ON THE ANALYTIC THEORY OF TERNARY CUBIC FORMS

OLGA BALKANOVA, WILLIAM DUKE, AND DMITRY FROLENKOV

Abstract. The analytic properties of a Dirichlet series whose coefficients count certain representations of integers by a ternary cubic form are studied. The form is a cubic number field version of the sum of three cubes. Three main results are proven. The first gives the analytic continuation of the series to the right half-plane. The second applies this, together with growth estimates, to give an asymptotic formula for a smoothed sum of the coefficients. The third main result refines this asymptotic formula, assuming greater smoothing, to get a second main term. In some cases the constant in this second term is given in terms of the arithmetic of an associated sextic number field. The proofs make use of certain Hecke-type zeta functions for a cubic number field.

1. Introduction

Consider the integral ternary cubic form,

$$F(x,y,z)=ax^3+by^3+cz^3+3dx^2y+3ex^2z+3fy^2x+3gy^2z+3hz^2x+3iz^2y+6jxyz, \tag{1.1}$$

where $a,b,c,\dots\in\mathbb{Z}$. Little is known in general about the number of solutions to

$$m=F(x,y,z)$$

with integers m,x,y,z. It can be infinite, so one usually restricts the x,y,z in some natural way so that this number, say $r_F(m)$, is finite. An analytic technique that is sometimes quite useful is to define the Dirichlet series

$$\psi_F(s)=\sum_{m\geq 1} r_F(m)m^{-s} \tag{1.2}$$

and relate its properties to $r_F(m)$, especially its averages, say

$$S_F(X)=\sum_{m\leq X} r_F(m),$$

and various smoothed versions.

This technique is well-known to be powerful in some cases where F is decomposable into a product of linear factors. For example, when $F(x,y,z)=xyz$ we have that

$$r_F(n)=d_3(n),$$

the generalized divisor function, when one restricts to positive x,y,z. The associated Dirichlet series is given by $\psi_F(s)=\zeta^3(s)$. Another paradigm case is when F is a norm form for a cubic number field $\mathbb{F}$, for instance

$$F(x,y,z)=(\alpha_1x+\alpha_2y+\alpha_3z)(\alpha_1'x+\alpha_2'y+\alpha_3'z)(\alpha_1''x+\alpha_2''y+\alpha_3''z), \tag{1.3}$$

where $\alpha_1,\alpha_2,\alpha_3$ is an integral basis for an integral ideal $\mathfrak{a}$ in $\mathbb{F}$ and where

$$\alpha=\alpha^{(1)},\alpha'=\alpha^{(2)},\alpha''=\alpha^{(3)}$$

denote the Galois conjugates of $\alpha\in\mathbb{F}$. After Dirichlet/Dedekind, one restricts to counting integer triples (x,y,z) for which $\alpha_1x+\alpha_2y+\alpha_3z$ have a given norm m but are not associated to one another by a unit of $\mathbb{F}$; $r_F(m)$ is then finite. If $\mathfrak{a}$ is the ring of integers $\mathcal{O}$ of $\mathbb{F}$ and $\mathbb{F}$ has class number one then $\psi_F(s)$ is simply the Dedekind zeta function of $\mathbb{F}$.

At the other extreme is the example

$$F(x,y,z) = x^3 + y^3 + z^3.$$

Here it is natural to define $r_F(m)$ by restricting $x, y, z > 0$. For this form F, "elementary" techniques can be applied to study averages of $r_F(m)$. In 2020 Vaughan [19] applied estimates for exponential sums in an ingenious way to show that for $F(x,y,z) = x^3+y^3+z^3$ we have

$$S_F(X) = \sum_{m\le X} r_F(m) = \Gamma^3(\tfrac{4}{3})X - \frac{\Gamma^2(\frac{4}{3})}{2\Gamma(\frac{5}{3})}X^{\frac{2}{3}} + O(X^{\frac{5}{9}}\log^{\frac{1}{3}} X). \tag{1.4}$$

Here $O(X^{\frac{2}{3}})$ represents the "trivial" estimate for the remainder after the main term. For this problem it likely seems unreasonable to expect that the Dirichlet series $\psi_F(s)$ from (1.2) can be very useful, since there is no known multiplicative structure behind the counting function for the sum of three cubes.

Thus it might be surprising that for the number field version of the sum of three cubes it is sometimes possible to adapt the ideas of Dirichlet/Dedekind's and utilize the units of the field to study $\psi_F(s)$. Suppose that $\{\alpha_1, \alpha_2, \alpha_3\}$ is an integral basis for an integral ideal $\mathfrak{a}$ in a totally real cubic number field $\mathbb{F}$ with discriminant D. Then

$$F(x,y,z) = (\alpha_1 x + \alpha_2 y + \alpha_3 z)^3 + (\alpha_1' x + \alpha_2' y + \alpha_3' z)^3 + (\alpha_1'' x + \alpha_2'' y + \alpha_3'' z)^3, \tag{1.5}$$

is a ternary cubic form with rational integral coefficients. Define the counting function

(1.6)

$$r_F(m) = \#\{(x,y,z) \in \mathbb{Z}^3;\ F(x,y,z) = m \text{ where } \alpha_1^{(j)}x + \alpha_2^{(j)}y + \alpha_3^{(j)}z > 0 \text{ for } j = 1,2,3\}$$

and let $\psi_F(s)$ be given by (1.2). It is easy to show by a lattice point count that $r_f(m)$ is finite and that the series $\psi_F(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$.

An example is the form

$$F(x,y,z) = 3x^3 - 4y^3 + 38z^3 - 3x^2y + 15x^2z + 15xy^2 + 39y^2z + 39z^2x - 48z^2y - 24xyz, \tag{1.7}$$

associated to the Galois extension $\mathbb{F} = \mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$. Here $\mathfrak{a}$ is the ring of integers in $\mathbb{F}$ with integral basis $\{1, \alpha, \alpha^2\}$ and discriminant $D = 49$.

Note that many arguments that apply elementary methods to a form rely on it having a simple structure, for example being diagonal. This comment applies to Vaughan's result.

We shall apply Hecke-like zeta functions with Grössencharakteren for $\mathbb{F}$ to show that for F in (1.5) the Dirichlet series $\psi_F(s)$ has an analytic continuation to $\mathrm{Re}(s) > 0$ except for a simple pole at $s = 1$. Our approach is inspired by the paper [10] of Hecke.

Theorem 1. *For a general totally real cubic number field $\mathbb{F}$ and an integral ideal $\mathfrak{a} \subset \mathbb{F}$, the function $\psi_F(s)$ has a simple pole at $s = 1$ with residue*

$$\Gamma^3(\tfrac{4}{3})N(\mathfrak{a})^{-1}D^{-\frac{1}{2}}$$

but is otherwise holomorphic for $\mathrm{Re}(s) > 0$.

To prove this we express $\psi_F(s)$ an an infinite series of terms involving the Hecke zeta functions multiplied by gamma factors. We will see that it is likely that the imaginary axis is a natural boundary for $\psi_F(s)$ coming from poles of the gamma factors. After estimating $\psi_F(s)$ in vertical strips to the right of the imaginary axis, we can then deduce the following asymptotic formula.

Theorem 2. *Fix F from (1.5). For $r \geq \frac{5}{2}$ and any $\epsilon > 0$ we have*

$$\sum_{1\leq m\leq X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}}X + O(X^\epsilon).$$

Despite the likely existence of a natural boundary, we are able to truncate the infinite series effectively and give a second main term in the asymptotic formula of Theorem 2, provided we require larger values of r. In the proof we apply a well-known result of Baker, that improves on an earlier result of Gelfond, and gives a lower bound for the difference between the logarithms of two algebraic numbers.

Theorem 3. *Fix F from (1.5). There exists constants R, K_1 depending on $\mathbb{F}$ such that for $r > R$ we have*

$$\sum_{1\leq m\leq X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}}X + K_1 \log X + O(1).$$

The constant K_1 is given explicitly below in terms of the values at $s = 0$ of certain completed Hecke zeta functions. If we assume that $\mathfrak{a}$ is the ring of integers $\mathcal{O}$ of $\mathbb{F}$ and that the wide class number of $\mathbb{F}$ is one, we can say more. In this case by [1] the narrow class number of $\mathbb{F}$ is either one or two. If it is one then it will be shown that $K_1 = 0$. For the example in (1.7) we have that $K_1 = 0$. If the narrow class number of $\mathbb{F}$ is two then $\mathbb{F}$ has a unique unramified (at the finite places) quadratic extension $\mathbb{K}$ and it will be shown that $K_1 > 0$ is given by

$$K_1 = \frac{R_{\mathbb{K}} h_{\mathbb{K}}}{2 w_{\mathbb{K}} R_{\mathbb{F}}^2}, \tag{1.8}$$

where $h_{\mathbb{K}}$ is the class number of $\mathbb{K}$, $R_{\mathbb{K}}$ is its regulator and $w_{\mathbb{K}}$ is the number of roots of unity in $\mathbb{K}$. The formula (1.8) applies, for instance, to those ternary cubics coming from $\mathbb{F}$ with $D = 229, 257, 697, 761, 788, 892, 985, \ldots$. When $D = 229$ the form is

$$F(x,y,z) = 3x^3 + 24x^2z + 24xy^2 + 18xyz + 96xz^2 + 3y^3 + 96y^2z + 60yz^2 + 131z^3,$$

which is associated to the non-Galois extension $\mathbb{F}$ determined by $x^3 - 4x - 1 = 0$, and the sextic field $\mathbb{K}$ is determined by $x^6 - x^4 - x^3 - x^2 + 1 = 0$. Here $K_1 = 0.0358081\ldots$. It seems remarkable that representations by these forms should be affected by the arithmetic of associated sextic number fields.

Remark. It is possible to apply invariant theory to give necessary conditions that our theorems apply to a given integral ternary cubic form F. There are two fundamental invariants of F as in (1.1) (See [16, §220,221] for the complete formulas):

$$S = abcj - (bcde + cafg + abhi) - j(agi + bhe + cdf) + \cdots$$

$$T = a^2b^2c^2 - 6abc(agi + bhe + cdf) - 20abcj^3 + 12(abcj(fh + id + eg) + \cdots.$$

with the discriminant being of degree twelve in the coefficients: $R = T^2 + 64S^3$. The Hessian H_F of F:

$$H_F = \det(\partial_{i,j} F),$$

is the most basic covariant of F. The necessary conditions are that $S = 0$, T be the cube of a positive integer and H_F be irreducible over $\mathbb{Q}$. These conditions can be verified by direct calculation.

2. A Hecke-type zeta function

In this section we define and state the basic properties of the Hecke-type zeta function that we need. Basic references are Hecke's papers [8],[9]. However, our zeta function does not seem to immediately reduce to one defined by Hecke and so we will prove all of the properties we state here in the section following this one.

Let $\mathbb{F} \subset \mathbb{R}$ be a totally real cubic number field. Let $\mathcal{O}$ be the ring of integers of $\mathbb{F}$, $\mathcal{O}^*$ be the group of units, and $\mathcal{O}^+$ be the subgroup of totally positive units.

Let $\mathfrak{a}$ be a (fractional) ideal in $\mathbb{F}$. Suppose that $\alpha_1, \alpha_2, \alpha_3$ is an integral basis for $\mathfrak{a}$. Let $\mathfrak{d}$ be the different and $D = N(\mathfrak{d})$ the discriminant of $\mathbb{F}$. For $a, b, c \in \mathbb{Z}$ write

$$\begin{aligned} \mu &= a\alpha_1 + b\alpha_2 + c\alpha_3 \\ \mu' &= a\alpha_1' + b\alpha_2' + c\alpha_3' \\ \mu'' &= a\alpha_1'' + b\alpha_2'' + c\alpha_3''. \end{aligned} \tag{2.1}$$

Here again $\mu = \mu^{(1)}, \mu' = \mu^{(2)}, \mu'' = \mu^{(3)}$ are the Galois conjugates of μ, ordered in a fixed way.

Sign Characters. A sign character is defined for $\mu \in \mathcal{O}$ and can be expressed uniquely by the formula

$$\nu = (\mathrm{sgn}^{a_1}\mu^{(1)})(\mathrm{sgn}^{a_2}\mu^{(2)})(sgn^{a_3}\mu^{(3)}), \tag{2.2}$$

where $a_j \in \{0, 1\}$. Call (a_1, a_2, a_3) the *signature* of ν. Note that $\nu(0) = 0$ for any ν. For $\mu_1, \mu_2 \in \mathbb{F}$ and $\eta \in \mathcal{O}^+$ we have

$$\nu(\mu_1)\nu(\mu_2) = \nu(\mu_1\mu_2) \text{ and } \nu(\eta\mu_1) = \nu(\mu_1).$$

For any function C summed over $\mathfrak{a}$ we can restrict to a sum over totally positive μ (to be denoted $\mu \succ 0$) using these characters:

$$\sum_{\substack{\mu\in\mathfrak{a}\\ \mu\succ 0}} C(\mu) = \tfrac{1}{8}\sum_{\nu}\Big(\sum_{\mu}\nu(\mu)C(\mu)\Big) \tag{2.3}$$

where the sum over ν is over all eight sign characters.

