_1(2s) - \frac{D^s}{8}\Psi_0(2s) + \frac{\sqrt{D}}{2}\zeta(2s-1). \tag{6.3}$$

Proof. First, consider the case $a \equiv 0 \pmod 4$. According to [28, p.59]

$$\psi(2s,\sqrt{\mathfrak{D}}) = 2^{2s-3}\mathfrak{D}^s\Psi_1(2s) - 2^{2s-3}\mathfrak{D}^s\Psi_0(2s) + \sqrt{\mathfrak{D}}\zeta(2s-1), \tag{6.4}$$

where $\mathfrak{D} = D/4 = (a^2-4)/4$. Furthermore, $\{n\alpha\} = \{n\sqrt{\mathfrak{D}}\}$ for $a \equiv 0 \pmod 4$, and thus $\psi(2s,\sqrt{\mathfrak{D}}) = \psi(2s,\alpha)$, which completes the proof of (6.3) in this case.

Next, consider the case of odd a. To handle this, we will combine Lemma 5.1 with the results of [28]. Applying Lemma 5.1 to study (6.1), we obtain

$$\begin{aligned}\Psi_0(s) - \Psi_1(s) &= 4\sum_{\mu>0,\mu'<0} \frac{1}{(|\mu|+|\mu'|)^s} = 4\sum_{d_1=1}^{\infty}\sum_{d_1/\alpha<d_2<d_1\alpha} \frac{1}{(d_1\sqrt{D})^s} = \\ &= 4\sum_{d_1=1}^{\infty}\frac{[\alpha d_1]-[d_1/\alpha]}{(d_1\sqrt{D})^s} = 4\sum_{d_1=1}^{\infty}\frac{d_1\sqrt{D}-\{\alpha d_1\}+\{d_1/\alpha\}}{(d_1\sqrt{D})^s}.\end{aligned} \tag{6.5}$$

The identity

$$\{\frac{a-\sqrt{D}}{2}d_1\} - \{\frac{a+\sqrt{D}}{2}d_1\} = 1 - 2\{\frac{a+\sqrt{D}}{2}d_1\} \tag{6.6}$$

implies that

$$\Psi_0(s) - \Psi_1(s) = 4D^{1/2-s/2}\zeta(s-1) - 8D^{-s/2}\psi(s,\alpha), \tag{6.7}$$

which completes the proof of (6.3). □

The function $\Psi_0(2s)$ has poles at the points:

$$1, \quad 0, \quad -1-2m \pm in\kappa \quad \text{for}\, m,n \in \mathbb{Z}_{\geq 0}, \tag{6.8}$$

and the function $\Psi_1(2s)$ has poles at the points: $-2m \pm in\kappa$ for $m,n \in \mathbb{Z}_{\geq 0}$. It follows from (6.2), (4.7) and (4.1) that

$$\operatorname{res}_{s=1}\Psi_0(2s) = \frac{2}{\sqrt{D}}.$$

Therefore, $\psi(2s,\alpha)$ is holomorphic at $s=1$.

Next, we evaluate the residues of $\psi(2s,\alpha)$ at $s=in\kappa$ with $n\neq 0$ (here once again we group together the summands with n and $-n$ in (6.2)).

Lemma 6.2. *We have*

$$\operatorname{res}_{s=in\kappa}\psi(2s,\alpha)=\operatorname{res}_{s=in\kappa}2^{-3}D^s\Psi_1(2s)=\\ =\frac{2^{2in\kappa-4}D^{in\kappa/2}}{\pi^{1-in\kappa}\log\alpha}\frac{\Gamma(in\kappa)}{\Gamma(2in\kappa)}\left(\xi(in\kappa,\nu_1\lambda^n)+\xi(in\kappa,\nu_1\lambda^{-n})\right). \tag{6.9}$$

Let

$$H(s)=\prod_{k=0}^{\infty}(1-\alpha^{-s-2k}). \tag{6.10}$$

Hecke proved (see [28, (8),(9)]) that for $s=\sigma+it$ we have [3]

$$H(2s)\psi(2s,\alpha)\ll 1+|t|^{m(\sigma)}, \tag{6.11}$$

where for $\sigma>1/2$ we have $m(\sigma)=0$, for $-\sigma_1\leq\sigma\leq 1/2$ we have $m(\sigma)=1-2\sigma+\epsilon$ with σ_1 being arbitrary positive half-integral number.

Remark. A different proof giving the meromorphic continuation of the Dirichlet series $\psi(s,\alpha)$ from (1.14) for certain real quadratic α can be found in [23]. Their proof is based on properties of the double zeta function of Barnes. *This* Hardy-Littlewood method was developed further in [38] and then in [16] (see also [17]), to cover much more general Dirichlet series. In fact, results of [16] yield another proof that our $\varphi(s,\alpha)$ has a meromorphic continuation to a function of finite order and even can be applied to determine the location of the poles, but without the explicit determination of their residues or the growth estimates needed in our proof of Theorem 1.5.

7. Asymptotic analysis of the hypergeometric function

In this section, we prove new estimates for the growth of $\phi(s,\alpha)$ by studying the asymptotic behavior of the hypergeometric function appearing in (5.15).

Using (4.5) and [40, 5.5.5] we have

$$\Phi_j(s)=\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(s+in\kappa)\Gamma(s-in\kappa)}{\Gamma(2s)\log\alpha}\times\\ \times {}_2F_1\left(s-in\kappa,s+in\kappa,1/2+s;\frac{2-a}{4}\right)(\zeta(s,\nu_j\lambda^n)+\zeta(s,\nu_j\lambda^{-n}))+\\ +\frac{\Gamma^2(s)\zeta(s,\nu_j)}{\Gamma(2s)\log\alpha}{}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right). \tag{7.1}$$

[3]This result was improved by Fujii [19, p.218].

For $s=\sigma+it$, $\beta\ge 0$ let

$$\mathcal{F}(\beta,\sigma,t,z)=H(2s)\frac{\Gamma(\sigma+it(1+\beta))\Gamma(\sigma+it(1-\beta))}{\Gamma(2\sigma+2it)}\times {}_2F_1\left(\sigma+it(1+\beta),\sigma+it(1-\beta),1/2+\sigma+it;-z\right), \tag{7.2}$$

where $H(s)$ is defined by (6.10). Applying (4.9) we get

$$H(2s)\Phi_j(s)\ll\sum_{n=0}^{\infty}\left|\mathcal{F}\left(\frac{n\kappa}{t},\sigma,t,\frac{a-2}{4}\right)\right|(1+|t-n\kappa|)^{k(\sigma)}(1+|t+n\kappa|)^{k(\sigma)}. \tag{7.3}$$

Let $\beta=n\kappa/t$. From now on, we assume that $\beta=n\kappa/t$. Note that if $n=0$ then $\beta=0$, and if $n\neq 0$ then $\beta\gg 1/t$. When $t=0$, we replace $t\beta$ by $n\kappa$.

Since $\mathcal{F}(\beta,\sigma,t,z)=\mathcal{F}(-\beta,\sigma,t,z)$, it is enough to study only the case $t\ge 0$. In our analysis, we will use several properties of the Gamma function. The first one is the following form of Stirling's formula:

$$\Gamma(\sigma+it)=\sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}e^{i\left(t\log|t|-t+\frac{\pi t(\sigma-1/2)}{2|t|}\right)}\times \times\left(1+\sum_{j=1}^{N-1}\frac{c_j}{|t|^j}+O(|t|^{-N})\right), \tag{7.4}$$

which is valid for $|t|\to\infty$ and a fixed σ (note that $c_j(\sigma)\ll 1$). Furthermore, we need some upper and lower bounds for "small" t (say $0<t\ll T^{\epsilon}$ and $T\to+\infty$). For $0\le\sigma<C$ and $\sigma+|t|>\delta$ we have [40, 5.6.9]

$$\Gamma(\sigma+it)\ll(1+|t|)^{\sigma-1/2}e^{-\pi|t|/2}. \tag{7.5}$$

In case when $\sigma<0$, using [40, 5.5.3] and [40, 5.6.7], we get for $\|\sigma+|t|\|>\delta$ (here $\|x\|$-is a distance to the nearest integer)

$$\Gamma(\sigma+it)\ll e^{-\pi|t|/2}. \tag{7.6}$$

For $\sigma\ge 1/2$ we have [40, 5.6.7]

$$\Gamma(\sigma+it)\gg e^{\pi|t|/2}, \tag{7.7}$$

and for $\sigma<1/2$, using [40, 5.5.3] and [40, 5.6.9], we obtain

$$\Gamma(\sigma+it)^{-1}\ll\Gamma(1-\sigma-it)e^{\pi|t|}\ll(1+|t|)^{1/2-\sigma}e^{\pi|t|/2}. \tag{7.8}$$

To sum up, for $0\le t\ll T^{\epsilon}$, $|\sigma|<C$ and $\|\sigma+|t|\|>\delta$ we have

$$T^{-\epsilon_1}e^{-\pi|t|/2}\ll\Gamma(\sigma+it)\ll e^{-\pi|t|/2}T^{\epsilon_1}. \tag{7.9}$$

From now on, we assume that

$$|\sigma|<C,\quad 0<\delta<z<Z, \tag{7.10}$$

where C and Z are some large constants. For simplicity, we omit the dependence on σ and z in $O(\cdot)$ and $\ll$ in this section.

Let us first consider the case $\beta=0$ (that is the term $n=0$ in (7.3)).