Grössencharaktere. Let (η_1, η_2) be a basis for the subgroup $\mathcal{O}^+$ of totally positive units ordered so that

$$\delta = \log\eta_1 \log\eta_2' - \log\eta_1' \log\eta_2 > 0. \tag{2.4}$$

From now on η always denotes a totally positive unit, so $\eta = \eta_1^{n_1}\eta_2^{n_2}$ for some $n_1, n_2 \in \mathbb{Z}$. For

$$M = \begin{pmatrix} 1 & \log\eta_1 & \log\eta_2 \\ 1 & \log\eta_1' & \log\eta_2' \\ 1 & \log\eta_1'' & \log\eta_2'' \end{pmatrix} \quad \text{define } e_i^{(j)} \text{ by} \quad M^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ e_1^{(1)} & e_2^{(1)} & e_3^{(1)} \\ e_1^{(2)} & e_2^{(2)} & e_3^{(2)} \end{pmatrix}. \tag{2.5}$$

To evaluate the first row of M^{-1} as well as to show that $\det M = 3\delta$ we have applied [12, Lemma, p. 123] to matrices

$$\begin{pmatrix} \log\eta_1 & \log\eta_1' & \log\eta_1'' \\ \log\eta_2 & \log\eta_2' & \log\eta_2'' \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix}$$

with the last row subsequently being equal to $(1,0,0), (0,1,0), (0,0,1)$. Thus a computation to eliminate η_1'' and η_2'' (since $\eta_j'\eta_j'\eta_j'' = 1$ for $j = 1, 2$) gives

$$3\delta e_1^{(1)} = 2\log\eta_2' + \log\eta_2 \qquad 3\delta e_2^{(1)} = -2\log\eta_2 - \log\eta_2' \qquad 3\delta e_3^{(1)} = \log\eta_2 - \log\eta_2' \tag{2.6}$$

$$3\delta e_1^{(2)} = -2\log\eta_1' - \log\eta_1 \qquad 3\delta e_2^{(2)} = 2\log\eta_1 + \log\eta_1' \qquad 3\delta e_3^{(2)} = \log\eta_1' - \log\eta_1. \tag{2.7}$$

The following relations hold:

$$\sum_{1\le k\le 3} e_k^{(r)} = 0 \ \text{ for } r = 1,2, \tag{2.8}$$

$$\tfrac{1}{3} + \sum_{1\le j\le 2} e_p^{(j)} \log \eta_j^{(q)} = \delta_{p,q} \ \text{ for } p,q = 1,2,3 \tag{2.9}$$

$$\sum_{1\le k\le 3} e_k^{(r)} \log \eta_s^{(k)} = \delta_{r,s} \ \text{ for } r,s = 1,2, \tag{2.10}$$

where $\delta_{i,j}$ is the Kronecker symbol. For $n = (n_1, n_2) \in \mathbb{Z}^2$ let

$$\lambda_n(\mu) = \exp\big(2\pi i \sum_{1\le i\le 2} n_i \sum_{1\le j\le 3} e_j^{(i)} \log|\mu^{(j)}|\big) \tag{2.11}$$

be a Grössencharakter for $\mathbb{F}$. Then, for $\mu_1, \mu_2 \in \mathbb{F}$, we have that

$$\lambda_n(\mu_1)\lambda_n(\mu_2) = \lambda_n(\mu_1\mu_2) \ \text{ and } \ \lambda_n(\eta\mu_1) = \lambda_n(\mu_1). \tag{2.12}$$

The Hecke-type zeta function. For $\mathfrak{a}$ a fractional ideal in $\mathbb{F}$ let $(\mu)_{\mathfrak{a}}$ denote a set of representatives for strict association classes of $\mu \in \mathfrak{a}$, where $\mu_1, \mu_2 \in \mathfrak{a}$ are strictly associated if there is a totally positive unit η so that $\mu_1 = \mu_2\eta$. The zeta function with sign character ν and Grössencharakter λ_n we need is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_{\mathfrak{a}}(s, \lambda_n\nu) = \sum_{(\mu)_{\mathfrak{a}}} \nu(\mu)\lambda_n(\mu)|N(\mu)|^{-s}. \tag{2.13}$$

If the signature of ν is (a_1, a_2, a_3) let

$$\sigma_\nu = \tfrac{1}{2}\sum_{j=1}^{3} a_j. \tag{2.14}$$

Clearly $\zeta_{\mathfrak{a}}(s, \nu\lambda_n)$ will vanish identically unless $\sigma \in \{0,1\}$. More generally, $\zeta_{\mathfrak{a}}(s, \nu\lambda_n)$ will vanish identically if there exists a unit ϵ with the property that $\nu(\epsilon)\lambda_n(\epsilon) \neq 1$.

Set for $j = 1,2,3$

$$\kappa_j = (e_j^{(1)}, e_j^{(2)}), \tag{2.15}$$

with $e_j^{(i)}$ given in (2.6).

Theorem 4. *Let $\mathbb{F}$ be a totally real cubic field.*
i) The zeta function $\zeta_{\mathfrak{a}}(s, \nu\lambda_n)$ from (2.13) is entire except for a simple pole at $s = 1$ when $\nu\lambda_n = \nu_0\lambda_0$.
ii) It satisfies the functional equation

$$\xi_{\mathfrak{a}}(s, \nu\lambda_n) = \Gamma_{\nu\lambda_n}(s)\zeta_{\mathfrak{a}}(s, \nu\lambda_n) = (-1)^{\sigma} N(\mathfrak{a})^{-1} D^{-\frac{1}{2}} \xi_{\frac{1}{\mathfrak{a}\mathfrak{d}}}(1-s, \nu\lambda_{-n}), \quad \text{where}$$

$\mathfrak{d}$ is the different of $\mathbb{F}$ and

$$\Gamma_{\nu\lambda_n}(s) = \pi^{-\frac{3s}{2}}\Gamma\big(\tfrac{s+a_1}{2} - \tfrac{2\pi i n\cdot\kappa_1}{2}\big)\Gamma\big(\tfrac{s+a_2}{2} - \tfrac{2\pi i n\cdot\kappa_2}{2}\big)\Gamma\big(\tfrac{s+a_3}{2} - \tfrac{2\pi i n\cdot\kappa_3}{2}\big), \tag{2.16}$$

with $\cdot$ the usual dot product.
iii) The zeta function has finite order. More precisely, for any $A > 1$ there is a $C > 0$ depending only on $\mathbb{F}$ and A so that for all s with $|s-1| \ge 1$

$$|\zeta_{\mathfrak{a}}(s, \nu\lambda_n)| \le Ce^{|s|^A}.$$

iv) We have the evaluation

$$\mathrm{res}_{s=1}\zeta_{\mathfrak{a}}(s, \nu_0\lambda_0) = 8\delta N(\mathfrak{a})^{-1} D^{-\frac{1}{2}}. \tag{2.17}$$

v) In any fixed vertical strip and away from a pole, the function $\xi(s,\nu\lambda_n)$ is bounded uniformly in the variables s and n.

vi) The following estimate holds for $s=\sigma+it$:

$$\zeta_{\mathfrak{a}}(s,\nu\lambda_n) \ll \prod_{j=1}^{3}(1+|t-2\pi(n\cdot\kappa_j)|)^{k(\sigma)}, \quad \textit{where} \tag{2.18}$$

$$k(\sigma)=\begin{cases} 0, & \textit{if} \quad \sigma>1, \\ (1-\sigma)/2+\epsilon, & \textit{if} \quad 0\le\sigma\le 1, \\ 1/2-\sigma, & \textit{if} \quad \sigma<0. \end{cases} \tag{2.19}$$

The implied constant does not depend on t or n.

3. An Epstein zeta function

In this section, we will prove Theorem 4. The main reference is Siegel's book [17]. For convenience and completeness, we have included an Appendix that contains detailed treatments of various long but somewhat routine calculations needed in the proof of Proposition 1.

The idea is to deduce the basic analytic properties i)–v) of $\zeta_{\mathfrak{a}}(s,\nu\lambda_n)$ from an identity that expresses it in terms of a Fourier coefficient of a certain Epstein zeta function. For $x=(x_1,x_2)\in\mathbb{R}^2$ and $j=1,2,3$ we let

$$w_j(x)=\exp(x_1\log\eta_1^{(j)}+x_2\log\eta_2^{(j)})=\eta_1^{(j)x_1}\eta_2^{(j)x_2}. \tag{3.1}$$

Define the Epstein zeta function for $\mathrm{Re}(s)>\frac{3}{2}$ by

$$Z_{\mathfrak{a}}(s,x)=\sum_{\substack{\mu\in\mathfrak{a}\\ \mu\neq 0}}\prod_{j=1}^{3}(w_j(x)\mu^{(j)})^{a_j}\left(\mu^2w_1^2(x)+\mu'^2w_2^2(x)+\mu''^2w_3^2(x)\right)^{-s-\sigma_\nu}, \tag{3.2}$$

where σ_ν is from (2.14).

Proposition 1. *The function $Z_{\mathfrak{a}}(s,x)$ is entire in s unless $(a_1,a_2,a_3)=(0,0,0)$, when it is holomorphic except for a simple pole at $s=\frac{3}{2}$ with residue $2\pi N(\mathfrak{a})^{-1}D^{-\frac{1}{2}}$. It has order at most one. The function*

$$Z^*_{\mathfrak{a}}(s,x)=\pi^{-s}\Gamma(s+\sigma_\nu)Z_{\mathfrak{a}}(s,x)$$

satisfies the functional equation

$$Z^*_{\mathfrak{a}}(s,x)=i^{-2\sigma_\nu}N(\mathfrak{a})^{-1}D^{-\frac{1}{2}}Z^*_{\frac{1}{\mathfrak{a}\mathfrak{d}}}(\tfrac{3}{2}-s,-x).$$

It is bounded away from a pole in any fixed vertical strip.

Proof. We will apply [17, Thm. 3. p.54] to prove this. Define the positive definite 3×3 matrix $Q_{\mathfrak{a},x}$ determined by

$$(a,b,c)Q_{\mathfrak{a},x}(a,b,c)^t=\mu^2w_1^2(x)+\mu'^2w_2^2(x)+\mu''^2w_3^2(x) \tag{3.3}$$

with entries a,b,c from (2.1). A calculation (see Lemma 12 in the Appendix) together with a standard result (see e.g. [11, p.88]) shows that

$$\det Q_{\mathfrak{a},x}=\det A^2=DN^2(\mathfrak{a}), \tag{3.4}$$

where

$$A=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1' & \alpha_2' & \alpha_3'\\ \alpha_1'' & \alpha_2'' & \alpha_3''\end{pmatrix}. \tag{3.5}$$

Let

$$P_{\mathfrak{a},x}(a,b,c) = \prod_{j=1}^{3}(w_j(x)\mu^{(j)})^{a_j} = \prod_{j=1}^{3} w_j^{a_j}(x)\prod_{j=1}^{3}(a\alpha_1^{(j)} + b\alpha_2^{(j)} + c\alpha_3^{(j)})^{a_j}. \tag{3.6}$$

Another calculation (Lemma 13 in the Appendix) now shows

$$\operatorname{tr} Q_{\mathfrak{a},x}^{-1} P_{\mathfrak{a},x}^{\dagger} = 0,$$

where $P_{\mathfrak{a},x}^{\dagger}$ is the 3×3 matrix of second partials of $P_{\mathfrak{a},x}$ in the variables a,b,c. This means that $P_{\mathfrak{a},x}$ is a spherical function with respect to $Q_{\mathfrak{a},x}$ (see [17, p.46]).

In the notation of [17, p.47] we have that

$$Z_{\mathfrak{a}}(s,x) = \zeta(s,0,0,Q_{\mathfrak{a},x},P_{\mathfrak{a},x}). \tag{3.7}$$

It is classical that the columns of A^{-1} for A from (3.5) give an integral basis for $\frac{1}{\mathfrak{a}\mathfrak{d}}$ (see [11, §36]). A calculation using this fact (Lemma 14 in the Appendix) shows that

$$Q_{\mathfrak{a},x}^{-1} = Q_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x}. \tag{3.8}$$

The dual polynomial is by definition (see [17, p.50.])

$$P_{\mathfrak{a},x}^{*}(a,b,c) = P_{\mathfrak{a},x}(Q_{\mathfrak{a},x}^{-1}(a,b,c)^t). \tag{3.9}$$

Using this, still another computation (Lemma 15 in the Appendix) gives that

$$P_{\mathfrak{a},x}^{*} = P_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x}. \tag{3.10}$$

Thus from (3.2), (3.8) and (3.10) we have

$$Z_{\frac{1}{\mathfrak{a}\mathfrak{d}}}(s,-x) = \zeta(s,0,0,Q_{\mathfrak{a},x}^{-1},P_{\mathfrak{a},x}^{*}). \tag{3.11}$$

Now using (3.7) and (3.11), all of Proposition 1 except the finite order statement, is a direct consequence of Theorem 3 of [17]. The finite order statement is a standard application of the theta function representation of $Z_{\mathfrak{a}}^{*}(s,x)$ that comes from [17, (60)]. □

In order to prove Theorem 4, we will express our zeta function as part of a Fourier coefficient of $Z_{\mathfrak{a}}(s,x)$. The function $Z_{\mathfrak{a}}(s,x)$ is invariant under $x_1 \mapsto x_1+1$ and under $x_2 \mapsto x_2+1$. Most of Theorem 4 follows immediately from Proposition 1 and the following result. The upper bound (2.18) is a consequence of the uniform Phragmén-Lindelöf result given in [15].

Recall the definition of $Z_{\mathfrak{a}}^{*}(s,x)$ from Proposition 1.

Proposition 2. *For $n=(n_1,n_2)\in\mathbb{Z}^2$ and $\mathrm{Re}(s)>1$*

$$\int_0^1\int_0^1 Z_{\mathfrak{a}}^{*}(\tfrac{3s}{2},x)e^{-2\pi i n\cdot x}dx_1\,dx_2 = \tfrac{1}{12\delta}\,\Gamma_{\nu\lambda_n}(s)\zeta_{\mathfrak{a}}(s,\nu\lambda_n),$$

where δ is from (2.4) and $\Gamma_{\nu\lambda_n}(s)$ is given in (2.16).

The proof of Proposition 2 reduces to that of the following Lemma by the change of variables $s\mapsto\frac{3s}{2}$, after an identification of the Gamma factors is made. Here we use $\frac{2a_1-a_2-a_3}{6}+\frac{\sigma_\nu}{6}=\frac{a_1}{2}$ and similar statements for the other two factors.

Lemma 1. *Assumptions as in Proposition 2,*

$$I(n) = \int_0^1\int_0^1 Z_{\mathfrak{a}}(s,x)e^{-2\pi i n\cdot x}dx_1\,dx_2 = \frac{\Gamma_{\nu,n}^{*}(s+\sigma_\nu)}{12\,\delta\,\Gamma(s+\sigma_\nu)}\zeta_{\mathfrak{a}}(\tfrac{2s}{3},\nu\lambda_n), \quad \textit{where}$$

$$\begin{aligned}\Gamma_{\nu\lambda_n}^{*}(s) = \Gamma\big(\tfrac{s}{3}+\tfrac{2a_1-a_2-a_3}{6}-\pi i\kappa_1\cdot n\big)\Gamma\big(\tfrac{s}{3}+\tfrac{2a_2-a_1-a_3}{6}-\pi i\kappa_2\cdot n\big)\\ \times\Gamma\big(\tfrac{s}{3}+\tfrac{2a_3-a_1-a_2}{6}-\pi i\kappa_3\cdot n\big).\end{aligned} \tag{3.12}$$

Proof. The integral $I(n)$ unfolds into the form

$$I(n)=\sum_{(\mu)_{\mathfrak{a}}} I_n(\mu,s), \tag{3.13}$$

where

$$I_n(\mu,s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{3}(w_j(x)\mu^{(j)})^{a_j} \times \left(\mu^2w_1^2(x)+\mu'^2w_2^2(x)+\mu''^2w_3^2(x)\right)^{-s-\sigma_\nu}e^{-2\pi in\cdot x}dx_1\,dx_2, \tag{3.14}$$

which is well defined on $(\mu)_{\mathfrak{a}}$. For a fixed representative μ make the following change of variables in this integral:

$$x\mapsto x-y,$$

where $y=(y_1,y_2)$ with

$$y_1=e_1^{(1)}\log|\mu|+e_2^{(1)}\log|\mu'|+e_3^{(1)}\log|\mu''| \tag{3.15}$$

$$y_2=e_1^{(2)}\log|\mu|+e_2^{(2)}\log|\mu'|+e_3^{(2)}\log|\mu''|. \tag{3.16}$$

Using (2.11) we prove that

$$I_n(\mu,s)=\lambda_n(\mu)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{3}(w_j(x-y)\mu^{(j)})^{a_j} \times \left(\mu^2w_1^2(x-y)+\mu'^2w_2^2(x-y)+\mu''^2w_3^2(x-y)\right)^{-s-\sigma_\nu}e^{-2\pi in\cdot x}dx_1\,dx_2. \tag{3.17}$$

It follows from (3.1) that

$$w_j(x-y)\mu^{(j)}=w_j(x)\mu^{(j)}\exp(-y_1\log\eta_1^{(j)}-y_2\log\eta_2^{(j)}). \tag{3.18}$$

Using (3.15) and (2.9), we obtain

$$y_1\log\eta_1^{(j)}+y_2\log\eta_2^{(j)}=\sum_{p=1}^{3}\log|\mu^{(p)}|\sum_{i=1}^{2}e_p^{(i)}\log\eta_i^{(j)}=\log|\mu^{(j)}|-\frac{1}{3}\log|N(\mu)|. \tag{3.19}$$

Substituting (3.19) to (3.18), we have

$$w_j(x-y)\mu^{(j)}=w_j(x)|N(\mu)|^{1/3}\frac{\mu^{(j)}}{|\mu^{(j)}|}=w_j(x)|N(\mu)|^{1/3}\operatorname{sgn}(\mu^{(j)}). \tag{3.20}$$

Substituting (3.20) to (3.17) and using (2.2), (2.14) we show that

$$I_n(\mu,s)=\nu(\mu)\lambda_n(\mu)|N(\mu)|^{-\frac{2s}{3}}\hat{I}(s+\sigma_\nu), \tag{3.21}$$

where

$$\hat{I}(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{3}w_j^{a_j}(x)\left(w_1^2(x)+w_2^2(x)+w_3^2(x)\right)^{-s}e^{-2\pi in\cdot x}dx_1\,dx_2. \tag{3.22}$$

Next we must compute the integral $\hat{I}(s)$. In view of (3.21) and (3.13), the proof of Lemma 1 will be complete once we show

$$\hat{I}(s)=\frac{\Gamma^*_{\nu\lambda_n}(s)}{12\,\delta\,\Gamma(s)}, \tag{3.23}$$

where $\Gamma^*_{\nu\lambda_n}(s)$ is from (3.12). Recall (3.1) and make the change of variables in (3.22) given for $j=1,2$ by $u_j = w_j(x)$. Solving for x_1, x_2 we have

$$\begin{aligned}\delta x_1 &= \log\eta_2' \log u_1 - \log\eta_2 \log u_2\\ \delta x_2 &= -\log\eta_1' \log u_1 + \log\eta_1 \log u_2.\end{aligned}$$

Since $w_1(x)w_2(x)w_3(x) = 1$ and the Jacobian (see (2.4)) is $(\delta u_1 u_2)^{-1}$ this yields

$$\hat{I}(s) = \frac{1}{\delta}\int_0^\infty\int_0^\infty \left(u_1^2 + u_2^2 + \tfrac{1}{(u_1u_2)^2}\right)^{-s} u_1^\alpha u_2^\beta \tfrac{du_1 du_2}{u_1 u_2}, \tag{3.24}$$

where

$$\begin{aligned}\alpha &= \tfrac{2\pi i}{\delta}(n_2\log\eta_1' - n_1\log\eta_2') + a_1 - a_3\\ \beta &= \tfrac{2\pi i}{\delta}(n_1\log\eta_2 - n_2\log\eta_1) + a_2 - a_3.\end{aligned} \tag{3.25}$$

We now apply the following double Mellin transform formula.

Lemma 2. *For* $\operatorname{Re} s > \max(\operatorname{Re}(\alpha+\beta), \operatorname{Re}(\alpha-2\beta), \operatorname{Re}(\beta-2\alpha))$ *one has*

$$\int_0^\infty\int_0^\infty (u_1 + u_2 + \tfrac{1}{u_1u_2})^{-s} u_1^\alpha u_2^\beta \tfrac{du_1du_2}{u_1u_2} = \tfrac{\Gamma(\frac{1}{3}(s-\alpha-\beta))\Gamma(\frac{1}{3}(s+2\alpha-\beta))\Gamma(\frac{1}{3}(s-\alpha+2\beta))}{3\Gamma(s)}. \tag{3.26}$$

Proof. Make the following substitution:

$$u_1 = yt, \quad u_2 = t(1-y), \quad 0\le y\le 1, \quad 0\le t<\infty,$$

getting

$$\int_0^1 y^{s+\alpha-1}(1-y)^{s+\beta-1}\int_0^\infty \left(1 + t^3 y(1-y)\right)^{-s} t^{2s+\alpha+\beta-1} dt dy.$$

To evaluate the t-integral, we apply [5, Sec. 6.2. (30)]. As a result, we obtain for $\operatorname{Re} s > \max(\operatorname{Re}(-\alpha/2-\beta/2), \operatorname{Re}(\alpha+\beta))$

$$\tfrac{\Gamma(\frac{1}{3}(2s+\alpha+\beta))\Gamma(\frac{1}{3}(s-\alpha-\beta))}{3\Gamma(s)}\int_0^1 y^{(s+2\alpha-\beta)/3-1}(1-y)^{(s-\alpha+2\beta)/3-1}dy.$$

Finally, using [14, 5.12.1] we obtain (3.26). [1] □

To verify (3.23), make the obvious change of variables in (3.24) so we can apply (3.26)

$$\hat{I}(s) = \tfrac{\Gamma(\frac{1}{6}(2s-\alpha-\beta))\Gamma(\frac{1}{6}(2s+2\alpha-\beta))\Gamma(\frac{1}{6}(2s-\alpha+2\beta))}{12\,\delta\,\Gamma(s)}.$$

Then use the next identities, which follow from (3.25) and the formulas for the $e's$ from (2.6),(2.7):

$$2\alpha - \beta = -6\pi i(n_1 e_1^{(1)} + n_2 e_1^{(2)}) + 2a_1 - a_2 - a_3, \tag{3.27}$$

$$2\beta - \alpha = -6\pi i(n_1 e_2^{(1)} + n_2 e_2^{(2)}) + 2a_2 - a_1 - a_3, \tag{3.28}$$

$$\alpha + \beta = 6\pi i(n_1 e_3^{(1)} + n_2 e_3^{(2)}) + a_1 + a_2 - 2a_3. \tag{3.29}$$

This completes the proof of Lemma 1 hence of Proposition 2 and Theorem 4. □

---

[1] We have for any $n\ge 1$ the following elegant formula

$$\begin{aligned}&(n+1)\Gamma(s)\int_0^\infty\cdots\int_0^\infty (u_1 + u_2 + \cdots + u_n + \tfrac{1}{u_1\cdots u_n})^{-s} u_1^{\alpha_1}\cdots u_n^{\alpha_n}\tfrac{du_1\cdots du_n}{u_1\cdots u_n}\\ &= \Gamma\big(\tfrac{1}{n+1}(s-\alpha_1-\cdots-\alpha_n)\big)\Gamma\big(\tfrac{1}{n+1}(s+n\alpha_1-\alpha_2-\cdots-\alpha_n)\big)\cdots\Gamma\big(\tfrac{1}{n+1}(s-\alpha_1-\cdots+n\alpha_n)\big).\end{aligned}$$

4. The Dirichlet series $\psi_F(s)$

In this section we prove the following result, which includes Theorem 1. When not specified, the dependence of an implied constant will become clear from the context of the argument.

Proposition 3. *The function $\psi_F(s)$ has a simple pole at $s=1$ with residue*

$$\frac{\Gamma^3(\frac{4}{3})}{N(\mathfrak{a})D^{\frac{1}{2}}}$$

but is otherwise holomorphic for $\mathrm{Re}(s)>0$. *For any fixed* $0<\delta_1<\delta_2$ *there is* $A>0$ *so that uniformly in* $\delta_1\le \mathrm{Re}(s)\le\delta_2$ *we have*

$$\psi_F(s)\ll e^{|\mathrm{Im}(t)|^A}.$$

For the ternary cubic form

$$F(x,y,z)=(\alpha_1x+\alpha_2y+\alpha_3z)^3+(\alpha_1'x+\alpha_2'y+\alpha_3'z)^3+(\alpha_1''x+\alpha_2''y+\alpha_3''z)^3,$$

where $\{\alpha_1,\alpha_2,\alpha_3\}$ is an integral basis for the ideal $\mathfrak{a}$, we have the identity

$$\psi_F(s)=\sum_{m\ge1}r_F(m)m^{-s}=\sum_{\substack{\mu\in\mathfrak{a}\\ \mu\succ0}}\big(\mu^3+\mu'^3+\mu''^3\big)^{-s}, \tag{4.1}$$

where $r_F(m)$ is defined by (1.6).

By (2.3) we can write

$$\psi_F(s)=\tfrac{1}{8}\sum_{\nu}\psi_F(s;\nu), \tag{4.2}$$

where for $\mathrm{Re}(s)>1$

$$\psi_F(s;\nu)=\sum_{\mu\in\mathfrak{a}}\nu(\mu)\big(|\mu|^3+|\mu'|^3+|\mu''|^3\big)^{-s}$$

Recall (3.1) and define for $x=(x_1,x_2)$ and $\mathrm{Re}(s)>1$

$$\psi_F(s,x;\nu)=\sum_{\mu\in\mathfrak{a}}\nu(\mu)\big(|\mu|^3w_1^3(x)+|\mu'|^3w_2^3(x)+|\mu''|^3w_3^3(x)\big)^{-s}. \tag{4.3}$$

Observe that this function is invariant in x under translations from $\mathbb{Z}^2$. We will analyze $\psi_F(s;\nu)$ by expanding $\psi_F(s,x;\nu)$ in a Fourier series and specialize x to be zero. For this we will follow the basic technique from above. Then we can apply (4.2).

Lemma 3. *For* $n=(n_1,n_2)\in\mathbb{Z}^2$ *and* $\mathrm{Re}(s)>1$

$$A_F(s,n;\nu)=\int_0^1\int_0^1\psi_F(s,x;\nu)e^{-2\pi i n\cdot x}dx_1\,dx_2=\tfrac{1}{27\delta}\Gamma^{**}_{\lambda_n}(s)\zeta_{\mathfrak{a}}(s,\nu\lambda_n),$$

where the Gamma factor is

$$\Gamma^{**}_{\lambda_n}(s)=\Gamma(s)^{-1}\Gamma\big(\tfrac{s}{3}-\tfrac{2\pi i n\cdot\kappa_1}{3}\big)\Gamma\big(\tfrac{s}{3}-\tfrac{2\pi i n\cdot\kappa_2}{3}\big)\Gamma\big(\tfrac{s}{3}-\tfrac{2\pi i n\cdot\kappa_3}{3}\big). \tag{4.4}$$

Proof. The integral $A_F(s,n;\nu)$ of Lemma 3 unfolds into the form

$$A_F(s,n;\nu)=\sum_{(\mu)_{\mathfrak{a}}}\nu(\mu)I_F(\mu,s), \tag{4.5}$$

where

$$I_F(\mu,s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(|\mu|^3w_1^3(x)+|\mu'|^3w_2^3(x)+|\mu''|^3w_3^3(x)\big)^{-s}e^{-2\pi i n\cdot x}dx_1\,dx_2, \tag{4.6}$$

which is well defined on $(\mu)_{\mathfrak{a}}$. For a fixed representative μ again make the change of variables $x \mapsto x - y$, where $y = (y_1, y_2)$ with

$$y_1 = e_1^{(1)} \log|\mu| + e_2^{(1)} \log|\mu'| + e_3^{(1)} \log|\mu''|$$
$$y_2 = e_1^{(2)} \log|\mu| + e_2^{(2)} \log|\mu'| + e_3^{(2)} \log|\mu''|.$$

Now, similarly to the proof of Lemma 1, we obtain an analogue of (3.21):

$$I_F(\mu, s) = \frac{\lambda_n(\mu)}{|N(\mu)|^s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(w_1^3(x) + w_2^3(x) + w_3^3(x)\right)^{-s} e^{-2\pi i n\cdot x} dx_1\, dx_2. \tag{4.7}$$

Next, the change of variables $u_j = w_j(x)$, $j = 1, 2$ allows us to prove an analogue of (3.24):

$$I_F(\mu, s) = \frac{\lambda_n(\mu)}{\delta|N(\mu)|^s} \int_0^{\infty} \int_0^{\infty} \left(u_1^3 + u_2^3 + \tfrac{1}{(u_1u_2)^3}\right)^{-s} u_1^{\alpha_0} u_2^{\beta_0} \tfrac{du_1 du_2}{u_1 u_2}, \tag{4.8}$$

where (see (3.25))

$$\alpha_0 = \tfrac{2\pi i}{\delta}(n_2 \log\eta_1' - n_1 \log\eta_2'), \quad \beta_0 = \tfrac{2\pi i}{\delta}(n_1 \log\eta_2 - n_2 \log\eta_1). \tag{4.9}$$

By making another change of variable $v_j = u_j^3$, $j = 1, 2$ and applying Lemma 2, we obtain (4.10)

$$I_F(\mu, s) = \frac{\lambda_n(\mu)}{27\delta|N(\mu)|^s\Gamma(s)} \Gamma\left(\frac{3s - \alpha_0 - \beta_0}{9}\right) \Gamma\left(\frac{3s + 2\alpha_0 - \beta_0}{9}\right) \Gamma\left(\frac{3s - \alpha_0 + 2\beta_0}{9}\right).$$

It follows from (2.6), (2.7) that (see also (3.27), (3.28), (3.29))

$$2\alpha_0 - \beta_0 = -6\pi i(n_1 e_1^{(1)} + n_2 e_1^{(2)}), \quad 2\beta_0 - \alpha_0 = -6\pi i(n_1 e_2^{(1)} + n_2 e_2^{(2)}) \tag{4.11}$$

$$\alpha_0 + \beta_0 = 6\pi i(n_1 e_3^{(1)} + n_2 e_3^{(2)}). \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10) and then using (4.5) we prove the lemma. □

The first statement of Proposition 3 follows from (4.2) and the next result, after the residue at $s = 1$ is computed using iv) of Theorem 4. The second statement also follows in view of iii) of Theorem 4.