Lemma 7.1. *For $T\to+\infty$ we have*

$$\mathcal{F}(0,\sigma,t,z)\ll T^{\epsilon_0},\quad \textit{for}\quad 0\le t\le T^{\epsilon}, \tag{7.11}$$

$$\mathcal{F}(0,\sigma,t,z)\ll t^{-1/2},\quad \textit{for}\quad t>T^{\epsilon}. \tag{7.12}$$

Proof. Applying [40, 15.8.1] and then writing the Mellin integral representation [40, 15.6.6], we obtain

$$\begin{aligned}{}_2F_1\left(\sigma+it,\sigma+it,1/2+\sigma+it;-z\right)&=\\ =(z+1)^{1/2-\sigma-it}&\frac{\Gamma(1/2+\sigma+it)}{\Gamma^2(1/2)2\pi i}\int_{(c)}\frac{\Gamma^2(1/2+w)\Gamma(-w)}{\Gamma(1/2+\sigma+it+w)}z^w dw,\end{aligned} \tag{7.13}$$

where $-1/2<c<0$. We first consider the case of large values of $t>T^{\epsilon}$. To do this, we move the line of integration to $\Re(w)=1/2$, passing the pole at $w=0$. Let $w=1/2+iy$. Using Stirling's formula (7.4), one can easily deduce that the contribution of $|y|\gg t^{\epsilon}$ is negligible. To estimate the remaining part, we apply [40, 5.4.3, 5.4.4], getting

$${}_2F_1\left(\sigma+it,\sigma+it,1/2+\sigma+it;-z\right)\ll 1+t^{-1/2}\int_{-t^{\epsilon}}^{t^{\epsilon}}e^{-\pi|y|}dy\ll 1. \tag{7.14}$$

Substituting (7.14) to (7.2) and using (7.4) we prove (7.12).

Now, let $0\le t\le T^{\epsilon}$. In this case, we again move the line of integration to $\Re(w)=1/2$. Using (7.4) and (7.9), we show that the contribution of $|y|\gg T^{\epsilon}$ is negligible and that the remaining integral can be estimated as

$${}_2F_1\left(\sigma+it,\sigma+it,1/2+\sigma+it;-z\right)\ll 1+T^{\epsilon_1}\int_{-T^{\epsilon}}^{T^{\epsilon}}e^{-\pi|y|}dy\ll T^{\epsilon_0}. \tag{7.15}$$

Substituting (7.15) into (7.2) and using (7.9), we prove (7.11). $\square$

From now on $\beta\neq 0$. Let us begin the analysis of $\mathcal{F}(\beta,\sigma,t,z)$ by transforming the hypergeometric function in (7.2) using [3, (1), (9), (13), p.105] and [3, (34), p. 107]:

$$\begin{aligned}{}_2F_1\left(\sigma+it(1-\beta),\sigma+it(1+\beta),1/2+\sigma+it;-z\right)&=\\ =z^{-\sigma-it(1-\beta)}\frac{\Gamma(1/2+\sigma+it)\Gamma(2it\beta)}{\Gamma(1/2+it\beta)\Gamma(\sigma+it(1+\beta))}&\times\\ \times{}_2F_1\left(\sigma+it(1-\beta),1/2-it\beta,1-2it\beta;\frac{-1}{z}\right)&+\\ +z^{-\sigma-it(1+\beta)}\frac{\Gamma(1/2+\sigma+it)\Gamma(-2it\beta)}{\Gamma(1/2-it\beta)\Gamma(\sigma+it(1-\beta))}&\times\\ \times{}_2F_1\left(\sigma+it(1+\beta),1/2+it\beta,1+2it\beta;\frac{-1}{z}\right)&.\end{aligned} \tag{7.16}$$

Next, in order to study asymptotic properties of the hypergeometric functions on the right hand side of (7.16) we use the integral representation [40, 15.6.1]:

$$(7.17)\quad {}_2F_1\left(\sigma+it(1-\beta),1/2-it\beta,1-2it\beta;\frac{-1}{z}\right)= \\ =\frac{\Gamma(1-2it\beta)}{\Gamma^2(1/2-it\beta)}\int_0^1\frac{y^{-1/2-it\beta}(1-y)^{-1/2-it\beta}}{(1+y/z)^{\sigma+it(1-\beta)}}dy.$$

Substituting (7.16) and (7.17) to (7.2) we obtain

$$(7.18)\qquad \mathcal{F}(\beta,\sigma,t,z)=\mathcal{F}_1(\beta,\sigma,t,z)+\mathcal{F}_1(-\beta,\sigma,t,z),$$

where

$$(7.19)\qquad \mathcal{F}_1(\beta,\sigma,t,z)=H(2s)\mathcal{P}(\beta,\sigma,t)\mathcal{I}(\beta,\sigma,t,z),$$

$$(7.20)\quad \mathcal{P}(\beta,\sigma,t)=\frac{\Gamma(\sigma+it(1+\beta))}{\Gamma(2\sigma+2it)}\frac{\Gamma(1/2+\sigma+it)\Gamma(-2it\beta)\Gamma(1+2it\beta)}{\Gamma(1/2-it\beta)\Gamma^2(1/2+it\beta)},$$

$$(7.21)\qquad \mathcal{I}(\beta,\sigma,t,z)=\int_0^1\frac{y^{-1/2+it\beta}(1-y)^{-1/2+it\beta}}{(z+y)^{\sigma+it(1+\beta)}}dy.$$

Consider the product of the Gamma functions in (7.20). Using [40, 5.5.3] and [40, 5.5.5] we show that

$$(7.22)\quad \mathcal{P}(-\beta,\sigma,t)=\frac{1}{2i\sinh(\pi t\beta)}\frac{\Gamma(\sigma+it(1-\beta))\Gamma(1/2+\sigma+it)}{\Gamma(2\sigma+2it)\Gamma(1/2-it\beta)}= \\ =\frac{-i\cosh(\pi t\beta)}{2^{2\sigma+2it}\sqrt{\pi}\sinh(\pi t\beta)}\frac{\Gamma(\sigma+it(1-\beta))\Gamma(1/2+it\beta)}{\Gamma(\sigma+it)}.$$

Note that estimating the integral (7.21) by absolute value one gets the trivial estimate:

$$(7.23)\qquad \mathcal{I}(\pm\beta,\sigma,t,z)\ll\int_0^1\frac{y^{-1/2}(1-y)^{-1/2}}{(z+y)^{\sigma}}dy\ll z^{-\sigma}+(1+z)^{-\sigma}.$$

Let us consider the case of small t, say $0\le t\ll T^{\epsilon}$ as $T\to+\infty$.

Lemma 7.2. *For $0\le t\ll T^{\epsilon_0}$ as $T\to+\infty$ we have*

$$(7.24)\qquad \mathcal{F}(\beta,\sigma,t,z)\ll T^{\epsilon}(\beta-1)^{\sigma-1/2}e^{-\pi t(\beta-1)},\quad \textit{for}\quad t\beta>T^{\epsilon_1}>2t,$$

$$(7.25)\qquad \mathcal{F}(\beta,\sigma,t,z)\ll T^{\epsilon},\quad \textit{for}\quad 0<t\beta\le T^{\epsilon_1}.$$

Proof. Using (7.22), (7.4) and (7.9) we show that

$$(7.26)\qquad \mathcal{P}(\beta,\sigma,t)\ll T^{\epsilon}(t(1+\beta))^{\sigma-1/2}e^{-\pi t\beta}\quad\text{for}\quad t\beta>T^{\epsilon_1},$$

and for $0<t\beta\le T^{\epsilon_1}$ we have $\mathcal{P}(\beta,\sigma,t)\ll T^{\epsilon}e^{-\pi t\beta}$. In the same way,

$$(7.27)\qquad \mathcal{P}(-\beta,\sigma,t)\ll T^{\epsilon}(t(\beta-1))^{\sigma-1/2}e^{-\pi t(\beta-1)}\quad\text{for}\quad t\beta\ge T^{\epsilon_1}>2t,$$

$$(7.28)\qquad \mathcal{P}(-\beta,\sigma,t)\ll T^{\epsilon}e^{-\pi t(\beta-1+|\beta-1|)/2}\quad\text{for}\quad 0<t\beta<T^{\epsilon_1}.$$

Combining these estimates with (7.18), (7.19), (7.23), we complete the proof. □

From now on, we assume that $t \gg T^{\epsilon}$ as $T \to +\infty$.

Lemma 7.3. *For $t \to +\infty$ and $\beta > 1 + t^{-1+\epsilon}$ the estimate holds:*

$$\mathcal{F}(\beta, \sigma, t, z) \ll (\beta-1)^{\sigma-1/2} e^{-\pi|t|(\beta-1)}. \tag{7.29}$$

Proof. It turns out that the main contribution to (7.18) comes from $\mathcal{F}_1(-\beta, \sigma, t, z)$, so let us start by studying this function. By Stirling's formula (7.4) for $\min(|1-\beta|, \beta) \gg |t|^{-1+\epsilon}$ we have

$$\frac{\Gamma(\sigma+it(1-\beta))\Gamma(1/2+it\beta)}{\Gamma(\sigma+it)} = \sqrt{2\pi}|1-\beta|^{\sigma-1/2} e^{-\frac{\pi|t|}{2}(|1-\beta|+|\beta|-1)} \times \\ \times e^{it((1-\beta)\log|1-\beta|+\beta\log\beta)} e^{\frac{\pi it(\sigma-1/2)}{2|t|}\left(\frac{1-\beta}{|1-\beta|}-1\right)} \times \\ \times \left(1 + \sum_{j=1}^{N-1} \frac{c_j}{|t\min(\beta, 1-\beta)|^j} + O(|t\min(\beta, 1-\beta)|^{-N})\right). \tag{7.30}$$

It follows from (7.22) and (7.30) that for $\beta > 1 + t^{-1+\epsilon}$ the following estimate holds:

$$\mathcal{P}(-\beta, \sigma, t) \ll (\beta-1)^{\sigma-1/2} e^{-\pi|t|(\beta-1)}. \tag{7.31}$$

Changing β to $-\beta$ in (7.30), for $\beta > t^{-1+\epsilon}$ we deduce that

$$\mathcal{P}(\beta, \sigma, t) \ll e^{-\pi|t|\beta}. \tag{7.32}$$

Now the estimate (7.29) follows from (7.18), (7.19), (7.31), (7.32) and (7.23). ∎

Next, we consider the case $t^{-1} \ll \beta < t^{-1+\epsilon_0}$.

Lemma 7.4. *For $t^{-1} \ll \beta < t^{-1+\epsilon_0}$ we have*

$$\mathcal{F}(\beta, \sigma, t, z) = H(2s) e^{-\frac{\pi it}{4|t|}} 2^{1-2\sigma-2it} \sqrt{\pi} (1-\beta)^{\sigma-1/2+it(1-\beta)} \times \\ \times (1+\beta)^{\sigma-1/2+it(1+\beta)} \frac{(z+1)^{1/2-\sigma-it}}{\sqrt{|t|}} \left(1 + \sum_{j=1}^{n} e_j \cdot (t\beta^2)^j\right) + O\left(\frac{t^{\epsilon}}{t^{n+1}}\right), \tag{7.33}$$

where $e_j \ll_{z,\sigma} 1$.