Lemma 4. *The series*

$$\psi_F(s;\nu) = \sum_{n\in\mathbb{Z}^2} \frac{\Gamma\left(\frac{s}{3} - \frac{2\pi i n\cdot\kappa_1}{3}\right)\Gamma\left(\frac{s}{3} - \frac{2\pi i n\cdot\kappa_2}{3}\right)\Gamma\left(\frac{s}{3} - \frac{2\pi i n\cdot\kappa_3}{3}\right)}{27\delta\,\Gamma(s)} \zeta_{\mathfrak{a}}(s, \nu\lambda_n). \tag{4.13}$$

converges uniformly on compact subsets in $\mathrm{Re}(s) > 0$ *that do not contain the possible pole at* $s = 1$.

This follows immediately from Lemma 3, vi) of Theorem 4, Stirling's formula and the following crucial lemma. Write $\|n\| = \sqrt{n_1^2 + n_2^2} = \sqrt{n \cdot n}$.

Lemma 5. *There is a constant* $C > 0$*, depending only on* η_1 *and* η_2*, such that for at least two of* $i = 1, 2, 3$ *we have*

$$|n \cdot \kappa_i| \geq C\|n\|. \tag{4.14}$$

Proof. For $(i, j) = (1, 2)$, $(i, j) = (1, 3)$ and $(i, j) = (2, 3)$ we define the binary quadratic form

$$q_{i,j}(n) = (n\cdot\kappa_i)^2 + (n\cdot\kappa_i)(n\cdot\kappa_j) + (n\cdot\kappa_j)^2 = a_{i,j}n_1^2 + b_{i,j}n_1n_2 + c_{i,j}n_2^2. \tag{4.15}$$

Using (2.6), (2.7) we obtain

$$3\delta^2 a_{1,2} = \log^2\eta_2 + \log\eta_2 \log\eta_2' + \log^2\eta_2', \tag{4.16}$$

$$3\delta^2 b_{1,2} = -\log\eta_1 \log\eta_2' - 2\log\eta_1' \log\eta_2' - 2\log\eta_1 \log\eta_2 - \log\eta_1' \log\eta_2, \tag{4.17}$$

$$3\delta^2 c_{1,2} = \log^2\eta_1 + \log\eta_1 \log\eta_1' + \log^2\eta_1', \quad b_{1,2}^2 - 4a_{1,2}c_{1,2} = -\tfrac{1}{3\delta^2}. \tag{4.18}$$

Applying (2.19) and (2.8), we show that $\kappa_3 = -\kappa_2 - \kappa_1$, and thus $q_{1,3}(n) = q_{2,3}(n) = q_{1,2}(n)$. Since $a_{1,2} > 0$ and $b_{1,2}^2 - 4a_{1,2}c_{1,2} < 0$, the quadratic form is positive definite. Therefore, for some $C_1 > 0$, depending only on η_1 and η_2, for all $n \neq (0,0)$

$$q_{i,j}(n) \geq C_1 \|n\|^2.$$

Consequently, for $(i,j) = (1,2)$, $(i,j) = (1,3)$ and $(i,j) = (2,3)$ at least one of $|n \cdot \kappa_i|$, $|n \cdot \kappa_j|$ satisfies (4.14). Hence, at least two of the three $|n \cdot \kappa_i|$ satisfy (4.14). □

Remark. It can be shown (see Lemma 16 in the Appendix) that the set of numbers

$$\{n \cdot \kappa_1,\, n \cdot \kappa_2,\, n \cdot \kappa_3;\ \|n\| \leq N\}$$

becomes dense in $\mathbb{R}$ as $N \to \infty$. These points will give rise to singularities of $\psi_F(s)$ when the corresponding zeta values are not zero. It seems likely that as a result $\psi_F(s)$ has a natural boundary on the imaginary axis.

The next result shows that these troublesome poles can only come from zetas with non-trivial sign characters.

Proposition 4. *The function $\psi_F(s;\nu_0)$ is regular for* $\operatorname{Re} s > -3$.

Proof. It follows from Lemma 4 and Theorem 4 (we take each $a_j = 0$) that

$$\psi_F(s;\nu_0) = \frac{\pi^{3s/2}}{27\delta\,\Gamma(s)} \sum_{n\in\mathbb{Z}^2} \xi_{\mathfrak{a}}(s,\nu_0\lambda_n) H(s,n), \quad H(s,n) = \prod_{j=1}^{3} \frac{\Gamma\big((s-2\pi i n\cdot\kappa_j)/3\big)}{\Gamma\big((s-2\pi i n\cdot\kappa_j)/2\big)}. \tag{4.19}$$

The function $\Gamma(z/3)\Gamma^{-1}(z/2)$ is regular at the points $z = -3k$ when k is a positive even number, but not if k is odd. Thus the poles of $H(s,n)$ are

$$s(j,m) = -3 - 6m + 2\pi i n \cdot \kappa_j, \quad m \in \mathbb{Z}_+,$$

and so $\psi_F(s;\nu_0)$ is regular for $\operatorname{Re} s > -3$. □

5. Growth in vertical strips

In this section we prove the following estimate, which is applied in the next sections to derive Theorems 2 and 3.

Proposition 5. *Fix $\epsilon, \delta > 0$. Then for $\delta \leq \operatorname{Re}(s) \leq 1$ we have*

$$\psi_F(s) \ll (1+|t|)^{\frac{5}{2}(1-\sigma)+\epsilon},$$

where as usual $s = \sigma + it$.

Proof. For $j = 1,2,3$ let $c_j = 2\pi \cdot (n \cdot \kappa_j)$. One has $c_1 + c_2 + c_3 = 0$. Therefore, combining Lemma 4 and vi) of Theorem 3 with the Stirling formula we obtain for $s = \sigma + it$ with $0 < \sigma < 1$ the estimate

$$\psi_F(s) \ll \sum_{n\in\mathbb{Z}^2} (1+|t|)^{1/2-\sigma} e^{-\frac{\pi}{6} f(n,t)} \prod_{j=1}^{3} (1+|t-c_j|)^{-\sigma/6+\epsilon}, \tag{5.1}$$

where

$$f(n,t) = |t-c_1| + |t-c_2| + |t+c_1+c_2| - 3|t|. \tag{5.2}$$

Without loss of generality, we assume that $c_1 \leq c_2$. Lemma 5 implies that

$$\|n\| \ll \max(|c_1|,|c_2|) \ll \|n\|.$$

To analyze $f(n,t)$, it is necessary to study six different cases depending on how the points $0, c_1, c_2, -c_1-c_2$ are located on the real line. First, there are three cases concerning the locations of c_1 and c_2. That is, $c_1 \le c_2 \le 0$ (case a), $c_1 \le 0 \le c_2$ (case b) or $0 \le c_1 \le c_2$ (case c). In case b, there are also four possibilities for the point $-c_1-c_2$. As a result,

Case a: $c_1 \le c_2 \le 0 \le -c_1 - c_2$,

Case b1: $c_1 \le 0 \le c_2 \le -c_1 - c_2$, if $\quad 0 \le 2c_2 \le -c_1$

Case b2: $c_1 \le 0 \le -c_1 - c_2 \le c_2$, if $\quad 0 \le c_2 \le -c_1 \le 2c_2$

Case b3: $c_1 \le -c_1 - c_2 \le 0 \le c_2$, if $\quad 0 \le c_2/2 \le -c_1 \le c_2$

Case b4: $-c_1 - c_2 \le c_1 \le 0 \le c_2$, if $\quad 0 \le -c_1 \le c_2/2$

Case c: $-c_1 - c_2 \le 0 \le c_1 \le c_2$.

Since all six cases can be treated in the same way, we give details only for case c. In this settings, we have

$$\begin{cases} f(n,t) = 0, & \text{if } \ t \le -c_1 - c_2 \\ f(n,t) = 2t + 2c_1 + 2c_2, & \text{if } \ -c_1 - c_2 \le t \le 0 \\ f(n,t) = -4t + 2c_1 + 2c_2, & \text{if } \ 0 \le t \le c_1 \\ f(n,t) = -2t + 2c_2, & \text{if } \ c_1 \le t \le c_2 \\ f(n,t) = 0, & \text{if } \ c_2 \le t. \end{cases} \tag{5.3}$$

First, we should mention $f(n,t) \ge 0$. Note that in case c we sum in (5.1) only over n such that $0 \le c_1 \le c_2$. However, finally we will expand the sum to the whole $\mathbb{Z}^2$. To estimate (5.1) in case c, we divide the sum into two parts: $\|n\| > N_0|t|$ and $\|n\| \le N_0|t|$, where N_0 is a large number depending on C from Lemma 5 and κ_1, κ_2. The idea is to estimate $f(n,t)$ from below so that $e^{-f(n,t)}$ becomes small enough. If $|t| < \|n\|/N_0$, then $|t| < c_2/(CN_0)$. Thus

$$|t| \le c_2 - \log^2 t - \log^2 c_2$$

and

$$f(n,t) \gg \log^2 t + \log^2 \|n\|. \tag{5.4}$$

Therefore, since for an arbitrary $c > 0$ and $A > 1$ one has $e^{-c\log^2 t} \ll (1+t)^{-A}$ the terms with $\|n\| > N_0|t|$ are negligible and their contribution to $\psi_F(s)$ can be estimated as $O((1+|t|)^{-A})$ for arbitrarily large A.

In the remaining case $\|n\| \le N_0|t|$, the trivial estimate $f(n,t) \ge 0$ yields

$$\psi_F(s) \ll \sum_{\|n\| \le N_0|t|} (1+|t|)^{1/2-\sigma}(1+|t|)^{-\sigma/2+\epsilon} \ll (1+|t|)^{5/2-3\sigma/2+\epsilon}. \tag{5.5}$$

This estimate can be improved by applying the Phragmen-Lindelof principle (see Theorem 2 of [15]). Since for $\delta > 0$ we have $\psi_F(1+\delta+it) \ll 1$, we have

$$\psi_F(s) \ll (1+|t|)^{5/2(1-\sigma)+\epsilon}. \tag{5.6}$$

□

6. Proof of Theorem 2

The next lemma is a modification of [6, Lemma on p.105].

Lemma 6. *For $r, c, x, T > 0$ we have*

$$\chi_{(1,\infty)}(x)\left(1-\frac{1}{x}\right)^r = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}x^s ds + O\left(\frac{x^c}{T^r}\min\left(1, \frac{1}{T|\log x|}\right)\right), \tag{6.1}$$

where $\chi_{(1,\infty)}(x)$ is the characteristic function of the interval $(1,\infty)$.

Applying Lemma 6, we prove the following statement.