Proof. Using [40, 15.8.1] and then applying the Mellin-Barnes integral representation [40, 15.6.6], we obtain

$$ {}_2F_1(\sigma+it(1+\beta), \sigma+it(1-\beta), 1/2+\sigma+it; -z) = \\ = (z+1)^{1/2-\sigma-it} {}_2F_1(1/2-it\beta, 1/2+it\beta, 1/2+\sigma+it; -z) = \\ = \frac{(z+1)^{1/2-\sigma-it}\Gamma(1/2+\sigma+it)}{\Gamma(1/2-it\beta)\Gamma(1/2+it\beta)2\pi i} \times \\ \times \int_{(c)} \frac{\Gamma(1/2-it\beta+w)\Gamma(1/2+it\beta+w)}{\Gamma(1/2+\sigma+it+w)} \Gamma(-w) z^w dw, \tag{7.34}$$

where $-1/2 < c < 0$. Since $t^{-1} \ll \beta < t^{-1+\epsilon_0}$, the third parameter of the hypergeometric function is much larger than the first and second, so we can simply move the line of integration to the right on $\Re w = 1/2 + n$, crossing the poles at $w = j$. Let $w = 1/2 + n + iv$. Applying (7.4), [40, 5.4.3, 5.4.4] we obtain

(7.35)
$$\frac{\Gamma(1/2+\sigma+it)}{\Gamma(1/2-it\beta)\Gamma(1/2+it\beta)}\frac{\Gamma(1/2-it\beta+w)\Gamma(1/2+it\beta+w)}{\Gamma(1/2+\sigma+it+w)}\Gamma(-w) \ll \\ \ll |t|^{\sigma}\frac{(1+|v+t\beta|)^{n+1/2}(1+|v-t\beta|)^{n+1/2}}{(1+|v+t|)^{1/2+n+\sigma}(1+|v|)^{1+n}}e^{-\pi g(t,\beta,v)/2},$$

where

$$g(t,\beta,v) = |v+t\beta| + |v-t\beta| + |v| - |v+t| + t - 2t\beta. \tag{7.36}$$

Using (7.36) one can easily deduce that the part of the integral (7.34) with $|v| \gg t\beta + \log^2 t$ is negligible. Estimating the remaining integral trivially, we obtain

$$\begin{aligned}(7.37)\quad {}_2F_1\left(\sigma+it(1+\beta), \sigma+it(1-\beta), 1/2+\sigma+it; -z\right) = \\ = (z+1)^{1/2-\sigma-it}\sum_{j=0}^{n}\frac{(-1)^j z^j}{j!}\frac{\Gamma(1/2+\sigma+it)}{\Gamma(1/2-it\beta)\Gamma(1/2+it\beta)}\times \\ \times \frac{\Gamma(1/2-it\beta+j)\Gamma(1/2+it\beta+j)}{\Gamma(1/2+\sigma+it+j)} + O\left(\frac{t^{\epsilon}}{t^{n+1/2}}\right).\end{aligned}$$

Using (7.4) we deduce that

$$\begin{aligned}(7.38)\quad \frac{\Gamma(\sigma+it(1+\beta))\Gamma(\sigma+it(1-\beta))}{\Gamma(2\sigma+2it)} = \frac{\sqrt{2\pi}}{\sqrt{|t|}}2^{1/2-2\sigma}(1-\beta^2)^{\sigma-1/2} \\ \times e^{it((1+\beta)\log(1+\beta)+(1-\beta)\log(1-\beta)-2\log 2)}e^{-\frac{\pi it}{4|t|}}\left(1+\sum_{j=1}^{N-1}\frac{c_j}{|t|^j}+O(|t|^{-N})\right).\end{aligned}$$

Substituting (7.37) and (7.38) into (7.2), we prove the lemma. □

Our next goal is to obtain an asymptotic formula in the range

$$t^{-1+\epsilon_0} < \beta < 1 + t^{-1+\epsilon}$$

as $t \to +\infty$. In view of (7.32), the part $\mathcal{F}_1(\beta,\sigma,t,z)$ of (7.18) is negligible and therefore we consider further only $\mathcal{F}_1(-\beta,\sigma,t,z)$. Accordingly, it is required to obtain an asymptotic expansion for the integral $\mathcal{I}(-\beta,\sigma,t,z)$ via the saddle point method. To this end, we rewrite the integral in the form:

$$\mathcal{I}(-\beta,\sigma,t,z) = \int_0^1 \frac{y^{-1/2}(1-y)^{-1/2}}{(z+y)^{\sigma}}e^{-itf_1(\beta,z,y)}dy, \tag{7.39}$$

$$f_1(\beta,z,y) = \beta\log y + \beta\log(1-y) + (1-\beta)\log(y+z). \tag{7.40}$$

The saddle point defined by $\frac{\partial}{\partial y}f_1(\beta,z,y)=0$ is equal to

$$y_+=\frac{1-2\beta z+\sqrt{4\beta^2z^2+4\beta^2z+1}}{2(1+\beta)}. \tag{7.41}$$

As we will see in the following lemma, $1/2<y_+<1$ and as β tends to zero the saddle point tends to one. Accordingly, we need a version of the saddle point method which allows the saddle point to be located close to the end point. Such version was proved in [39, Theorem 1.3]. To apply this method, we also need to split the integral $\mathcal{I}(-\beta,\sigma,t,z)$ as a sum of several parts in order to localize the saddle point and to have good control over the growth of the function under the integral. Let us define four smoothed functions: $\chi_0(x)$, $\chi_{1/2}(x)$, $\chi_{sp}(x)$ $\chi_1(x)$, such that

$$\chi_0(x)+\chi_{1/2}(x)+\chi_{sp}(x)+\chi_1(x)=1 \quad \text{for} \quad 0<x<1, \tag{7.42}$$

$$\chi_0(x)=1 \quad \text{for} \quad 0<x<\frac{1-y_+}{8}, \quad \chi_0(x)=0 \quad \text{for} \quad x>\frac{1-y_+}{4}, \tag{7.43}$$

$$\chi_{1/2}(x)=1 \quad \text{for} \quad \frac{1-y_+}{4}<x<\frac{3y_+-1}{2}, \tag{7.44}$$
$$\chi_{1/2}(x)=0 \quad \text{for} \quad x<\frac{1-y_+}{8},\ x>\frac{5y_+-1}{4},$$

$$\chi_{sp}(x)=1 \quad \text{for} \quad |x-y_+|<\frac{1-y_+}{4}, \tag{7.45}$$
$$\chi_{sp}(x)=0 \quad \text{for} \quad |x-y_+|>\frac{1-y_+}{2},$$

(7.46)
$$\chi_1(x)=1 \quad \text{for} \quad \frac{1+y_+}{2}<x<1, \quad \chi_1(x)=0 \quad \text{for} \quad 0<x<\frac{1+3y_+}{4}.$$

For $\mathfrak{a}\in\{0,1/2,sp,1\}$ let
(7.47)
$$\mathcal{I}_{\mathfrak{a}}(-\beta,\sigma,t,z)=\int_0^1 g_{\mathfrak{a}}(y)e^{-itf_1(y)}dy, \quad g_{\mathfrak{a}}(y)=\frac{\chi_{\mathfrak{a}}(y)y^{-1/2}(1-y)^{-1/2}}{(z+y)^{\sigma}},$$

where for simplicity, we write $f_1(y)$ instead of $f_1(\beta,z,y)$.

Lemma 7.5. *For $t^{-1+\epsilon_0}<\beta<1+t^{-1+\epsilon}$ we have*

$$\mathcal{I}(-\beta,\sigma,t,z)=e^{\frac{\pi it}{4|t|}}\sqrt{2\pi}\frac{y_+^{1/2-it\beta}(1-y_+)^{1/2-it\beta}}{\sqrt{t\beta}(y_++z)^{\sigma+it(1-\beta)}} \tag{7.48}$$
$$\times\left(1+\frac{(2z+1)\beta}{\sqrt{4\beta^2z^2+4\beta^2z+1}}\right)^{1/2}\left(1+\sum_{j=1}^{n}\frac{c_j}{(t\beta)^j}\right)+O\left(\frac{\sqrt{\beta}}{(t\beta)^{n+1}}\right),$$

where $c_j\ll_{z,\sigma}1$.

Proof. First, we prove an asymptotic formula for $\mathcal{I}_{sp}(-\beta,\sigma,t,z)$ using the saddle point method. The saddle point is a solution of the equation $f_1'(y)=0$, which yields the quadratic equation:

$$(1+\beta)y^2+(2\beta z-1)y-\beta z=0. \tag{7.49}$$

At the point $y=0$ the quadratic polynomial above is equal to $-\beta z$, and at the point $y=1$ to $\beta(1+z)$, therefore only the largest root of (7.49) belongs to the interval $(0,1)$, being the only relevant saddle point:

$$y_+=\frac{1-2\beta z+\sqrt{(2\beta z-1)^2+4\beta(1+\beta)z}}{2(1+\beta)}. \tag{7.50}$$

Furthermore, since the left hand side of (7.49) at the point $y=1/2$ is equal to $\frac{\beta-1}{4}<0$, we have $1/2<y_+<1$ (and for $1<\beta<1+t^{-1+\epsilon}$ the point y_+ is close to $1/2$). As $\beta\to0$ the equality holds:

$$1-y_+=\frac{2\beta(1+z)}{1+2\beta(1+z)+\sqrt{4\beta^2z^2+4\beta^2z+1}}=\beta(1+z)+O(\beta^2). \tag{7.51}$$

From the equation $f_1'(y)=0$ one can deduce that

$$z+y_+=\frac{(1-\beta)y_+(1-y_+)}{\beta(2y_+-1)},$$

and thus

$$f_1''(y_+)=\frac{-\beta}{1-\beta}\frac{2(1+\beta)y_+^2-2(1+\beta)y_++1}{y_+^2(1-y_+)^2}. \tag{7.52}$$

Expressing y_+^2 via (7.49), and then substituting (7.50), we obtain

$$\begin{aligned}f_1''(y_+)&=\frac{-\beta}{1-\beta}\frac{1+2\beta z-2\beta(2z+1)y_+}{y_+^2(1-y_+)^2}=\\&=\frac{-\beta}{y_+^2(1-y_+)^2}\frac{\sqrt{4\beta^2z^2+4\beta^2z+1}}{\beta(1+2z)+\sqrt{4\beta^2z^2+4\beta^2z+1}}.\end{aligned} \tag{7.53}$$

To apply the saddle point method in the form proved in [39, Theorem 1.3], one needs to determine N,M,T,U such that for $s=0,\ldots,2n+1$ and $r=2,\ldots,2n+3$

$$|g_{sp}^{(s)}(y)|\ll\frac{U}{N^s},\quad |tf_1^{(r)}(y)|\ll\frac{T}{M^r}, \tag{7.54}$$

and $|tf_1^{(2)}(y)|\gg\frac{T}{M^2}$.

It follows from (7.45), (7.47) and (7.51) that $|g_{sp}(y)|\ll_{z,\sigma}\beta^{-1/2}$ and thus $U=\beta^{-1/2}$. Evaluating $g_{sp}^{(s)}(y)$ one can deduce that $N=\beta$. Using (7.52) and (7.51) we obtain $|tf_1^{(2)}(y)|\asymp\frac{t}{\beta}$. Evaluating $f_1^{(r)}(y)$ we have $|tf_1^{(r)}(y)|\ll\frac{t\beta}{\beta^r}$, and therefore $M=\beta$ and $T=t\beta$. Now we are ready to estimate all error terms in [39, (1.7)]. Due to the presence of $\chi_{sp}(y)$ in $g_{sp}(y)$ we immediately obtain that the term with functions $H_i(x)$ vanishes. The first error term of [39, (1.7)] can be estimated as

(7.55)
$$\frac{UM^{2n+5}}{NT^{n+2}}\sum_{j=1}^{[n/2+1/2]}\left(\frac{1}{y_+^{n+2+j}}+\frac{1}{(1-y_+)^{n+2+j}}\right)\sum_{i=j}^{n+1-j}\frac{(N/M)^i}{N^{n+1-j}}\ll\frac{\sqrt{\beta}}{(t\beta)^{n+2}},$$

$$\frac{UM^{2n+4}(M/N+1)}{T^{n+2}N^{n+1}}\left(\frac{1}{y_+^{n+2}}+\frac{1}{(1-y_+)^{n+2}}\right)\ll\frac{\sqrt{\beta}}{(t\beta)^{n+2}}, \tag{7.56}$$

(7.57)
$$\frac{UM^{2n+4}}{T^{n+2}}\sum_{j=1}^{n+1}\left(\frac{1}{y_+^{n+2+j}}+\frac{1}{(1-y_+)^{n+2+j}}\right)\sum_{i=0}^{n+1-j}\frac{(N/M)^i}{N^{n+1-j}}\ll\frac{\sqrt{\beta}}{(t\beta)^{n+2}},$$

$$\frac{U}{T^{n+1}}\left(\frac{M^{2n+2}}{N^{2n+1}}+M\right)\ll\frac{\sqrt{\beta}}{(t\beta)^{n+1}}. \tag{7.58}$$

Furthermore, we would like to simplify a bit the formula for the main term in [39, (1.7)]. Using [39, (3.4)], (7.53) and (7.51), we show that

$$\lambda_2=\frac{-tf_1''(y_+)}{4\pi}\asymp\frac{t\beta}{(1-y_+)^2}\asymp\frac{t}{\beta}\quad\text{and}\quad\lambda_k\asymp\frac{t\beta}{\beta^k}. \tag{7.59}$$

Also in [39, (3.21)] in our case we have $\eta_k\asymp\beta^{-1/2-k}$, and therefore, $\varpi_k\asymp\beta^{-1/2-k}$ (this also follows from [39, (4.8)]). Finally, [39, (1.7)] for $\mathcal{I}_{sp}(-\beta,\sigma,t,z)$ becomes

$$\mathcal{I}_{sp}(-\beta,\sigma,t,z)=e^{\frac{\pi it}{4|t|}}\frac{\sqrt{2\pi}e^{-itf_1(y_+)}}{\sqrt{|tf_1''(y_+)|}}\left(g_{sp}(y_+)+\sum_{j=1}^{n}\frac{c_j}{(t\beta)^j\sqrt{\beta}}\right)+O\left(\frac{\sqrt{\beta}}{(t\beta)^{n+1}}\right). \tag{7.60}$$

Using (7.47), (7.53) and (7.51), we infer

$$\mathcal{I}_{sp}(-\beta,\sigma,t,z)=e^{\frac{\pi it}{4|t|}}\frac{\sqrt{2\pi}}{\sqrt{\beta t}}\frac{y_+^{1/2-it\beta}(1-y_+)^{1/2-it\beta}}{(y_++z)^{\sigma+it(1-\beta)}}\times\\ \times\left(1+\frac{(2z+1)\beta}{\sqrt{4\beta^2z^2+4\beta^z+1}}\right)^{1/2}\left(1+\sum_{j=1}^{n}\frac{c_j}{(t\beta)^j}\right)+O\left(\frac{\sqrt{\beta}}{(t\beta)^{n+1}}\right). \tag{7.61}$$

In order to prove (7.48) it remains to show that the three remaining integrals $\mathcal{I}_{\mathfrak{a}}(-\beta,\sigma,t,z)$ are small. To do this, we will simply integrate by parts. Consider the case $\mathfrak{a}=1$ (other cases can be treated in a similar manner).

Since $f_1'(y) = \frac{\beta}{y} - \frac{\beta}{1-y} + \frac{1-\beta}{z+y} \asymp \frac{\beta}{1-y}$, we get

$$\mathcal{I}_1(-\beta,\sigma,t,z) \ll \frac{1}{t}\int_0^1 \frac{\partial}{\partial y}\left(\frac{g_1(y)}{f_1'(y)}\right) e^{-itf_1(y)}dy \ll \frac{1}{t\beta}\int_0^1 \frac{\chi_1(y)y^{-1/2}(1-y)^{-1/2}}{(z+y)^\sigma}dy \ll \frac{\sqrt{\beta}}{t\beta}. \tag{7.62}$$

Integrating by parts n-times and arguing similarly, we have

$$\mathcal{I}_1(-\beta,\sigma,t,z) \ll \frac{\sqrt{\beta}}{(t\beta)^n}. \tag{7.63}$$

Thus (7.48) follows. □

Using Lemma 7.5 we obtain the following estimate.

Lemma 7.6. *For $t \to +\infty$ and $|1-\beta| < t^{-1+\epsilon}$ the estimate holds:*

$$\mathcal{F}(\beta,\sigma,t,z) \ll |t|^{\epsilon_1-\sigma} e^{-\frac{\pi|t|}{2}(|1-\beta|+|\beta|-1)}. \tag{7.64}$$

Proof. Using (7.9), we obtain

$$\mathcal{P}(-\beta,\sigma,t) \ll t^{1/2-\sigma+\epsilon_1} e^{-\frac{\pi|t|}{2}(|1-\beta|+|\beta|-1)}. \tag{7.65}$$

Using this estimate and Lemma 7.5 we estimate $\mathcal{F}_1(-\beta,\sigma,t,z)$. Since $\mathcal{F}_1(\beta,\sigma,t,z)$ is negligible (due to (7.32)) we (7.18) complete the proof of the Lemma. □

Combining Lemma 7.5 and computations performed in Lemma 7.3, we obtain the following asymptotic formula.

Lemma 7.7. *For $t^{-1+\epsilon_0} < \beta < 1 - t^{-1+\epsilon}$ we have*

$$\begin{aligned}\mathcal{F}(\beta,\sigma,t,z) &= H(2s)e^{-\frac{\pi it}{4|t|}} 2^{1-2\sigma-2it}\sqrt{\pi}(1-\beta)^{\sigma-1/2+it(1-\beta)}\beta^{-1/2+it\beta} \\ &\times \frac{y_+^{1/2-it\beta}(1-y_+)^{1/2-it\beta}}{\sqrt{|t|}(y_++z)^{\sigma+it(1-\beta)}}\left(1+\sum_{j=1}^n \frac{c_j}{(t\beta)^j} + \sum_{j=1}^n \frac{d_j}{(t(1-\beta))^j}\right)\times \\ &\times\left(1+\frac{(2z+1)\beta}{\sqrt{4\beta^2z^2+4\beta^2z+1}}\right)^{1/2} + \\ &+ O\left(\frac{(1-\beta)^{\sigma-1/2}\sqrt{\beta}}{(t\beta)^{n+1}} + \frac{(1-\beta)^{\sigma-1/2}}{(t(1-\beta))^{n+3/2}\sqrt{t}}\right),\end{aligned} \tag{7.66}$$

where $c_j, d_j \ll_{z,\sigma} 1$.