Lemma 7. *For any $c > 1$ and $T \le X$ we have*

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\psi_F(s)X^s ds + O\left(\frac{X^c}{T^r}\right). \tag{6.2}$$

Proof. Since $\sum_{m\le x} r_F(m) \ll x^{1+\epsilon}$ and $r_F(m) \ge 0$, the series $\sum_{m=1}^{\infty} r_F(m)m^{-s}$ converges absolutely for $\mathrm{Re}(s) > 1$. Therefore, using (6.1) we have for $c > 1$ that

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\psi_F(s)X^s ds + O\left(\sum_{m=1}^{X}\frac{r_F(m)X^c}{m^cT^r}\right).$$

□

If we move the line of integration to $\mathrm{Re}\, s = \delta_0$, where δ_0 is an arbitrary small positive number, we will cross the pole at the point $s = 1$. Using Propositions 3 and 5, we show that

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})X}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}} + O\left(X^{\delta_0}\int_{-T}^{T}\frac{(1+|t|)^{5(1-\delta_0)/2+\epsilon}}{(1+|t|)^{r+1}}dt\right) + O\left(\int_{\delta_0}^{1+\epsilon}\frac{T^{5(1-\sigma)/2+\epsilon}}{T^{r+1}}X^\sigma d\sigma\right) + O\left(\frac{X^{1+\epsilon}}{T^r}\right). \tag{6.3}$$

By estimating these integrals trivially , we prove that the error term in (6.3) is less than

$$X^{\delta_0}\max\left(1, T^{5(1-\delta_0)/2-r+\epsilon}\right) + T^{3/2-r+\epsilon}\max\left(\left(\frac{X}{T^{5/2}}\right)^{1+\epsilon}, \left(\frac{X}{T^{5/2}}\right)^{\delta_0}\right) + \frac{X^{1+\epsilon}}{T^r}. \tag{6.4}$$

Therefore, choosing T sufficiently large (say $T = X$), we conclude that for $r \ge 5/2$

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})X}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}} + O\left(X^{\delta_0+\epsilon}\right), \tag{6.5}$$

thus proving Theorem 2. □

7. Proof of Theorem 3

In this section, we will improve (6.5), find the second main term in the asymptotic formula, prove Theorem 3 and (1.8). To this end, we will move the line of integration in (6.2) to $\mathrm{Re}\, s = -a$ with some $a > 0$. To overcome the difficulty of having a set of poles on $\mathrm{Re}\, s = 0$ that becomes dense, we will first truncate the sum in (4.13) at $\|n\| = dT$, where d is some large constant such that (5.4) holds, and thus the contribution of terms with

$\|n\| > dT$ is negligible. By doing this and using (4.2) and (4.13), we prove an analogue of (6.3), namely,

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})X}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}} + \frac{1}{8}\sum_{\nu}\frac{1}{27\delta}\sum_{\|n\|\le dT} J(n,\nu,X,T) + O\left(\int_{\epsilon}^{1+\epsilon}\frac{T^{5(1-\sigma)/2+\epsilon}}{T^{r+1}}X^{\sigma}d\sigma\right) + O\left(\frac{X^{1+\epsilon}}{T^r}\right), \tag{7.1}$$

where

$$J(n,\nu,X,T) = \frac{1}{2\pi i}\int_{\epsilon-iT}^{\epsilon+iT} g(n,\nu,X,s)ds, \tag{7.2}$$

$$g(n,\nu,X,s) = \frac{\Gamma(1+r)}{\Gamma(1+r+s)}\Gamma\left(\tfrac{s-2\pi in\cdot\kappa_1}{3}\right)\Gamma\left(\tfrac{s-2\pi in\cdot\kappa_2}{3}\right)\Gamma\left(\tfrac{s-2\pi in\cdot\kappa_3}{3}\right)\zeta_{\mathfrak{a}}(s,\nu\lambda_n)X^s. \tag{7.3}$$

Now we move the line of integration in (7.2) to $\operatorname{Re} s = -\epsilon$ with some $0<\epsilon<1$, crossing poles at the points

$$s_j(n) = 2\pi in\cdot\kappa_j \quad \text{for} \quad j=1,2,3. \tag{7.4}$$

Lemma 8. *For $r > 5/2+\epsilon_0$ and $T = X^{2/5}$ one has*

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})X}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}} + S_{res} + O(X^{-\epsilon}), \tag{7.5}$$

where

$$S_{res} = \frac{1}{8}\sum_{\nu}\frac{1}{27\delta}\sum_{\substack{\|n\|\le dT\\ |s_j(n)|\le T}} \operatorname{res}_{s_j(n)} g(n,\nu,X,s). \tag{7.6}$$

Proof. Since the number of poles (after taking the sum over $\|n\|\le dT$) is finite, it is possible to change T slightly so that $\operatorname{Im} s_j(n) \ne T$. To this end, we consider the interval $(T,2T)$ and divide it into subintervals of length $(fT)^{-1}$ with a huge constant f. As a result, we have fT^2 intervals. Since the number of poles is less than $100(dT)^2$, choosing $f > 2025d^2$, it is possible to find an interval that does not contain any pole. By choosing a new T as the center of this interval, we ensure that $|T-\operatorname{Im} s_j(n)| \gg T^{-1}$. With this new T we have

$$J(n,\nu,X,T) = \sum_{|s_j(n)|\le T}\operatorname{res}_{s_j(n)} g(n,\nu,X,s) + J_{\epsilon}(n,\nu,X,T) + \sum_{\pm} J_0(n,\nu,X,\pm T), \tag{7.7}$$

$$J_{\epsilon}(n,\nu,X,T) = \frac{1}{2\pi i}\int_{-\epsilon-iT}^{-\epsilon+iT} g(n,\nu,X,s)ds, \tag{7.8}$$

$$J_0(n,\nu,X,T) = \frac{1}{2\pi i}\int_{-\epsilon+iT}^{\epsilon+iT} g(n,\nu,X,s)ds. \tag{7.9}$$

Substituting (7.7) into (7.1), we obtain

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})X}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}} + S_{\epsilon} + S_0 + S_{res} + O\left(T^{3/2-r+\epsilon}X^{\epsilon}\max\left(1,\left(\frac{X}{T^{5/2}}\right)\right)\right) + O\left(\frac{X^{1+\epsilon}}{T^r}\right), \tag{7.10}$$

where S_{res} is defined by (7.6) and
(7.11)
$$S_\epsilon = \frac{1}{8}\sum_\nu \frac{1}{27\delta}\sum_{\|n\|\le dT}\sum_{\pm} J_\epsilon(n,\nu,X,\pm T), \quad S_0 = \frac{1}{8}\sum_\nu \frac{1}{27\delta}\sum_{\|n\|\le dT}\sum_{\pm} J_0(n,\nu,X,\pm T).$$

To estimate S_ϵ, we substitute (7.8) to (7.11), and after that interchange the order of summation and integration. Next, we increase the summation over n back to the whole $\mathbb{Z}^2$ with a negligibly small error term (since all estimates from Proposition 5 still hold). As a result,

$$S_\epsilon = \frac{1}{8}\sum_\nu \frac{1}{2\pi i}\int_{-\epsilon-iT}^{-\epsilon+iT} \frac{\Gamma(1+r)\Gamma(s)}{\Gamma(1+r+s)}\psi_F^-(s;\nu)X^s ds, \tag{7.12}$$

where for $\operatorname{Re} s \le 0$

$$\psi_F^-(s;\nu) = \frac{1}{27\delta\,\Gamma(s)}\sum_{n\in\mathbb{Z}^2}\Gamma\Big(\tfrac{s}{3}-\tfrac{2\pi i n\cdot\kappa_1}{3}\Big)\Gamma\Big(\tfrac{s}{3}-\tfrac{2\pi i n\cdot\kappa_2}{3}\Big)\Gamma\Big(\tfrac{s}{3}-\tfrac{2\pi i n\cdot\kappa_3}{3}\Big)\zeta_{\mathfrak{a}}(s,\nu\lambda_n). \tag{7.13}$$

Arguing as in Proposition 5, one can prove an analogue of (5.5). Note that now it is required to use $k(\sigma) = 1/2 - \sigma$ from (2.19) instead of $k(\sigma) = (1-\sigma)/2 + \epsilon$, as we did in Proposition 5. Accordingly, for $\operatorname{Re} s < 0$ and far away from possible poles of the Gamma functions, we obtain the following result

$$\psi_F^-(s;\nu) \ll \sum_{\|n\|\le N_0|t|} (1+|t|)^{1/2-\sigma}(1+|t|)^{-2\sigma} \ll (1+|t|)^{5/2-3\sigma}. \tag{7.14}$$

Estimating (7.12) using (7.14) and assuming that $r > 5/2 + \epsilon_0$, we have

$$S_\epsilon \ll \int_{-T}^{T} (1+|t|)^{3/2-r+3\epsilon} X^{-\epsilon} dt \ll X^{-\epsilon}. \tag{7.15}$$

To estimate S_0 from (7.11) we first let (as in Proposition 5)

$$c_j = 2\pi\cdot(n\cdot\kappa_j) \quad \text{for} \quad j = 1,2,3. \tag{7.16}$$

Note that for $\|n\| \le dT$ we have

$$|c_j| \ll \|n\| \ll T. \tag{7.17}$$

Since $c_1 + c_2 + c_3 = 0$, the product of three Gamma functions with κ_j from (7.3) with $s = \sigma + iT$ is equal to

$$\Gamma\Big(\tfrac{\sigma+i(T-c_1)}{3}\Big)\Gamma\Big(\tfrac{\sigma+i(T-c_2)}{3}\Big)\Gamma\Big(\tfrac{\sigma+i(T+c_1+c_2)}{3}\Big). \tag{7.18}$$

We divide the sum over n in S_0 (see (7.11)) into several parts (for simplicity, we will continue to denote each part as S_0) depending on whether the imaginary parts of the arguments of the Gamma functions in (7.18) are small or not. If none of them is small, that is

$$|T-c_1|, |T-c_2|, |T+c_1+c_2| \gg 1, \tag{7.19}$$

we can argue as before. More precisely, we apply Stirling's formula to all Gamma functions, change the order of summation over n and integration over σ, and apply (7.14). As a result, we have

$$S_0 \ll \int_{-\epsilon}^{+\epsilon} T^{3/2-r+\epsilon} X^\sigma d\sigma \ll X^\epsilon T^{3/2-r+\epsilon}. \tag{7.20}$$

Now, consider the case when at least one of the absolute values in (7.19) is small. Note that all three of them cannot be small. Suppose that $|T-c_1|$ is small and that the other two are not. Using Stirling's formula, we deduce from (7.3) that for $|\sigma|<\epsilon$

$$g(n,\nu,X,s) \ll \frac{\zeta_{\mathfrak{a}}(\sigma+iT,\nu\lambda_n)X^{\sigma}}{T^{1/2+r}(|\sigma|+|T-c_1|)}e^{-\frac{\pi}{6}(|T-c_2|+|T+c_1+c_2|-3T)} \times |T-c_2|^{\sigma/3-1/2}|T+c_1+c_2|^{\sigma/3-1/2}. \tag{7.21}$$

Since

$$|T-c_2|+|T+c_1+c_2|-3T \geq c_1-T \gg -1,$$

the exponential factor in (7.21) is less than some constant. Applying (2.18) and (2.19), we show that

$$g(n,\nu,X,s) \ll \frac{T^{\epsilon}X^{\sigma}}{T^{1/2+r}(|\sigma|+|T-c_1|)}|T-c_2|^{-2\sigma/3}|T+c_1+c_2|^{-2\sigma/3}. \tag{7.22}$$

Since we assume that only $|T-c_1|$ is small, substituting (7.22) into (7.9) and taking into account (7.17), we infer that

$$J_0(n,\nu,X,T) \ll \frac{T^{\epsilon}X^{\epsilon}}{T^{1/2+r}}\int_{-\epsilon}^{\epsilon}\frac{d\sigma}{|\sigma|+|T-c_1|} \ll \frac{T^{\epsilon}X^{\epsilon}}{T^{1/2+r}}. \tag{7.23}$$

Substituting (7.23) into (7.11) and extending the summation over n from $|T-c_1|\ll 1$ to the entire $\|n\|\ll T$, we prove that

$$S_0 \ll T^{3/2-r}(XT)^{\epsilon}. \tag{7.24}$$

Now suppose that $|T-c_1|$, $|T-c_2|\ll 1$. Using Stirling's formula, we deduce from (7.3) that for $|\sigma|<\epsilon$

$$g(n,\nu,X,s) \ll \frac{\zeta_{\mathfrak{a}}(\sigma+iT,\nu\lambda_n)X^{\sigma}|T+c_1+c_2|^{\sigma/3-1/2}e^{-\frac{\pi}{6}(|T+c_1+c_2|-3T)}}{(|\sigma|+|T-c_1|)(|\sigma|+|T-c_2|)T^{1/2+r}}. \tag{7.25}$$

Similarly to the previous case, the exponential factor is bounded by some constant. Since $|T+c_1+c_2|\gg T$, applying (2.18) and (2.19) we show that

$$g(n,\nu,X,s) \ll \frac{X^{\sigma}T^{-1/2-r-2\sigma/3}}{(|\sigma|+|T-c_1|)(|\sigma|+|T-c_2|)}. \tag{7.26}$$

Substituting (7.26) into (7.9), we obtain

$$J_0(n,\nu,X,T) \ll \int_{-\epsilon}^{\epsilon}\frac{X^{\epsilon}T^{-1/2-r+\epsilon}d\sigma}{(|\sigma|+|T-c_1|)(|\sigma|+|T-c_2|)} \ll \frac{X^{\epsilon}T^{-1/2-r+\epsilon}}{\max(|T-c_1|,|T-c_2|)}. \tag{7.27}$$

Substituting (7.27) into (7.11), using the fact that we choose T such that $|T-c_j|\gg T^{-1}$, and estimating the sum over n trivially, we infer that

$$S_0 \ll \sum_{\substack{\|n\|\ll T\\ T^{-1}\ll|T-c_1|,|T-c_2|\ll 1}}\frac{X^{\epsilon}T^{-1/2-r+\epsilon}}{\max(|T-c_1|,|T-c_2|)} \ll X^{\epsilon}T^{5/2-r+\epsilon}. \tag{7.28}$$

Note that with a little more work one can prove an estimate $X^{\epsilon}T^{3/2-r+\epsilon}$ in (7.28). It follows from (7.10), (7.15), (7.20), (7.24), and (7.28) that for $r>5/2+\epsilon_0$

$$\sum_{1\leq m\leq X} r_F(m)(1-\tfrac{m}{X})^r = \frac{\Gamma^3(\frac{4}{3})X}{(r+1)N(\mathfrak{a})D^{\frac{1}{2}}} + S_{res} + E(X,T), \tag{7.29}$$

where the error term is

$$E(X,T) \ll \frac{X^{1+\epsilon}}{T^r} + T^{3/2-r+\epsilon}X^{\epsilon}\max\left(1, \frac{X}{T^{5/2}}\right) + T^{5/2-r+\epsilon}X^{\epsilon} + X^{-\epsilon}. \tag{7.30}$$

Taking $T = X^{2/5}$ we obtain

$$E(X,T) \ll X^{\frac{5-2r}{5}+\epsilon} + X^{-\epsilon} \ll X^{-\epsilon}, \tag{7.31}$$

provided that $r > 5/2 + \epsilon_0$. □

Our next goal is to show that the contribution to (7.5) of the residue at zero point is $K_1 \log X + O(1)$ with some constant K_1, and the contribution of all other residues is $O(1)$. Note that to prove this we have to assume that r is very large. This is the reason why did not try to move the integration line in Lemma 8 , say, to $\operatorname{Re} s = -a$ and optimize the value of a.

Note that, as in Proposition 4, some poles (7.4) may be canceled by zeros of $\zeta(s, \nu\lambda_n)$.

Lemma 9. *All poles of the function $g(n, \nu, X, s)$ (see (7.3)) are simple, except one at the point $s = 0$.*

Proof. To prove the lemma, we show that for any fixed $n \neq (0, 0)$ one has

$$n \cdot \kappa_i \neq n \cdot \kappa_j \quad \text{for any} \quad i, j = 1, 2, 3,\ i \neq j. \tag{7.32}$$

Since $\kappa_3 = -\kappa_1 - \kappa_2$ (see (2.15),(2.8)), it is required to prove that

$$n \cdot (\kappa_1 - \kappa_2),\ n \cdot (2\kappa_1 + \kappa_2),\ n \cdot (\kappa_1 + 2\kappa_2) \neq 0. \tag{7.33}$$

Suppose that $n \cdot (\kappa_1 - \kappa_2) = 0$ for some $n = (n_1, n_2) \neq (0, 0)$. Then using (2.15) and (2.6), (2.7) we have

$$n_1 \log(\eta_2\eta_2') - n_2 \log(\eta_1\eta_1') = 0 \Rightarrow (\eta_2\eta_2')^{n_1} = (\eta_1\eta_1')^{n_2} \Rightarrow (\eta_2'')^{n_1} = (\eta_1'')^{n_2}.$$

Since $\eta_j'' = \sigma_2(\eta_j)$, where σ_2 is an embedding, we obtain $\sigma_2(\eta_2^{n_1} - \eta_1^{n_2}) = 0$, and therefore $\eta_2^{n_1} = \eta_1^{n_2}$, which contradicts the fact that η_1, η_2 is a basis for the subgroup of totally positive units. The two remaining statements in (7.33) can be proved in the same way. □

Lemma 10. *There exists some R such that for $T = X^{2/5}$ and $r > R$ we have*

$$\frac{1}{8}\sum_{\nu}\frac{1}{27\delta}\sum_{\substack{\|n\| \leq dT \\ 0 \neq |s_j(n)| \leq T}} \operatorname{res}_{s_j(n)} g(n, \nu, X, s) \ll 1. \tag{7.34}$$

Proof. Consider the residues at the points $s_1(n)$. Other residues can be treated similarly. It follows from (7.3) and (7.4) that

$$\operatorname{res}_{s_1(n)} g(n, \nu, X, s) \ll \frac{\zeta_{\mathfrak{a}}(ic_1, \nu\lambda_n)}{\Gamma(1 + r + ic_1)}\Gamma\left(\tfrac{i(c_1-c_2)}{3}\right)\Gamma\left(\tfrac{i(c_1-c_3)}{3}\right). \tag{7.35}$$

By Lemma 9 we have $c_1 \neq c_2$, $c_1 \neq c_3$. Nevertheless, it may happen that either $|c_1 - c_2| \ll 1$ or $|c_1 - c_3| = |2c_1 + c_2| \ll 1$ or both $|c_1 - c_2| \ll 1$, $|c_1 - c_3| \ll 1$.

Suppose that $|c_1 - c_2| \ll 1$ and $|c_1| \gg 1$. In this case, applying Stirling's formula and (2.18) we have

$$\begin{aligned} \operatorname{res}_{s_1(n)} g(n, \nu, X, s) &\ll \frac{\zeta_{\mathfrak{a}}(ic_1, \nu\lambda_n)|2c_1 + c_2|^{-1/2}}{|c_1 - c_2| \cdot |c_1|^{1/2+r}} e^{\frac{-\pi}{6}(|2c_1+c_2|-3|c_1|)} \\ &\ll \frac{|2c_1 + c_2|^{\epsilon}|c_1|^{-1/2-r}}{|c_1 - c_2|} \ll \frac{|c_1|^{\epsilon-1/2-r}}{|c_1 - c_2|}, \end{aligned} \tag{7.36}$$

since $|2c_1 + c_2| - 3|c_1| > -1$. From Lemma 5 it follows that $\|n\| \ll |c_1|, |c_2| \ll \|n\|$. Consequently, to show (7.34) it is required to prove that

$$\sum_{\substack{\|n\|\le dT \\ 0<|c_1-c_2|\ll 1}} \frac{\|n\|^{-1/2-r+\epsilon}}{|c_1 - c_2|} \ll 1. \tag{7.37}$$

Fortunately, the quantity $|c_1 - c_2|$ is separated from zero. Using (2.6) and (2.15) we have

$$|c_1 - c_2| \gg |n \cdot (\kappa_1 - \kappa_2)| \gg |n_1 \log(\eta_2\eta_2') - n_2 \log(\eta_1\eta_1')| = |n_1 \log \eta_2'' - n_2 \log \eta_1''|.$$

Since η_1'' and η_2'' are multiplicatively independent, we can apply Baker's result [2, Theorem 3.1] on linear forms in logarithms, getting

$$|c_1 - c_2| \gg \|n\|^{-C} \tag{7.38}$$

with some absolute constant C depending on η_1'', η_2'' and the field $\mathbb{F}$. There are several results on a precise value of the constant C in (7.38), to mention a few: [3], [4], [7], [13]. But even the best known upper bounds on the constant C are very large. Nevertheless, an estimate (7.38) allows us to prove (7.37) for $r > C + 2$. An interesting question is whether it is possible to obtain (7.38) with some small value of C (say, $C = 1/2$) using the fact that we are investigating a linear form in logarithms of totally positive units.

Suppose that $|c_1 - c_2| \ll 1$ and $|c_1| \ll 1$. Since $c_1 + c_2 + c_3 = 0$, we conclude that $|c_3| \ll 1$. Thus $|c_1 - c_3| \ll 1$ and $\|n\| \ll 1$ by Lemma 5. In this case,

$$\operatorname{res}_{s_1(n)} g(n, \nu, X, s) \ll \frac{1}{|c_1 - c_2| \cdot |c_1 - c_3|},$$

and it is left to show that

$$\sum_{\|n\|\ll 1} \frac{1}{|c_1 - c_2| \cdot |c_1 - c_3|} \ll 1,$$

which immediately follows from (7.38). Note that the case $|c_1 - c_2| \ll 1$ and $|c_1 - c_3| \ll 1$ can be treated in the same way.

From now on, we may assume that $|c_1 - c_2| \gg 1$ and $|c_1 - c_3| \gg 1$. Accordingly, applying Stirling's formula and (2.18) we deduce from (7.35) that

$$\begin{aligned} \operatorname{res}_{s_1(n)} g(n, \nu, X, s) &\ll \frac{\zeta_{\mathfrak{a}}(ic_1, \nu\lambda_n) e^{-\pi f_{1,2}(n)/6}}{(1 + |c_1|)^{1/2+r} \sqrt{|2c_1 + c_2|}\sqrt{|c_1 - c_2|}} \\ &\ll \frac{|2c_1 + c_2|^{\epsilon} |c_1 - c_2|^{\epsilon}}{(1 + |c_1|)^{1/2+r}} e^{-\pi f_{1,2}(n)/6}, \end{aligned} \tag{7.39}$$

where $f_{1,2}(n) = |c_1 - c_2| + |2c_1 + c_2| - 3|c_1|$. Since $|c_1|, |c_2| \ll \|n\|$, to complete the proof of (7.34) it remains to show that

$$\sum_{\substack{\|n\|\le dT \\ 0\ne|s_j(n)|\le T}} \operatorname{res}_{s_1(n)} g(n, \nu, X, s) \ll \sum_{\|n\|\le dT} \frac{\|n\|^{\epsilon} e^{-\pi f_{1,2}(n)/6}}{(1 + |c_1|)^{1/2+r}} \ll 1. \tag{7.40}$$

Suppose that $c_2 > 0$ (the case $c_2 < 0$ can be treated similarly). In this case,

$$\begin{cases} f_{1,2}(n) = 0, & \text{if } \; c_1 \ge c_2 \\ f_{1,2}(n) = 2c_2 - 2c_1, & \text{if } \; 0 \le c_1 \le c_2 \\ f_{1,2}(n) = 2c_2 + 4c_1, & \text{if } \; -c_2/2 \le c_1 \le 0 \\ f_{1,2}(n) = 0, & \text{if } \; c_1 \le -c_2/2 \end{cases}. \tag{7.41}$$

Note that $f_{1,2}(n) \geq 0$. If $|c_1| \leq |c_2|/4$, then it follows from Lemma 5 and (7.41) that $f_{1,2}(n) \gg |c_2| \gg \|n\|$, and for some absolute $k_0 > 0$

$$\sum_{\substack{\|n\|\leq dT \\ |c_1|\leq|c_2|/4}} \frac{\|n\|^{\epsilon}e^{-\pi f_{1,2}(n)/6}}{(1+|c_1|)^{1/2+r}} \ll \sum_{\substack{\|n\|\leq dT \\ |c_1|\leq|c_2|/4}} \frac{\|n\|^{\epsilon}e^{-k_0\|n\|}}{(1+|c_1|)^{1/2+r}} \ll 1. \tag{7.42}$$

If $|c_2|/4 < |c_1| \leq |c_2|$, then by Lemma 5 we have $\|n\| \ll |c_1|, |c_2| \ll \|n\|$. Consequently,

$$\sum_{\substack{\|n\|\leq dT \\ |c_2|/4<|c_1|\leq|c_2|}} \frac{\|n\|^{\epsilon}e^{-\pi f_{1,2}(n)/6}}{(1+|c_1|)^{1/2+r}} \ll \sum_{\|n\|\leq dT} \frac{1}{\|n\|^{1/2+r-\epsilon}} \ll 1 \tag{7.43}$$

for $r > 3/2+\epsilon$. If $|c_1| > |c_2|$, then using Lemma 5 we infer that $\|n\| \ll |c_1| \ll \|n\|$. As a result,

$$\sum_{\substack{\|n\|\leq dT \\ |c_1|>|c_2|}} \frac{\|n\|^{\epsilon}e^{-\pi f_{1,2}(n)/6}}{(1+|c_1|)^{1/2+r}} \ll \sum_{\|n\|\leq dT} \frac{1}{\|n\|^{1/2+r-\epsilon}} \ll 1 \tag{7.44}$$

for $r > 3/2+\epsilon$. Combining (7.42)-(7.44), we show that (7.40) holds, which completes the proof of the lemma. □

Finally, let us compute the residue of the function (7.3) at the point $s = 0$. To emphasize the dependence of the sign character ν (defined by (2.2)) on a_1, a_2, a_3 we denote it by ν_{a_1,a_2,a_3}.

Lemma 11. *The following asymptotic formula holds:*

$$\frac{1}{8}\sum_{\nu}\frac{1}{27\delta}\text{res}_{s=0}\, g(n,\nu,X,s) = K_1 \log X + O(1), \tag{7.45}$$

where

$$K_1 = \frac{1}{16\pi\delta}\left(\xi_{\mathfrak{a}}(0,\nu_{1,1,0}\lambda_0) + \xi_{\mathfrak{a}}(0,\nu_{1,0,1}\lambda_0) + \xi_{\mathfrak{a}}(0,\nu_{0,1,1}\lambda_0)\right). \tag{7.46}$$

Proof. The function $g(n,\nu,X,s)$ has a pole at $s = 0$ only if $n = (0,0)$. In this case, it is reduced to

$$g(0,\nu,X,s) = \frac{\Gamma(1+r)X^s}{\Gamma(1+r+s)}\zeta_{\mathfrak{a}}(s,\nu\lambda_0)\Gamma^3\left(\tfrac{s}{3}\right). \tag{7.47}$$

Note that (see (2.14)) $\zeta_{\mathfrak{a}}(s,\nu_{a_1,a_2,a_3}\lambda_0) \equiv 0$ if $\sigma_\nu = 1/2$ or $\sigma_\nu = 3/2$. Hence it is needed to consider only four cases of ν_{a_1,a_2,a_3}, namely $\nu_{1,1,0}$, $\nu_{1,0,1}$, $\nu_{0,1,1}$ and $\nu_{0,0,0}$.

In the last case, the function $g(0,\nu_{0,0,0},X,s)$ is regular at $s = 0$ (see Proposition 4).

Consider the case $\nu = \nu_{1,1,0}$. By Theorem 4

$$g(0,\nu_{1,1,0},X,s) = \frac{\Gamma^3\big(s/3\big)}{\Gamma\big(s/2\big)}f(s),$$

where

$$f(s) = \frac{\Gamma(1+r)X^s\pi^{3s/2}}{\Gamma(1+r+s)\Gamma^2(\frac{s+1}{2})}\xi_{\mathfrak{a}}(s,\nu_{1,1,0}\lambda_0)$$

Therefore,

$$\text{res}_{s=0}g(0,\nu_{1,1,0},X,s) = \text{res}_{s=0}\frac{\Gamma^3\big(s/3\big)}{\Gamma\big(s/2\big)}f(s) = \frac{27}{2}f'(0) - \frac{27}{4}\gamma f(0),$$

Calculating $f'(0)$, we have

$$\mathrm{res}_{s=0}g(0,\nu_{1,1,0},X,s)=\frac{27}{2\pi}\xi_{\mathfrak{a}}(0,\nu_{1,1,0}\lambda_0)\log X+O(1). \tag{7.48}$$

The cases $\nu=\nu_{1,0,1}$ and $\nu=\nu_{0,1,1}$ lead to formulas analogous to (7.48) with the only change of ν on the right-hand side. Combining them, we prove (7.45). □

Substituting (7.45) into (7.5) and using (7.34), we conclude that there exists some huge R such that the asymptotic formula

$$\sum_{1\le m\le X} r_F(m)(1-\tfrac{m}{X})^r=\frac{\Gamma^3(\frac{4}{3})X}{(k+1)N(\mathfrak{a})D^{\frac12}}+K_1\log X+O(1) \tag{7.49}$$

holds for $r>R$, thus proving theorem 3. □

Turning finally to the proof of the statements made after Theorem 3, if we assume that $\mathfrak{a}$ is the ring of integers $\mathcal{O}$ of $\mathbb{F}$ and that the wide class number of $\mathbb{F}$ is one then by [1, p.94] we have that $\#\mathcal{O}^*/\mathcal{O}^+=8$ or 4 according to whether the narrow class number of $\mathbb{F}$ is one or two. In the first case it can be checked that the only sign character ν for which $\nu(\epsilon)=1$ for all $\epsilon\in\mathcal{O}^*$ is the trivial one, so $K_1=0$. In the second case there is a unique ν with $\sigma_\nu=1$ for which $\nu(\epsilon)=1$ for all ϵ. For this ν we have that

$$\zeta_{\mathcal{O}}(s,\nu)=4L_{\mathbb{F}}(s,\chi), \tag{7.50}$$

where $L_{\mathbb{F}}(s,\chi)$ is the usual Hecke L-function for the narrow class character of $\mathbb{F}$ induced by ν. The Dedekind zeta function for $\mathbb{K}$ factors as

$$\zeta_{\mathbb{K}}(s)=L_{\mathbb{F}}(s,\chi)\zeta_{\mathbb{F}}(s). \tag{7.51}$$