Proof. We substitute (7.30) to (7.22), which yields

$$\begin{aligned}\mathcal{P}(-\beta,\sigma,t) &= \frac{-it}{|t|}2^{1/2-2\sigma-2it}(1-\beta)^{\sigma-1/2+it(1-\beta)}\beta^{it\beta} \\ &\times\left(1+\sum_{j=1}^N \frac{c_j}{|t\min(\beta,1-\beta)|^j} + O(|t\min(\beta,1-\beta)|^{-N-1})\right).\end{aligned} \tag{7.67}$$

It follows from (7.32) and (7.23) that the contribution of $\mathcal{F}_1(\beta,\sigma,t,z)$ to (7.18) is negligible. Substituting (7.67) and (7.48) to (7.19) and using (7.18), we prove (7.66). $\square$

Let us compare the main terms of asymptotic expansions (7.66) and (7.33). For $t^{-1} \ll \beta < t^{-1+\epsilon_0}$ we have

$$e^{it((1+\beta)\log(1+\beta)+(1-\beta)\log(1-\beta))} = e^{O(t\beta^2)} = 1+O(t\beta^2), \tag{7.68}$$

$$(1-\beta^2)^{\sigma-1/2} = 1+O(\beta^2). \tag{7.69}$$

Therefore, the main term of (7.33) is

$$H(2s)e^{-\frac{\pi it}{4|t|}}2^{1-2\sigma-2it}\sqrt{\pi}\frac{(z+1)^{1/2-\sigma-it}}{\sqrt{|t|}}. \tag{7.70}$$

Next, we simplify the main term of (7.66). Using (7.51) we have

$$y_+^{1/2-it\beta} = y_+^{1/2}e^{-it\beta\log y_+} = y_+^{1/2}e^{O(t\beta^2)} = 1+O(\beta)+O(t\beta^2), \tag{7.71}$$

$$(1-y_+)^{1/2-it\beta} = \beta^{1/2-it\beta}(1+z)^{1/2-it\beta}\left(1+O(\beta)+O(t\beta^2)\right), \tag{7.72}$$

$$(y_++z)^{\sigma+it(1-\beta)} = (1+z)^{\sigma+it(1-\beta)}(1-\beta)^{\sigma+it(1-\beta)}\left(1+O(t\beta^2)\right). \tag{7.73}$$

Therefore, the main term of (7.66) is equal to

$$H(2s)e^{-\frac{\pi it}{4|t|}}2^{1-2\sigma-2it}\sqrt{\pi}\frac{(1+z)^{1/2-\sigma-it}}{\sqrt{|t|}}, \tag{7.74}$$

which coincides with (7.70).

As a corollary of the obtained results on the hypergeometric function, we prove the following estimate on the growth of $\phi(s,\alpha)$.

Lemma 7.8. *For $s=\sigma+it$, $|t|\ll T^\epsilon$ and $|\sigma|<C$ we have*

$$H(2s)\phi(s,\alpha) \ll T^\epsilon, \tag{7.75}$$

and for $|t|\to+\infty$ we have

$$H(2s)\phi(s,\alpha) \ll |t|^{\mathfrak{k}(\sigma)}, \tag{7.76}$$

where

$$\mathfrak{k}(\sigma) = \begin{cases} 0, & \textit{if} \quad \sigma>1,\\ 3(1-\sigma)/2+\epsilon, & \textit{if} \quad 0\le\sigma\le 1,\\ 3/2-2\sigma, & \textit{if} \quad \sigma<0. \end{cases} \tag{7.77}$$

Proof. For $\sigma>1$ the estimates (7.75) and (7.76) hold since $\phi(s,\alpha)$ converges absolutely. From now on, let $\sigma\le 1$. Due to (5.18) and the following estimate on the Riemann-zeta function

$$|\zeta(s)| \ll |t|^{1/2-\sigma}\,\text{for}\,\sigma<0, \quad |\zeta(s)|\ll|t|^{1/2-\sigma/2}\,\text{for}\,0<\sigma<1,$$

it is enough to prove (7.75), (7.76) for $\Phi_j(s)$ instead of $\phi(s,\alpha)$, i.e.

$$H(2s)\Phi_j(s) \ll T^\epsilon \quad \text{for} \quad t\ll T^\epsilon, \tag{7.78}$$

$$H(2s)\Phi_j(s) \ll |t|^{\mathfrak{k}(\sigma)}. \tag{7.79}$$

Let us consider the case $t > 0$ (the opposite case $t < 0$ can be treated in the same way). The estimate (7.78) follows immediately from (7.3), (7.11) and Lemma 7.2. Next, we prove (7.79). Recall that $n\kappa = t\beta$. To estimate the sum (7.3) we split it into several sub-sums: $n = 0$, $0 < \beta \le t^{-1+\epsilon_0}$, $t^{-1+\epsilon_0} < \beta < 1 - t^{-1+\epsilon}$, $|\beta - 1| < t^{-1+\epsilon}$ and $\beta > 1 + t^{-1+\epsilon}$. If $n = 0$ we apply (7.12), for $n \ll t^{\epsilon_0}$ we apply Lemma 7.4, getting

$$\sum_{0 \le n \ll t^{\epsilon_0}} |\mathcal{F}(\beta, \sigma, t, z)| \left(1 + |t - n\kappa|\right)^{k(\sigma)} \left(1 + |t - n\kappa|\right)^{k(\sigma)} \ll |t|^{2k(\sigma) - 1/2 + \epsilon_0}. \tag{7.80}$$

The part of (7.3) with $\beta > 1 + t^{-1+\epsilon}$ is negligible in view of Lemma 7.3. In case when $|\beta - 1| < t^{-1+\epsilon}$, we use (7.64) to show that

$$\begin{aligned}\sum_{|n\kappa - t| < t^{\epsilon}} |\mathcal{F}(\beta, \sigma, t, z)| \left(1 + |t||1-\beta|\right)^{k(\sigma)} \left(1 + |t||1+\beta|\right)^{k(\sigma)} \\ \ll \sum_{|n\kappa - t| < t^{\epsilon}} |t|^{k(\sigma) - \sigma + \epsilon} \ll |t|^{k(\sigma) - \sigma + \epsilon_1}.\end{aligned} \tag{7.81}$$

We are left to consider the range $t^{-1+\epsilon_0} < \beta < 1 - t^{-1+\epsilon}$. Using (7.66) and (7.51), we have

$$\begin{aligned}\sum_{t^{\epsilon_0} < n\kappa < t - t^{\epsilon}} |\mathcal{F}(\beta, \sigma, t, z)| \left(1 + |t||1-\beta|\right)^{k(\sigma)} \left(1 + |t||1+\beta|\right)^{k(\sigma)} \ll \\ \ll \sum_{t^{\epsilon_0} < n\kappa < t - t^{\epsilon}} |t|^{2k(\sigma) - 1/2} (1-\beta)^{k(\sigma) + \sigma - 1/2} \ll |t|^{2k(\sigma) + 1/2},\end{aligned} \tag{7.82}$$

since $k(\sigma) + \sigma > -1/2$ due to (4.10). Combining (7.80), (7.81) and (7.82), we obtain

$$H(2s)\Phi_j(s) \ll |t|^{2k(\sigma) + 1/2}, \tag{7.83}$$

where $k(\sigma)$ is defined by (4.10). To improve this result in case when $0 \le \sigma \le 1$, we apply the Phragmen-Lindelöf theorem, which completes the proof of (7.76). □

8. Proof of Theorems 1.3, 1.4 and 1.5

A possible way to relate the sums (1.15) and (1.16) with (1.13) and (1.14) is to apply the following statement (this is an easy modification of the lemma on p. 105 of [9]).

Lemma 8.1. *For $r, c, x, T > 0$ we have*

$$\begin{aligned}\chi_{(1,\infty)}(x) \left(1 - \frac{1}{x}\right)^{r} = \frac{1}{2\pi i} \int_{c - iT}^{c + iT} \frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)} x^{s} ds + \\ + O\left(\frac{x^{c}}{T^{r}} \min\left(1, \frac{1}{T|\log x|}\right)\right),\end{aligned} \tag{8.1}$$

where $\chi_{(1,\infty)}(x)$ *is the characteristic function of the interval* $(1,\infty)$.

We begin the proofs by applying Lemma 8.1 to (1.15) and (1.16), which yields the identity.

Lemma 8.2. *We have*

$$\Delta_r(x,\alpha)+2S_r(\sqrt{\alpha x},\alpha)= \tag{8.2}$$
$$=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)x^s ds+O\left(\frac{x^c}{T^{r+1}}\right).$$

Proof. Using (8.1), we immediately obtain the main term of (8.2) plus the following error term:

(8.3)
$$\sum_{n\le x}\frac{x^c n^{\epsilon}}{n^c T^r}\min\left(1,\frac{1}{T|\log(x/n)|}\right)+\sum_{n\le\sqrt{\alpha x}}\frac{x^c}{n^{2c}T^r}\min\left(1,\frac{1}{T|\log(\alpha x/n^2)|}\right).$$

Let us estimate the first sum. Assume that $c>1+\epsilon_0$. We have

$$\sum_{n\le x}\frac{x^c n^{\epsilon}}{n^c T^r}\min\left(1,\frac{1}{T|\log(x/n)|}\right) \tag{8.4}$$
$$\ll\sum_{0\le x-n\ll x/T}\frac{x^{c+\epsilon}}{n^c T^r}+\sum_{x-n\gg x/T}\frac{x^c}{n^{c-\epsilon}T^{r+1}\log(x/n)}.$$

These sums can be estimated as

$$\frac{x^{c+\epsilon}}{x^c T^r}\frac{x}{T}+\frac{x^c}{T^{r+1}}\int_1^{x(1-1/T)}\frac{dy}{y^{c-\epsilon}\log(x/y)} \tag{8.5}$$
$$\ll\frac{x^{1+\epsilon}}{T^{r+1}}+\frac{x^c}{T^{r+1}}+\frac{x^c}{T^{r+1}}\int_{x(1-0.01)}^{x(1-1/T)}\frac{dy}{y^{c-\epsilon}\log(x/y)}.$$

Finally,

$$\frac{x^{1+\epsilon}}{T^{r+1}}+\frac{x^c}{T^{r+1}}+\frac{x^c}{T^{r+1}}\int_{x(1-0.01)}^{x(1-1/T)}\frac{dy}{y^{c-\epsilon}\log(x/y)} \tag{8.6}$$
$$\ll\frac{x^c}{T^{r+1}}+\frac{x^c}{T^{r+1}}\frac{x}{x^{c-\epsilon}}\log T\ll\frac{x^c}{T^{r+1}}.$$

□

Next, we move the line of integration in (8.2) to $\Re s=-\delta$ with $\delta=-1/2+2k$ and k - some large (but fixed) positive integer. Doing this, we cross poles at (see (5.16), (5.17)) $s=0$ and

$$-2m\pm in\kappa,\quad -1-2m\pm in\kappa,\quad \text{for}\, m,n\in\mathbb{Z}_{\ge0}, m\le k-1. \tag{8.7}$$

Note that these poles includes poles of $\Gamma(s)$ in (8.2).

Our goal now is to show that the combination $\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)$ is holomorphic at $s=in\kappa$.

Lemma 8.3. *For $n \neq 0$ we have*

$$\operatorname{res}_{s=in\kappa}\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)=0. \tag{8.8}$$

Proof. It follows from (4.1) that $\alpha^{in\kappa}=e^{\pi in}=(-1)^n$. Using this together with (5.23), (6.9) we conclude that

$$\begin{aligned}\operatorname{res}_{s=in\kappa}&\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)=\\ &=\left(\frac{2^{2in\kappa-3}D^{in\kappa/2}\alpha^{in\kappa}}{\pi^{1-in\kappa}\log\alpha}-\frac{(2\sqrt{\pi})^{2in\kappa-2}D^{in\kappa/2}(-1)^n}{2\log\alpha}\right)\times\\ &\qquad\times\frac{\Gamma(in\kappa)}{\Gamma(2in\kappa)}\left(\xi(in\kappa,\nu_1\lambda^n)+\xi(in\kappa,\nu_1\lambda^{-n})\right)=0.\end{aligned} \tag{8.9}$$

□

Next, we evaluate the residue at the point $s=0$.

Lemma 8.4. *We have*

$$\begin{aligned}\operatorname{res}_{s=0}\frac{2\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}(\alpha x)^s\psi(2s,\alpha)&=\frac{1}{4}-\frac{\sqrt{D}}{12}+\frac{\zeta'(0,\nu_1)}{2\pi\log\alpha}+\\ &+\frac{a-3}{12\log\alpha}\left(\log x-\psi(1+r)-\psi(1/2)-\gamma+\log\frac{\alpha D}{A}\right).\end{aligned} \tag{8.10}$$

Proof. Using (6.3) we obtain

$$\begin{aligned}\operatorname{res}_{s=0}&\frac{2\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}(\alpha x)^s\psi(2s,\alpha)=\\ &=2\operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{8\Gamma(1+r+s)}(\alpha xD)^s\Psi_1(2s)-\\ &\quad-2\operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{8\Gamma(1+r+s)}(\alpha xD)^s\Psi_0(2s)+\sqrt{D}\zeta(-1).\end{aligned} \tag{8.11}$$

In order to evaluate these two residues, we factor out from (6.2) the term with $n=0$, getting

$$\frac{\Gamma(1+r)\Gamma(s)}{8\Gamma(1+r+s)}(\alpha xD)^s\Psi_1(2s)=\frac{\Gamma(s)\Gamma^2(s/2)}{\Gamma(2s)}g_{1,0}(s)+\frac{\Gamma(s)}{\Gamma(2s)}g_{1,1}(s), \tag{8.12}$$

$$\frac{\Gamma(1+r)\Gamma(s)}{8\Gamma(1+r+s)}(\alpha xD)^s\Psi_0(2s)=\frac{\Gamma(s)\xi(s,\nu_0)}{\Gamma(2s)}g_{0,0}(s)+\frac{\Gamma(s)}{\Gamma(2s)}g_{0,1}(s), \tag{8.13}$$

where

$$g_{0,0}(s)=\frac{\Gamma(1+r)(\alpha xD)^s}{\Gamma(1+r+s)}\frac{2^{2s-5}A^{-s}}{\pi\log\alpha}\Gamma^2\left(\frac{1+s}{2}\right), \tag{8.14}$$

$$g_{0,1}(s)=\frac{\Gamma(1+r)(\alpha xD)^s}{\Gamma(1+r+s)}\frac{2^{2s-5}A^{-s}}{\pi\log\alpha}\times\sum_{n\neq0}\xi(s,\nu_0\lambda^n)\Gamma\left(\frac{s+1+in\kappa}{2}\right)\Gamma\left(\frac{s+1-in\kappa}{2}\right),\tag{8.15}$$

$$g_{1,0}(s)=\frac{\Gamma(1+r)(\alpha xD)^s}{\Gamma(1+r+s)}\frac{2^{2s-5}A^{-s}}{\pi\log\alpha}\xi(s,\nu_1),\tag{8.16}$$

$$g_{1,1}(s)=\frac{\Gamma(1+r)(\alpha xD)^s}{\Gamma(1+r+s)}\frac{2^{2s-5}A^{-s}}{\pi\log\alpha}\times\sum_{n\neq0}\xi(s,\nu_1\lambda^n)\Gamma\left(\frac{s+in\kappa}{2}\right)\Gamma\left(\frac{s-in\kappa}{2}\right).\tag{8.17}$$

Substituting (8.12) and (8.13) to (8.11), we have

$$\operatorname{res}_{s=0}\frac{2\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}(\alpha x)^s\psi(2s,\alpha)=\\=2\operatorname{res}_{s=0}\frac{\Gamma^2(s/2)\Gamma(s)}{\Gamma(2s)}g_{1,0}(s)-2\operatorname{res}_{s=0}\frac{\Gamma(s)\xi(s,\nu_0)}{\Gamma(2s)}g_{0,0}(s)+\sqrt{D}\zeta(-1).\tag{8.18}$$

It follows from [40, 5.5.5] and [40, 5.7.1] that

$$\frac{\Gamma^2(s/2)\Gamma(s)}{\Gamma(2s)}=\frac{\Gamma^2(s/2)\sqrt{\pi}}{2^{2s-1}\Gamma(s+1/2)},\quad\Gamma^2(z)=\frac{1}{z^2}-\frac{2\gamma}{z}+O(1).\tag{8.19}$$

Let

$$\tilde{g}_{1,0}(s)=\frac{\sqrt{\pi}}{2^{2s-1}\Gamma(s+1/2)}g_{1,0}(s),\tag{8.20}$$

then using (8.19) we obtain

$$\operatorname{res}_{s=0}\frac{\Gamma^2(s/2)\Gamma(s)}{\Gamma(2s)}g_{1,0}(s)=4\tilde{g}'_{1,0}(0)-4\gamma\tilde{g}_{1,0}(0).\tag{8.21}$$

Therefore, applying (4.7) and (8.21), we can rewrite (8.18) as

$$\operatorname{res}_{s=0}\frac{2\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}(\alpha x)^s\psi(2s,\alpha)=\\=8\tilde{g}'_{1,0}(0)-8\gamma\tilde{g}_{1,0}(0+8g_{0,0}(0)\log\alpha+\sqrt{D}\zeta(-1).\tag{8.22}$$

To evaluate $g_{0,0}(0)$ we simply use (8.14)

$$g_{0,0}(0)=\frac{2^{-5}}{\log\alpha}.\tag{8.23}$$

To compute $\tilde{g}'_{1,0}(0)$ and $\tilde{g}_{1,0}(0)$ we apply (8.16),(8.20) together with (4.8), showing that

$$\tilde{g}_{1,0}(0)=\frac{a-3}{96\log\alpha},\tag{8.24}$$

$$\tilde{g}'_{1,0}(0) = \frac{a-3}{96\log\alpha}\left(\log\frac{x\alpha D}{A} - \psi(1+r) - \psi(1/2)\right) + \frac{\zeta'(0,\nu_1)}{16\pi\log\alpha}. \tag{8.25}$$

Substituting (8.23), (8.24) and (8.25) to (8.22) we prove (8.10). □

In the same manner we prove the following result.

Lemma 8.5. *We have*

$$\operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\phi(s,\alpha)x^s = \log\sqrt{\alpha} - \frac{1}{4} - \frac{\zeta'(0,\nu_1)}{2\pi\log\alpha} - \tag{8.26}$$
$$- \frac{a-3}{12\log\alpha}\left(\log x - \psi(1+r) - \psi(1/2) - \gamma + \log(\pi\sqrt{D})\right).$$

Proof. Applying (5.18) we obtain

$$\operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\phi(s,\alpha)x^s = \operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{4\Gamma(1+r+s)}(xD)^s\Phi_0(s) - \tag{8.27}$$
$$- \operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{4\Gamma(1+r+s)}(xD)^s\Phi_1(s) - \zeta(0)\log\alpha.$$

In order to evaluate these two residues, we factor out from (5.2) the term with $n=0$, getting

$$\frac{\Gamma(1+r)\Gamma(s)}{4\Gamma(1+r+s)}(xD)^s\Phi_0(s) = \frac{\Gamma(s)\xi(s,\nu_0)}{\Gamma(2s)}f_{0,0}(s) + \frac{\Gamma(s)}{\Gamma(2s)}f_{0,1}(s), \tag{8.28}$$

$$\frac{\Gamma(1+r)\Gamma(s)}{4\Gamma(1+r+s)}(xD)^s\Phi_1(s) = \frac{\Gamma(s)\Gamma^2(s/2)}{\Gamma(2s)}f_{1,0}(s) + \frac{\Gamma(s)}{\Gamma(2s)}f_{1,1}(s), \tag{8.29}$$

where

$$f_{0,0}(s) = \frac{\Gamma(1+r)x^s}{\Gamma(1+r+s)}\frac{(2\sqrt{\pi})^{2s-2}D^{s/2}}{4\log\alpha}\Gamma^2\left(\frac{s+1}{2}\right)\times \tag{8.30}$$
$$\times {}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right),$$

$$f_{0,1}(s) = \frac{\Gamma(1+r)x^s}{\Gamma(1+r+s)}\frac{(2\sqrt{\pi})^{2s-2}D^{s/2}}{4\log\alpha}\times \tag{8.31}$$
$$\times\sum_{n\neq0}(-1)^n\xi(s,\nu_0\lambda^n)\Gamma\left(\frac{s+1+in\kappa}{2}\right)\Gamma\left(\frac{s+1-in\kappa}{2}\right)\times$$
$$\times {}_2F_1\left(s-in\kappa,s+in\kappa,1/2+s;\frac{2-a}{4}\right),$$

$$f_{1,0}(s) = \frac{\Gamma(1+r)x^s}{\Gamma(1+r+s)}\frac{(2\sqrt{\pi})^{2s-2}D^{s/2}}{4\log\alpha}\xi(s,\nu_1)\times \tag{8.32}$$
$$\times {}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right),$$

$$f_{1,1}(s)=\frac{\Gamma(1+r)x^s}{\Gamma(1+r+s)}\frac{(2\sqrt{\pi})^{2s-2}D^{s/2}}{4\log\alpha}\times \\ \times\sum_{n=-\infty}^{\infty}(-1)^n\xi(s,\nu_1\lambda^n)\Gamma\left(\frac{s+in\kappa}{2}\right)\Gamma\left(\frac{s-in\kappa}{2}\right)\times \\ \times {}_2F_1\left(s-in\kappa,s+in\kappa,1/2+s;\frac{2-a}{4}\right). \tag{8.33}$$