Since $\mathbb{K}/\mathbb{F}$ is ramified at two real places we have that $L_{\mathbb{F}}(s,\chi)$ has a zero of order 1 at $s=0$. Also by (7.50)

$$\xi_{\mathcal{O}}(0,\nu)=8\pi L'_{\mathbb{F}}(0,\chi).$$

Since $\delta=2R_{\mathbb{F}}$ it follows from (7.46) and (7.51) that

$$K_1=\frac{L'_{\mathbb{F}}(0,\chi)}{4R_{\mathbb{F}}}=\frac{1}{4R_{\mathbb{F}}}\lim_{s\to0}\frac{\zeta_{\mathbb{K}}(s)}{s\zeta_{\mathbb{F}}(s)}. \tag{7.52}$$

To evaluate the limit we apply the class number formula formulated at $s=0$ (see e.g. [18, (2) p.71]). Since $\mathbb{F}$ is a totally real cubic field and $\mathbb{K}$ has two real and two pairs of complex embeddings, we have

$$\lim_{s\to0}\frac{\zeta_{\mathbb{F}}(s)}{s^2}=-\frac{h_{\mathbb{F}}R_{\mathbb{F}}}{w_{\mathbb{F}}},\quad \lim_{s\to0}\frac{\zeta_{\mathbb{K}}(s)}{s^3}=-\frac{h_{\mathbb{K}}R_{\mathbb{K}}}{w_{\mathbb{K}}}. \tag{7.53}$$

Note that the number of roots of unity $w_{\mathbb{F}}$ is two and the class number $h_{\mathbb{F}}$ is one. Therefore, substituting (7.53) to (7.52), we prove (1.8).

8. Appendix

Lemma 12. *The following identity holds:*

$$\det Q_{\mathfrak{a},x}=\det A^2, \tag{8.1}$$

where matrices $Q_{\mathfrak{a},x}$ *and* A *are defined by* (3.3) *and* (3.5), *respectively.*

Proof. Let

$$A_{\mathfrak{a},x}=\begin{pmatrix}\alpha_1^{(1)}w_1(x) & \alpha_2^{(1)}w_1(x) & \alpha_3^{(1)}w_1(x)\\ \alpha_1^{(2)}w_2(x) & \alpha_2^{(2)}w_2(x) & \alpha_3^{(2)}w_2(x)\\ \alpha_1^{(3)}w_3(x) & \alpha_2^{(3)}w_3(x) & \alpha_3^{(3)}w_3(x)\end{pmatrix}, \tag{8.2}$$

where $w_j(x)$ are defined by (3.1). Using the definition of matrix $Q_{\mathfrak{a},x}$ (see (3.3)) together with (2.1) we deduce that

$$Q_{\mathfrak{a},x} = A^T_{\mathfrak{a},x} A_{\mathfrak{a},x}. \tag{8.3}$$

Therefore,

$$\det Q_{\mathfrak{a},x} = (w_1(x)w_2(x)w_3(x))^2 \begin{vmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} \\ \alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} \\ \alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} \end{vmatrix}^2. \tag{8.4}$$

Since η_j are units we have $\eta_j^{(1)}\eta_j^{(2)}\eta_j^{(3)} = 1$, and thus (see (3.1)) $w_1(x)w_2(x)w_3(x) = 1$. Therefore, for A defined by (3.5) we have

$$\det Q_{\mathfrak{a},x} = \det A^2.$$

□

Let

$$P_{\mathfrak{a},x}(a,b,c) = \prod_{j=1}^{3} w_j^{a_j}(x) \prod_{j=1}^{3} (a\alpha_1^{(j)} + b\alpha_2^{(j)} + c\alpha_3^{(j)})^{a_j}. \tag{8.5}$$

Lemma 13. *The following identity holds:*

$$\operatorname{tr} Q^{-1}_{\mathfrak{a},x} P^{\dagger}_{\mathfrak{a},x} = 0, \tag{8.6}$$

where $P^{\dagger}_{\mathfrak{a},x}$ is the Hessian matrix $P_{\mathfrak{a},x}$ in the variables a, b, c.

Proof. First, note that the multiple $\prod_{j=1}^{3} w_j^{a_j}(x)$ does not depend on a, b, c, and thus

$$P^{\dagger}_{\mathfrak{a},x} = \prod_{j=1}^{3} w_j^{a_j}(x) P^{\dagger\dagger}_{a_1,a_2,a_3}, \quad tr(Q^{-1}_{\mathfrak{a},x} P^{\dagger}_{\mathfrak{a},x}) = \prod_{j=1}^{3} w_j^{a_j}(x) tr(Q^{-1}_{\mathfrak{a},x} P^{\dagger\dagger}_{a_1,a_2,a_3}), \tag{8.7}$$

where $P^{\dagger\dagger}_{a_1,a_2,a_3}$ is the Hessian matrix for the polynomial

$$P_{a_1,a_2,a_3}(a,b,c) = \prod_{j=1}^{3} (a\alpha_1^{(j)} + a\alpha_2^{(j)} + a\alpha_3^{(j)})^{a_j}. \tag{8.8}$$

Note that the Hessian matrix is zero if more than one a_j is zero. Also for $a_1 = a_2 = a_3 = 1$ the following decomposition takes place:

$$P^{\dagger\dagger}_{1,1,1} = (a\alpha_1^{(3)} + a\alpha_2^{(3)} + a\alpha_3^{(3)}) P^{\dagger\dagger}_{1,1,0} + (a\alpha_1^{(2)} + a\alpha_2^{(2)} + a\alpha_3^{(2)}) P^{\dagger\dagger}_{1,0,1} + \\ + (a\alpha_1^{(1)} + a\alpha_2^{(1)} + a\alpha_3^{(1)}) P^{\dagger\dagger}_{0,1,1}. \tag{8.9}$$

Therefore, if we prove $tr(Q^{-1}_{\mathfrak{a},x} P^{\dagger\dagger}_{a_1,a_2,a_3}) = 0$ in the case of exactly one of $a_j = 0$, we will prove the same for the case of all $a_j = 1$.

It is sufficient to consider only the case $a_1 = a_2 = 1, a_3 = 0$ since the remaining cases can be treated in a similar way. Then

$$P^{\dagger\dagger}_{1,1,0} = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix} = \begin{pmatrix} 2\alpha_1^{(1)}\alpha_1^{(2)} & \alpha_1^{(1)}\alpha_2^{(2)} + \alpha_1^{(2)}\alpha_2^{(1)} & \alpha_1^{(1)}\alpha_3^{(2)} + \alpha_1^{(2)}\alpha_3^{(1)} \\ \alpha_1^{(1)}\alpha_2^{(2)} + \alpha_1^{(2)}\alpha_2^{(1)} & 2\alpha_2^{(1)}\alpha_2^{(2)} & \alpha_2^{(1)}\alpha_3^{(2)} + \alpha_2^{(2)}\alpha_3^{(1)} \\ \alpha_1^{(1)}\alpha_3^{(2)} + \alpha_1^{(2)}\alpha_3^{(1)} & \alpha_2^{(1)}\alpha_3^{(2)} + \alpha_2^{(2)}\alpha_3^{(1)} & 2\alpha_3^{(1)}\alpha_3^{(2)} \end{pmatrix}. \tag{8.10}$$

In view of (8.3), in order to show that $tr(Q^{-1}_{\mathfrak{a},x} P^{\dagger\dagger}_{1,1,0}) = 0$ it is required to prove that

$$tr(A^{-1}_{\mathfrak{a},x} (A^{-1}_{\mathfrak{a},x})^T P^{\dagger\dagger}_{1,1,0}) = 0. \tag{8.11}$$

To calculate the inverse of $A_{\mathfrak{a},x}$ (defined by (8.2)) we will use the cofactor matrix given by

$$A_{\mathfrak{a},x}^{-1} = \frac{1}{\det A_{\mathfrak{a},x}} C_{\mathfrak{a},x}^T. \tag{8.12}$$

Therefore, (8.11) is equivalent to

$$tr(C_{\mathfrak{a},x}^T C_{\mathfrak{a},x} P_{1,1,0}^{\dagger\dagger}) = 0. \tag{8.13}$$

We have

$$C_{\mathfrak{a},x} = \begin{pmatrix} w_2(x)w_3(x)b_{1,1} & w_2(x)w_3(x)b_{1,2} & w_2(x)w_3(x)b_{1,3} \\ w_1(x)w_3(x)b_{2,1} & w_1(x)w_3(x)b_{2,2} & w_1(x)w_3(x)b_{2,3} \\ w_1(x)w_2(x)b_{3,1} & w_1(x)w_2(x)b_{3,2} & w_1(x)w_2(x)b_{3,3} \end{pmatrix} = D_x B_{\mathfrak{a}}, \tag{8.14}$$

where

$$D_x = \begin{pmatrix} w_2(x)w_3(x) & 0 & 0 \\ 0 & w_1(x)w_3(x) & 0 \\ 0 & 0 & w_1(x)w_2(x) \end{pmatrix}, \quad B_{\mathfrak{a}} = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}. \tag{8.15}$$

Furthermore,

$$B_{\mathfrak{a}} = \begin{pmatrix} \alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)} & \alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)} & \alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)} \\ \alpha_2^{(3)}\alpha_3^{(1)} - \alpha_2^{(1)}\alpha_3^{(3)} & \alpha_1^{(1)}\alpha_3^{(3)} - \alpha_1^{(3)}\alpha_3^{(1)} & \alpha_1^{(3)}\alpha_2^{(1)} - \alpha_1^{(1)}\alpha_2^{(3)} \\ \alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)} & \alpha_3^{(1)}\alpha_1^{(2)} - \alpha_1^{(1)}\alpha_3^{(2)} & \alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)} \end{pmatrix}. \tag{8.16}$$

Hence (8.13) is equivalent to

$$tr(B_{\mathfrak{a}}^T D_x^2 B_{\mathfrak{a}} P_{1,1,0}^{\dagger\dagger}) = 0. \tag{8.17}$$

To simplify the notation, we denote the diagonal entries of D_x^2 by d_1, d_2, d_3, namely

$$d_1 := w_2^2(x)w_3^2(x), \quad d_2 := w_1^2(x)w_3^2(x), \quad d_3 := w_1^2(x)w_2^2(x).$$

Then

$$B_{\mathfrak{a}}^T D_x^2 = \begin{pmatrix} d_1b_{1,1} & d_2b_{2,1} & d_3b_{3,1} \\ d_1b_{1,2} & d_2b_{2,2} & d_3b_{3,2} \\ d_1b_{1,3} & d_2b_{2,3} & d_3b_{3,3} \end{pmatrix}, \tag{8.18}$$

(8.19)

$$B_{\mathfrak{a}} P_{1,1,0}^{\dagger\dagger} = \begin{pmatrix} b_{1,1}p_{1,1} + b_{1,2}p_{2,1} + b_{1,3}p_{3,1} & b_{1,1}p_{1,2} + b_{1,2}p_{2,2} + b_{1,3}p_{3,2} & b_{1,1}p_{1,3} + b_{1,2}p_{2,3} + b_{1,3}p_{3,3} \\ b_{2,1}p_{1,1} + b_{2,2}p_{2,1} + b_{2,3}p_{3,1} & b_{2,1}p_{1,2} + b_{2,2}p_{2,2} + b_{2,3}p_{3,2} & b_{2,1}p_{1,3} + b_{2,2}p_{2,3} + b_{2,3}p_{3,3} \\ b_{3,1}p_{1,1} + b_{3,2}p_{2,1} + b_{3,3}p_{3,1} & b_{3,1}p_{1,2} + b_{3,2}p_{2,2} + b_{3,3}p_{3,2} & b_{3,1}p_{1,3} + b_{3,2}p_{2,3} + b_{3,3}p_{3,3} \end{pmatrix}.$$

Using (8.18), (8.19) and the fact that the matrix $P_{1,1,0}^{\dagger\dagger}$ is symmetric we obtain

$$tr(B_{\mathfrak{a}}^T D_x^2 B_{\mathfrak{a}} P_{1,1,0}^{\dagger\dagger}) = \sum_{i=1}^{3} d_i S_i, \tag{8.20}$$

$$S_i = b_{i,1}^2 p_{1,1} + b_{i,2}^2 p_{2,2} + b_{i,3}^2 p_{3,3} + 2b_{i,1}b_{i,2}p_{1,2} + 2b_{i,1}b_{i,3}p_{1,3} + 2b_{i,2}b_{i,3}p_{2,3} \tag{8.21}$$

Now to prove that trace (8.20) is zero it is needed to show that $S_i = 0$. Consider the case $i = 1$. Using (8.10) and (8.16) we obtain

$$\frac{S_1}{2} = (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})^2\alpha_1^{(1)}\alpha_1^{(2)} + (\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})^2\alpha_2^{(1)}\alpha_2^{(2)} + (\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})^2\alpha_3^{(1)}\alpha_3^{(2)} + (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})(\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})(\alpha_1^{(1)}\alpha_2^{(2)} + \alpha_1^{(2)}\alpha_2^{(1)}) + (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})(\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})(\alpha_1^{(1)}\alpha_3^{(2)} + \alpha_1^{(2)}\alpha_3^{(1)}) + (\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})(\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})(\alpha_2^{(1)}\alpha_3^{(2)} + \alpha_2^{(2)}\alpha_3^{(1)}). \tag{8.22}$$

To simplify (8.22), we subsequently group together the terms with $\alpha_1^{(1)}$, $\alpha_2^{(1)}$, $\alpha_3^{(1)}$, getting

$$\frac{S_1}{2} = (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})^2\alpha_1^{(1)}S_{1,1} + (\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})\alpha_2^{(1)}S_{1,2} + (\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})^2\alpha_3^{(1)}S_{1,3}, \tag{8.23}$$

where

$$S_{1,1} = (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})\alpha_1^{(2)} + (\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})\alpha_2^{(2)} + (\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})\alpha_3^{(2)} = 0, \tag{8.24}$$

$$S_{1,2} = (\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})\alpha_2^{(2)} + (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})\alpha_1^{(2)} + (\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})\alpha_3^{(2)} = 0, \tag{8.25}$$

$$S_{1,3} = (\alpha_1^{(2)}\alpha_2^{(3)} - \alpha_1^{(3)}\alpha_2^{(2)})\alpha_3^{(2)} + (\alpha_2^{(2)}\alpha_3^{(3)} - \alpha_2^{(3)}\alpha_3^{(2)})\alpha_1^{(2)} + (\alpha_1^{(3)}\alpha_3^{(2)} - \alpha_1^{(2)}\alpha_3^{(3)})\alpha_2^{(2)} = 0. \tag{8.26}$$

Therefore, $S_1 = 0$. In the same way we can verify that $S_2 = 0$. In this case, we group together the terms with $\alpha_1^{(2)}$, $\alpha_2^{(2)}$, $\alpha_3^{(2)}$. The case of S_3 differs slightly from the previous two since there is no $\alpha_1^{(3)}$, $\alpha_2^{(3)}$, $\alpha_3^{(3)}$ in S_3

$$\frac{S_3}{2} = (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})^2\alpha_1^{(1)}\alpha_1^{(2)} + (\alpha_1^{(2)}\alpha_3^{(1)} - \alpha_1^{(1)}\alpha_3^{(2)})^2\alpha_2^{(1)}\alpha_2^{(2)} + (\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})^2\alpha_3^{(1)}\alpha_3^{(2)} + (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})(\alpha_3^{(1)}\alpha_1^{(2)} - \alpha_1^{(1)}\alpha_3^{(2)})(\alpha_1^{(1)}\alpha_2^{(2)} + \alpha_1^{(2)}\alpha_2^{(1)}) + (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})(\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})(\alpha_1^{(1)}\alpha_3^{(2)} + \alpha_1^{(2)}\alpha_3^{(1)}) + (\alpha_1^{(2)}\alpha_3^{(1)} - \alpha_1^{(1)}\alpha_3^{(2)})(\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})(\alpha_2^{(1)}\alpha_3^{(2)} + \alpha_2^{(2)}\alpha_3^{(1)}).$$

So we group together the terms with $\alpha_1^{(1)}$, $\alpha_2^{(1)}$, $\alpha_3^{(1)}$, getting

$$\frac{S_3}{2} = (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})\alpha_1^{(1)}S_{3,1} + (\alpha_3^{(1)}\alpha_1^{(2)} - \alpha_1^{(1)}\alpha_3^{(2)})\alpha_2^{(1)}S_{3,2} + (\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})\alpha_3^{(1)}S_{3,3}, \tag{8.27}$$

where

$$S_{3,1} = (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})\alpha_1^{(2)} + (\alpha_1^{(2)}\alpha_3^{(1)} - \alpha_1^{(1)}\alpha_3^{(2)})\alpha_2^{(2)} + (\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})\alpha_3^{(2)} = 0,$$

$$S_{3,2} = (\alpha_1^{(2)}\alpha_3^{(1)} - \alpha_1^{(1)}\alpha_3^{(2)})\alpha_2^{(2)} + (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})\alpha_1^{(2)} + (\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})\alpha_3^{(2)} = 0,$$

$$S_{3,3} = (\alpha_1^{(1)}\alpha_2^{(2)} - \alpha_1^{(2)}\alpha_2^{(1)})\alpha_3^{(2)} + (\alpha_2^{(1)}\alpha_3^{(2)} - \alpha_2^{(2)}\alpha_3^{(1)})\alpha_1^{(2)} + (\alpha_3^{(1)}\alpha_1^{(2)} - \alpha_1^{(1)}\alpha_3^{(2)})\alpha_2^{(2)} = 0.$$

Therefore, $S_3 = 0$, and the trace (8.20) is also equal to zero. This concludes the proof. □

Lemma 14. *The following identity holds:*

$$Q^{-1}_{\mathfrak{a},x} = Q_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x}. \tag{8.28}$$

where matrices $Q_{\mathfrak{a},x}$ is defined by (3.3).

Proof. Let $\beta_1, \beta_2, \beta_3$ be an integral basis for $\frac{1}{\mathfrak{a}\mathfrak{d}}$. Then (see Lemma 12)

$$Q_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} = B^T_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} B_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x}, \tag{8.29}$$

where

$$B_{\frac{1}{\mathfrak{a}\mathfrak{d}},x} = \begin{pmatrix} \beta_1^{(1)} w_1(x) & \beta_2^{(1)} w_1(x) & \beta_3^{(1)} w_1(x) \\ \beta_1^{(2)} w_2(x) & \beta_2^{(2)} w_2(x) & \beta_3^{(2)} w_2(x) \\ \beta_1^{(3)} w_3(x) & \beta_2^{(3)} w_3(x) & \beta_3^{(3)} w_3(x) \end{pmatrix}. \tag{8.30}$$

In accordance with (3.5) we define

$$B = \begin{pmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} \end{pmatrix}. \tag{8.31}$$

Since the columns of A^{-1} (for A from (3.5)) form an integral basis for $\frac{1}{\mathfrak{a}\mathfrak{d}}$ (see [11, §36]), we infer that

$$B^T A = I, \text{ or equivalently } AB^T = I, \tag{8.32}$$

where I is the 3×3 identity matrix. Let $[A]_i$ and $[B]_i$ be vectors consisting of the elements of the i-th row of matrices A and B, respectively. Let $(A)_i$ and $(B)_i$ be the column vectors of A and B. Using this notation, we can rewrite the second equality in (8.32) as

$$AB^T = \begin{pmatrix} [A]_1\cdot[B]_1 & [A]_1\cdot[B]_2 & [A]_1\cdot[B]_3 \\ [A]_2\cdot[B]_1 & [A]_2\cdot[B]_2 & [A]_2\cdot[B]_3 \\ [A]_3\cdot[B]_1 & [A]_3\cdot[B]_2 & [A]_3\cdot[B]_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.33}$$

Using (8.2), (8.30), (8.33) and (3.1), we show that

$$B_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} A^T_{\mathfrak{a},x} = \begin{pmatrix} [A]_1\cdot[B]_1 & [A]_2\cdot[B]_1 w_1(-x)w_2(x) & [A]_3\cdot[B]_1 w_1(-x)w_3(x) \\ [A]_1\cdot[B]_2 w_1(x)w_2(-x) & [A]_2\cdot[B]_2 & [A]_3\cdot[B]_2 w_2(-x)w_3(x) \\ [A]_1\cdot[B]_3 w_1(x)w_3(-x) & [A]_2\cdot[B]_3 w_2(x)w_3(-x) & [A]_3\cdot[B]_3 \end{pmatrix} = I. \tag{8.34}$$

This immediately implies that $B^T_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} A_{\mathfrak{a},x} = I$. Finally, applying (8.3), (8.29), (8.34) we conclude that

$$Q_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} \cdot Q_{\mathfrak{a},x} = B^T_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} \left(B_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} \cdot A^T_{\mathfrak{a},x} \right) A_{\mathfrak{a},x} = B^T_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} \cdot A_{\mathfrak{a},x} = I,$$

thus proving (8.28). □

Lemma 15. *The following identity holds:*

$$P^*_{\mathfrak{a},x} = P_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x}, \tag{8.35}$$

where $P_{\mathfrak{a},x}$ and $P^*_{\mathfrak{a},x}$ are defined by (3.6) *and* (3.9), *respectively.*

Proof. Using (8.28), (8.29) and (8.30), we have

$$Q^{-1}_{\mathfrak{a},x} = B^T_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} B_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x} = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{pmatrix}, \tag{8.36}$$

where for $1 \le i, j \le 3$

$$q_{i,j} = \sum_{k=1}^{3} \beta_i^{(k)} \beta_j^{(k)} w_k^2(-x). \tag{8.37}$$

Applying (3.6), (3.9) and (8.36), we show that

$$P^*_{\mathfrak{a},x}(a,b,c)=\prod_{j=1}^{3}w_j^{a_j}(x)\prod_{j=1}^{3}\left(\sum_{m=1}^{3}(aq_{m,1}+bq_{m,2}+cq_{m,3})\alpha_m^{(j)}\right)^{a_j}$$
$$=\prod_{j=1}^{3}w_j^{a_j}(x)\prod_{j=1}^{3}\left(a\sum_{m=1}^{3}q_{m,1}\alpha_m^{(j)}+b\sum_{m=1}^{3}q_{m,2}\alpha_m^{(j)}+c\sum_{m=1}^{3}q_{m,3}\alpha_m^{(j)}\right)^{a_j}. \tag{8.38}$$

It follows from (8.37) and (8.33) that for $1\le l\le 3$

$$\sum_{m=1}^{3}q_{m,l}\alpha_m^{(j)}=\sum_{k=1}^{3}w_k^2(-x)\beta_l^{(k)}\sum_{m=1}^{3}\beta_m^{(k)}\alpha_m^{(j)}$$
$$=\sum_{k=1}^{3}w_k^2(-x)\beta_l^{(k)}[B_k][A_j]=w_j^2(-x)\beta_l^{(j)}. \tag{8.39}$$

Substituting (8.39) to (8.38), applying the relation $w_j(x)w_j(-x)=1$, we have

$$P^*_{\mathfrak{a},x}(a,b,c)=\prod_{j=1}^{3}w_j^{a_j}(-x)\prod_{j=1}^{3}\left(a\beta_1^{(j)}+b\beta_2^{(j)}+c\beta_3^{(j)}\right)^{a_j}=P_{\frac{1}{\mathfrak{a}\mathfrak{d}},-x}(a,b,c).$$

This completes the proof of (8.35). □

Lemma 16. *The set of numbers*

$$\{n\cdot\kappa_1,\,n\cdot\kappa_2,\,n\cdot\kappa_3;\ \|n\|\le N\} \tag{8.40}$$

becomes dense in $\mathbb{R}$ *as* $N\to\infty$.

Proof. According to (2.15), we have to show that

$$\{n_1e_1^{(1)}+n_2e_1^{(2)},\,n_1e_2^{(1)}+n_2e_2^{(2)}\ n_1e_3^{(1)}+n_2e_3^{(2)}:n_1,n_2\in\mathbb{Z}\} \tag{8.41}$$

is dense in $\mathbb{R}$. It is known that for $\alpha\notin\mathbb{Q}$ the set $\{n_1+\alpha n_2\ :n_1,n_2\in\mathbb{Z}\}$ is dense in $\mathbb{R}$. Thus, if we show that at least one of the fractions

$$\frac{e_1^{(1)}}{e_1^{(2)}},\quad\frac{e_2^{(1)}}{e_2^{(2)}},\quad\frac{e_3^{(1)}}{e_3^{(2)}} \tag{8.42}$$

is not a rational number, we prove (8.41). Assume the opposite: all numbers in (8.42) are rationals. Then, since $e_3^{(j)}=-e_2^{(j)}-e_1^{(j)}$ there exist $a_1,a_2,a_3\in\mathbb{Q}$ such that

$$\frac{e_1^{(1)}}{e_1^{(2)}}=a_1,\ \frac{e_2^{(1)}}{e_2^{(2)}}=a_2,\ \frac{e_1^{(1)}+e_2^{(1)}}{e_1^{(2)}+e_2^{(2)}}=a_3. \tag{8.43}$$

Note that $a_i\neq a_j$ (otherwise the corresponding cofactor of the matrix (2.5) M^{-1} must be zero, which is impossible since the matrix M has no zero entries). Thus, it follows from (8.43) that there exist $b_{1,1},b_{1,2},b_{2,1}\in\mathbb{Q}$ such that

$$e_1^{(1)}=b_{1,1}e_2^{(2)},\quad e_2^{(1)}=b_{1,2}e_2^{(2)},\quad e_1^{(2)}=b_{2,1}e_2^{(2)}. \tag{8.44}$$

Furthermore, the following equalities hold

$$3\delta(e_1^{(2)}+2e_2^{(2)})=3\log\eta_1,\quad -3\delta(e_1^{(1)}+2e_2^{(1)})=3\log\eta_2. \tag{8.45}$$

Consider the trivial equation

$$3\delta(e_1^{(2)}+2e_2^{(2)})(e_1^{(1)}+2e_2^{(1)})-3\delta(e_1^{(2)}+2e_2^{(2)})(e_1^{(1)}+2e_2^{(1)})=0. \tag{8.46}$$

Substituting (8.44) and (8.45) into (8.46), we obtain

$$(b_{1,1} + 2b_{1,2}) \log \eta_1 + (b_{2,1} + 2) \log \eta_2 = 0. \tag{8.47}$$

Note that it is impossible that $b_{1,1} + 2b_{1,2} = b_{2,1} + 2 = 0$, since otherwise

$$\begin{vmatrix} e_1^{(1)} & e_2^{(1)} \\ e_1^{(2)} & e_2^{(2)} \end{vmatrix} = 0,$$

which contradicts the fact that the matrix M (2.5) does not have zero entries. Therefore, (8.47) yields the nontrivial relation $\eta_1^{q_1}\eta_2^{q_2} = 1$ with $q_1, q_2 \in \mathbb{Z}$, which is impossible since η_1, η_2 are multiplicatively independent as a basis of a group of totally positive units. Thus, at least one of the numbers (8.42) is irrational and the set (8.41) is dense in $\mathbb{R}$. □

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Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, 119991, Russia
Email address: balkanova@mi-ras.ru

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, U.S.A.
Email address: wdduke@g.ucla.edu

Higher School of Economics and Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, 119991, Russia
Email address: frolenkov@mi-ras.ru