Arguing as in Lemma 8.4 (i.e. applying (8.19)), we infer that

$$\mathrm{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\phi(s,\alpha)x^s= \\ =-4f_{0,0}(0)\log\alpha-4\tilde{f}'_{1,0}(0)+4\gamma\tilde{f}_{1,0}(0)-\zeta(0)\log\alpha, \tag{8.34}$$

where

$$\tilde{f}_{1,0}(s)=\frac{\sqrt{\pi}}{2^{2s-1}\Gamma(s+1/2)}f_{1,0}(s). \tag{8.35}$$

To evaluate the hypergeometric function in (8.30), (8.32), we apply (5.13):

$${}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right)=\left(\frac{a}{2}\right)^{-s}{}_2F_1\left(\frac{s+1}{2},\frac{s}{2},1/2+s;1-\frac{4}{a^2}\right). \tag{8.36}$$

The advantage of working with the hypergeometric function on the right hand side of (8.36) is that it can be represented as a series [40, 15.2.1]. Doing so, we obtain

$${}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right)\Big|_{s=0}=1. \tag{8.37}$$

To evaluate $f_{0,0}(0)$ we simply use (8.30) and (8.37), for $\tilde{f}_{1,0}(0)$ we use (8.32), (8.35), (8.37) together with (4.8), getting

$$f_{0,0}(0)=\frac{1}{16\log\alpha},\quad \tilde{f}_{1,0}(0)=\frac{a-3}{48\log\alpha}. \tag{8.38}$$

It is left to study $\tilde{f}'_{1,0}(0)$. To this end, it is required to find the derivative of the hypergeometric function in (8.32) with respect to s at the point $s=0$. It follows from (8.36) that

$$\frac{d}{ds}{}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right)\Big|_{s=0}= \\ =-\log\frac{a}{2}+\frac{d}{ds}{}_2F_1\left(\frac{s+1}{2},\frac{s}{2},1/2+s;\frac{D}{a^2}\right)\Big|_{s=0}. \tag{8.39}$$

Representing the last hypergeometric function via its series representation [40, 15.2.1], we find that

$$\begin{aligned}(8.40)\quad \frac{d}{ds}{}_2F_1\left(\frac{s+1}{2},\frac{s}{2},1/2+s;\frac{D}{a^2}\right)\bigg|_{s=0} &= \\ &= \frac{d}{ds}\left(1+\sum_{j=1}^{\infty}\frac{(s/2)_j\,(s/2+1/2)_j}{j!(s+1/2)_j}\left(\frac{D}{a^2}\right)^j\right)\Bigg|_{s=0}.\end{aligned}$$

Since $(s/2)_j = \frac{s}{2}(s/2+1)\cdot\ldots\cdot(s/2+j-1)$, the derivative is non-zero only when we differentiate the factor $s/2$. Thus

$$\begin{aligned}(8.41)\quad \frac{d}{ds}{}_2F_1\left(\frac{s+1}{2},\frac{s}{2},1/2+s;\frac{D}{a^2}\right)\bigg|_{s=0} &= \\ &= \frac{1}{2}\sum_{j=1}^{\infty}\frac{(j-1)!}{j!}\left(\frac{D}{a^2}\right)^j = \frac{-1}{2}\log\left(1-\frac{D}{a^2}\right) = \log\frac{a}{2}.\end{aligned}$$

It follows from (8.39) and (8.41) that

$$(8.42)\qquad \frac{d}{ds}{}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right)\bigg|_{s=0} = 0.$$

Using (8.35), (8.32), (8.42) and (4.8), we obtain

$$\begin{aligned}(8.43)\quad \tilde{f}'_{1,0}(0) &= \frac{1}{8\pi\log\alpha}\frac{d}{ds}\times \\ &\times\left(\frac{\Gamma(1+r)\Gamma(1/2)}{\Gamma(1+r+s)\Gamma(s+1/2)}(\pi x\sqrt{D})^s{}_2F_1\left(s,s,1/2+s;\frac{2-a}{4}\right)\xi(s,\nu_1)\right)\Bigg|_{s=0} \\ &= \frac{a-3}{48\log\alpha}\Big(\log(\pi x\sqrt{D})-\psi(1+r)-\psi(1/2)\Big)+\frac{\zeta'(0,\nu_1)}{8\pi\log\alpha}.\end{aligned}$$

Substituting (8.38) and (8.43) to (8.34), we prove (8.26). □

Lemma 8.6. *We have*

$$(8.44)\qquad \operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)x^s = C(a),$$

where $C(a)$ is defined by (1.22).

Proof. Summing (8.10) and (8.26), recalling that $A=\sqrt{D}/\pi$, we finally show that

$$\begin{aligned}(8.45)\quad \operatorname{res}_{s=0}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)x^s &= \\ &= \frac{a-3}{12\log\alpha}\log\frac{\alpha D}{A\pi\sqrt{D}}+\log\sqrt{\alpha}-\frac{\sqrt{D}}{12} = C(a).\end{aligned}$$

□

Now we are going to estimate residues at the points $-1-2m\pm in\kappa$, see (8.7).

Lemma 8.7. *For $n>0$ we have*

$$\operatorname{res}_{s=-1-2m\pm in\kappa}\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)\ll_m n^{5/2+4m}. \tag{8.46}$$

Proof. We estimate only the residues of the function $\phi(s,\alpha)$ since for $\psi(2s,\alpha)$ all computations are similar (compare (5.2) and (6.2)). Furthermore, we consider only the case of + sign in the pole. It follows from (5.18) and (5.2) that

$$\operatorname{res}_{s=-1-2m+in\kappa}\phi(s,\alpha)=\operatorname{res}_{s=-1-2m+in\kappa}\frac{D^s}{4}\Phi_0(s). \tag{8.47}$$

Using (8.47), (5.15) and the first identity of (5.13), we obtain

$$\operatorname{res}_{s=-1-2m+in\kappa}\phi(s,\alpha)\ll\frac{\Gamma\left(-m+in\kappa\right)}{\Gamma(-2-4m+2in\kappa)} \tag{8.48}$$
$$\times\,\xi(-1-2m+in\kappa,\nu_0\lambda^{\pm n})\,{}_2F_1\left(\frac{-1-2m}{2},-m,-1/2-2m+in\kappa;\frac{D}{a^2}\right).$$

Since the second parameter of the hypergeometric function is a negative integer it reduces [40, 15.2.4] to a polynomial and thus is $\ll_m 1$. To estimate the Hecke zeta function we apply (4.6), (4.5), getting

$$\xi(-1-2m+in\kappa,\nu_0\lambda^{\pm n})=\xi(2+2m-in\kappa,\nu_0\lambda^{\mp n})\ll_m\Gamma\left(1+m-in\kappa\right). \tag{8.49}$$

Substituting (8.49) to (8.48) and using the Stirling formula (7.4), we find

$$\operatorname{res}_{s=-1-2m+in\kappa}\phi(s,\alpha)\ll\frac{\Gamma\left(-m+in\kappa\right)\Gamma\left(1+m-in\kappa\right)}{\Gamma(-2-4m+2in\kappa)}\ll n^{5/2+4m}, \tag{8.50}$$

thus proving the lemma. ∎

Similarly, we estimate residues at the points $-2m\pm in\kappa$, see (8.7).

Lemma 8.8. *For $m>0$ and $n>0$ we have*

$$\operatorname{res}_{s=-2m\pm in\kappa}\left(\phi(s,\alpha)+2\alpha^s\psi(2s,\alpha)\right)\ll_m n^{1/2+4m}. \tag{8.51}$$

Proof. Once again we estimate only residues of the function $\phi(s,\alpha)$ and consider only the case of + sign in the pole. It follows from (5.18) and (5.2) that

$$\operatorname{res}_{s=-2m+in\kappa}\phi(s,\alpha)=\operatorname{res}_{s=-2m+in\kappa}\frac{-D^s}{4}\Phi_1(s). \tag{8.52}$$

Using (8.52), (5.15), the first identity of (5.13) followed by (4.6), (4.5), and the Stirling formula (7.4), we obtain

$$\operatorname{res}_{s=-2m+in\kappa}\phi(s,\alpha)\ll\frac{\Gamma\left(-m+in\kappa\right)}{\Gamma(-4m+2in\kappa)}\xi(-2m+in\kappa,\nu_1\lambda^{\pm n})\ll \tag{8.53}$$
$$\ll\frac{\Gamma\left(-m+in\kappa\right)\Gamma(1+m-in\kappa)}{\Gamma(-4m+2in\kappa)}\ll n^{1/2+4m}.$$

$\square$

8.1. **Proof of Theorem 1.3.** As has been already explained, we move the line of integration in (8.2) to $\Re s = -\delta$ with $\delta = -1/2 + 2k$ and k being some large (but fixed) positive integer, crossing the poles at (8.7). The residue at $s = 0$ is evaluated in Lemma 8.6, and we prove in Lemma 8.3 that residues at the points $s = in\kappa, n \neq 0$ are equal to zero. The contribution of the remaining residues was estimated in Lemmas 8.7 and 8.8. Consequently,

$$\begin{gathered} \Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x},\alpha) = C(a) + O\left(\frac{x^c}{T^{r+1}}\right) + \\ O\left(\sum_{m=0}^{k-1}\sum_{0<n\ll T}\frac{n^{4m+3/2-r}}{x^{1+2m}}\right) + O\left(\sum_{m=1}^{k-1}\sum_{0<n\ll T}\frac{n^{4m-1/2-r}}{x^{2m}}\right) + O(x^{-1+\epsilon}) + \\ O\left(\int_{-\delta-iT}^{c+iT}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}G(s)x^s ds\right) + O\left(\int_{-\delta\pm iT}^{c\pm iT}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}G(s)x^s d\sigma\right), \end{gathered} \tag{8.54}$$

where

$$G(s) = \phi(s,\alpha) + 2\alpha^s\psi(2s,\alpha). \tag{8.55}$$

Note that we used the following fact (see (8.47), (5.15),(5.13)):

$$\begin{aligned} &\operatorname{res}_{s=-1-2m}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\phi(s,\alpha)x^s = \\ &= \operatorname{res}_{s=-1-2m}\Gamma^2\left(\frac{s+1}{2}\right)\frac{x^s D^{s/2}}{4\log\alpha}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)\Gamma(2s)} \\ &\times (2\sqrt{\pi})^{2s-2}\xi(s,\nu_0)\left(\frac{a}{2}\right)^{-s}{}_2F_1\left(\frac{s}{2},\frac{1+s}{2},1/2+s;\frac{D}{a^2}\right) \ll_m x^{-1-2m+\epsilon}. \end{aligned} \tag{8.56}$$

Similarly, the residue at $s = -2m$ is bounded by $x^{-2m+\epsilon}$.

To estimate the integrals in (8.54), we apply (6.11), (7.76). In order to be able to do this, we choose among special values of T (see [19, p.227]):

$$\frac{\pi k}{\log\alpha} < T_k < \frac{\pi(k+1)}{\log\alpha}. \tag{8.57}$$

In this case, $H(2\sigma \pm 2iT_k) \gg 1$. For simplicity, we will write T instead of T_k.

Lemma 8.9. *For $c = 1+\epsilon$, $\delta > 0$, $0 < r < 3/2 + 2\delta$ we have*

$$\begin{aligned} &\frac{x^c}{T^{r+1}} + \int_{-\delta-iT}^{-\delta+iT}\frac{\Gamma(s)}{\Gamma(r+1+s)}G(s)x^s dt + \int_{-\delta\pm iT}^{c\pm iT}\frac{\Gamma(s)}{\Gamma(r+1+s)}G(s)x^s d\sigma \\ &\qquad \ll \frac{x^{1+\epsilon}}{T^{1+r}} + \frac{T^{3/2-r+2\delta}}{x^\delta}. \end{aligned} \tag{8.58}$$

Proof. Consider the first integral in (8.58). First, we treat the part over $|t| \ll T^{\epsilon}$. Using Lemma 7.2 we deduce that

$$\int_{-\delta-iT^{\epsilon}}^{-\delta+iT^{\epsilon}} \frac{\Gamma(s)}{\Gamma(r+1+s)} G(s) x^s dt \ll T^{\epsilon} x^{-\delta}. \tag{8.59}$$

In the remaining range, we apply (7.76), showing that

$$\int_{-\delta+iT^{\epsilon}}^{-\delta+iT} \frac{\Gamma(s)}{\Gamma(r+1+s)} G(s) x^s dt \ll \int_{-\delta+iT^{\epsilon}}^{-\delta+iT} t^{1/2+2\delta-r} x^{-\delta} dt \ll T^{3/2-r+2\delta} x^{-\delta} \tag{8.60}$$

if $r < 3/2 + 2\delta$. Using (6.11), (7.76), we obtain

$$\begin{aligned} \int_{-\delta+iT}^{c+iT} \frac{\Gamma(s)G(s)}{\Gamma(r+1+s)} x^{\sigma} d\sigma &\ll \int_{-\delta}^{c} T^{\mathfrak{k}(\sigma)-1-r} x^{\sigma} d\sigma \\ \ll \int_{-\delta}^{0} T^{1/2-2\sigma-r} x^{\sigma} d\sigma &+ \int_{0}^{1} T^{1/2-3\sigma/2-r+\epsilon} x^{\sigma} d\sigma + \int_{1}^{c} T^{-1-r} x^{\sigma} d\sigma \\ \ll T^{1/2-r} \max\left(\frac{T^{2\delta}}{x^{\delta}}, 1\right) &+ T^{1/2-r+\epsilon} \max\left(\frac{x}{T^{3/2}}, 1\right) + \frac{x^c}{T^{1+r}} \\ &\ll T^{1/2-r+\epsilon} \max\left(\frac{x}{T^{3/2}}, \frac{T^{2\delta}}{x^{\delta}}\right) + \frac{x^c}{T^{1+r}}. \end{aligned} \tag{8.61}$$

The same result holds for the integral over $(-\delta - iT, c - iT)$. Taking into account both (8.60) and (8.61) and choosing $c = 1+\epsilon$, we obtain (8.58). $\square$

Applying Lemma 8.9 to (8.54), we obtain for $0 < r < 5/2$

$$\begin{aligned} \Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x}, \alpha) &= C(a) + O\left(x^{-1+\epsilon} + \frac{x^{1+\epsilon}}{T^{1+r}} + \frac{T^{3/2-r+2\delta}}{x^{\delta}}\right) \\ &+ O\left(\sum_{m=0}^{k-1} \frac{T^{4m+5/2-r}}{x^{1+2m}}\right) + O\left(\sum_{m=1}^{k-1} \frac{T^{4m+1/2-r}}{x^{2m}}\right) \\ &= C(a) + O\left(x^{-1+\epsilon} + \frac{x^{1+\epsilon}}{T^{1+r}} + \frac{T^{3/2-r+2\delta}}{x^{\delta}}\right) \\ &+ O\left(\frac{T^{5/2-r}}{x} \max\left(1, \frac{T^{4k}}{x^{2k}}\right) + \frac{T^{9/2-r}}{x^2} \max\left(1, \frac{T^{4k-4}}{x^{2k-2}}\right)\right). \end{aligned} \tag{8.62}$$

The optimal choice of T is $T_{op} = x^{\frac{2+2\delta}{5+4\delta}}$, which yields (since $T_{op} < x^{1/2}$) the estimate

$$\Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x}, \alpha) = C(a) + O\left(x^{-1+\epsilon} + x^{1+\epsilon-(1+r)\frac{2+2\delta}{5+4\delta}}\right). \tag{8.63}$$

To ensure that the error term is smaller than the main term, one requires $(1+r)\frac{2+2\delta}{5+4\delta} > 1 + 2\epsilon$, which holds if $r > \frac{3/2+\delta}{1+\delta} + 5\epsilon$. In terms of δ we have

$\delta > \frac{3-2r+10\epsilon}{2r-2-8\epsilon}$. Since $\delta = -1/2 + 2k$, choosing k sufficiently large, we obtain

$$\Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x},\alpha) = C(a) + O_a(x^{-\epsilon}) \tag{8.64}$$

for $r \geq 1 + \epsilon_1$, thus proving Theorem 1.3.

8.2. **Proof of Theorems 1.4 and 1.5.** In case of Theorem 1.4, we have $r > 3$, and thus for $m = 0$ the sum over n in (8.54) is now $O(1)$ instead of $O(T^{5/2-r})$. Nevertheless, this does not affect (8.63). To get the error term $O\left(x^{-1+\epsilon}\right)$ in (8.63), one needs $(1+r)\frac{2+2\delta}{5+4\delta} > 2$, which holds if $r > \frac{4+3\delta}{1+\delta}$. The latter inequality is valid for $r > 3$ since we can choose $\delta = -1/2 + 2k$ with k being sufficiently large. This completes the proof of Theorem 1.4.

The proof of Theorem 1.5 differs only slightly from the proof of Theorems 1.3 and 1.4. Applying (8.1) to (1.15), we get

$$\Delta_r(x,\alpha) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\phi(s,\alpha)x^s ds + O\left(\frac{x^c}{T^{r+1}}\right). \tag{8.65}$$

Moving the line of integration to the left, we first pass poles at the points $s = 0$ and $s = in\kappa$. The residue at $s = 0$ gives (see (8.26)) the main term of (1.26), and the residues (5.23) at $s = in\kappa$ give rise to the series in (1.26). Note that replacing the sum of residues by the series (1.26) produces the error

$$\sum_{|n|\gg T} \frac{(-1)^n\Gamma(in\kappa)}{\Gamma(1+r+in\kappa)}(xD)^{in\kappa}\zeta(in\kappa,\nu_1\lambda^n) \ll \sum_{|n|\gg T}\frac{n^{1/2+\epsilon}}{n^{1+r}} \ll T^{1/2-r+\epsilon}, \tag{8.66}$$

which is smaller than $x^{1+\epsilon}/T^{1+r}$ for $T \ll \sqrt{x}$. Thus, this does not affect the error. The rest of the proof is identical to that given in Theorem 1.3, since we get an analogue of (8.54) with $G(s)$ replaced by $\phi(s,\alpha)$, and the function $\phi(s,\alpha)$ grows faster than $2\alpha^s\psi(2s,\alpha)$.

Remark. An interesting question is to determine the smallest value of r for which Theorems 1.3 and 1.4 still hold. Unfortunately, we were unable to obtain some reasonable conjectures based solely on numerical computations. So instead, we decided to speculate a little about the possibility of various improvements in the proofs of Theorems 1.3 and 1.4. One key ingredient is to improve the convexity estimate on $\phi(s,\alpha)$ for $\Re s < 0$, see Lemma 7.8. It seems reasonable to assume that there is a square-root cancellation in the sum (7.82). This would allow us to improve the estimate (7.77) from $\mathfrak{k}(\sigma) = 3/2 - 2\sigma$ to $\mathfrak{k}(\sigma) = 1 - 2\sigma$. Furthermore, if it were possible to obtain an asymptotic expansion for this sum, we could also assume a square-root cancellation in the integral (8.60). All these assumptions yield the following improvement of (8.62):

$$\Delta_r(x,\alpha)+2S_r(\sqrt{\alpha x},\alpha) = C(a)+O\left(x^{-\delta+\epsilon}+\frac{x^{1+\epsilon}}{T^{1+r}}+\frac{T^{1/2-r+2\delta}}{x^{\delta}}\right), \tag{8.67}$$

where $0 < \delta < 1$. The optimal choice of T now is $T_{op} = x^{\frac{2+2\delta}{3+4\delta}}$. As a result,

$$\Delta_r(x,\alpha) + 2S_r(\sqrt{\alpha x},\alpha) = C(a) + O\left(x^{-\delta+\epsilon} + x^{1+\epsilon-(1+r)\frac{2+2\delta}{3+4\delta}}\right). \tag{8.68}$$

The error term above is $O(x^{-\epsilon})$ if we assume that $r > \frac{1+2\delta}{2+2\delta}$. Taking δ a small positive number, we could conclude that Theorem 1.3 holds for $r > 1/2$, under these assumptions. Furthermore, it seems unlikely that further reduction of $1/2$ is possible.

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