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A Converse Theorem and the Saito-Kurokawa Lift

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1 Introduction

The study of liftings is of considerable importance in the theory of automorphic forms. A particular lifting of interest is that of Saito-Kurokawa. This is a correspondence between classical modular forms and a subspace, Maass's *spezialschar* of Siegel modular forms of degree 2. In a series of papers by Maass, Andrianov, and Zagier (see [EZ]), this correspondence has been shown to be the composition of three isomorphisms involving intermediately Kohnen's space and a space of Jacobi forms. The technique of theta liftings has also been successfully applied to this correspondence (see [Koj]), and it has been treated more generally from the point of view of automorphic representations in [PS].

Early examples of liftings, such as those of Doi and Naganuma [DN] and Shimura [S], were first proved using converse theorems. In this paper, we give another proof of the lifting from Kohnen's space to Maass's space in the spirit of Doi and Naganuma by using a converse theorem, thus bypassing the use of Jacobi modular forms. This converse theorem, due to Imai, calls for proving functional equations for certain Dirichlet series with GL(2) twists which, for lifts, reduce to Rankin-Selberg convolutions. This reduction is nontrivial and requires the identification of a sum over Heegner points of a weight-zero Maass form as Fourier coefficients of a form of weight $1/2$. Such an identification was first observed in a cocompact situation by Maass [M] and was later exploited in [D] to prove equidistribution results for the Heegner points and cycles. An important refinement (and a self-contained proof) of Maass's formula for the modular group was recently given by Katok and Sarnak [KS]. In the case of Eisenstein series, this identification is achieved by combining formulas of Goldfeld and Hoffstein [GH] and a classical result of Dirichlet.

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For the constant eigenfunction, the corresponding weight-$1/2$ form is not a Maass form but satisfies a certain inhomogeneous equation (see the end of Section 4). This interesting function has class numbers for its negative Fourier coefficients and is closely related (by a Maass operator) to a similar function introduced by Hirzebruch and Zagier [HZ].

After the fundamental work of Jacquet, Piatetski-Shapiro, and Shalika [JPSS] on the converse theorem for $GL(3)$ and its application to the lifting of Gelbart and Jacquet [GJ], there has been much work on higher-rank converse theorems (see, for example, [CPS1] and [CPS2]). The L-functions involved in these works are, naturally, Euler products. To our knowledge, our proof is the first application of a converse theorem to a lifting where the Dirichlet series are not Euler products.

Finally, we remark that it seems very likely that the technique of this paper can be adapted to handle the lift in the nonholomorphic case.

2 Statement of the main theorem

We shall assume that the reader is familiar with the basic properties of $1/2$-integral weight holomorphic modular forms, Maass forms, and Siegel modular forms. Some good references are [S], [Iw], and [Kl].

Recall that a Siegel modular form of degree 2 and weight k is a holomorphic function F defined on $\mathcal{H}_2 = \{Z = X + iY \in M(2, \mathbf{C}) \mid {}^tZ = Z, Y > 0\}$ such that, for all $M \in \Gamma_2 = Sp(2, \mathbf{Z})$,

$$F(MZ) = F((AZ + B)/(CZ + D)^{-1}) = |CZ + D|^k F(Z).$$

Such a form F has a Fourier series expansion $F(Z) = \sum_{T \geq 0} A(T)e(\mathrm{tr}\, TZ)$ where the sum runs over all 2×2 $1/2$-integral, positive semidefinite symmetric matrices T. Also, F is a cusp form exactly when $A(T) = 0$ for $|T| = 0$.

Suppose k is an even integer. Let $S_{k-1/2}$ denote the space of holomorphic cusp forms of weight $k - 1/2$ for $\Gamma_0(4)$. Let $S^+_{k-1/2}$ be the subspace of $S_{k-1/2}$ consisting of those forms whose nth Fourier coefficient vanishes unless $n \equiv 0, 3 \bmod 4$ (see [Koh]). Our main purpose is to provide a new direct proof of the following result.

Theorem 1. Suppose $f \in S^+_{k-1/2}$ has the Fourier expansion

$$f(z) = \sum_{n \equiv 0,3 \bmod 4} c(n)e(nz).$$

For each $T > 0$, define

$$A(T) = A\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{d|(n,r,m)} d^{k-1} c\left(\frac{4nm - r^2}{d^2}\right).$$

Then $F(Z) = \sum_{T>0} A(T)e(\mathrm{tr}\, TZ)$ is a Siegel cusp form of weight k and degree 2. □

3 Hecke correspondence for $Sp(2,\mathbf{Z})$

In this section we will give a generalization of Hecke's correspondence to Siegel modular forms of degree 2. The "converse" part, due to Imai, will be our main tool for proving Theorem 1.

The statement entails twisting by the spectral eigenfunctions of

$$\Delta = y^2(\partial_x^2 + \partial_y^2)$$

for $L^2(\Gamma\backslash\mathcal{H})$ where $\Gamma = SL(2,\mathbf{Z})$. There are three types, namely, the constant function $\phi_0(z) = \sqrt{3/\pi}$, an orthonormal basis of cusp forms $\phi_1(z), \phi_2(z), \ldots,$ and the unitary Eisenstein series $E(z, 1/2 + ir), r \geq 0$ (see [Iw]). We may assume that ϕ_j is real and either even or odd in the sense that $\phi_j(-\bar{z}) = \pm\phi_j(z)$. The Eisenstein series is given explicitly for Res > 1 by

$$E(z,s) = \frac{1}{2}\sum_{\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s$$

and has analytic continuation with only a simple pole at $s = 1$. We shall use ϕ to denote any of these eigenfunctions and denote the corresponding eigenvalue of Δ by $1/4 + r^2$.

Suppose we are given a set of complex numbers $A(T)$ for every 2×2 positive definite $1/2$-integral matrix T such that $A(T) << (t_{11}t_{22})^C$ for some C and $A({}^tUTU) = (\det U)^k A(T)$, $\forall U \in GL(2,\mathbf{Z})$. Define the function

$$F(Z) = \sum_{T>0} A(T)e(\mathrm{tr}\ TZ),$$

which is clearly holomorphic on $\mathcal{H}_2$. For ϕ as above, define the Dirichlet series

$$\Psi(s,\phi) = \sum_{T>0/GL(2,\mathbf{Z})}{}^{*} \frac{A(T)\phi(z_T)}{|T|^{s+(k-1)/2}}$$

which is well defined and absolutely convergent for Re(s) sufficiently large. Here the sum is weighted by the order of the stabilizer of T, and T is identified with a point $z_T \in \mathcal{H}$.

The following generalization of the Hecke correspondence to cusp forms of degree 2 is proved in Imai [Im].

Theorem 2. Suppose k is a positive integer. The following two statements are equivalent.

(1) $F(Z)$ is a Siegel cusp form of degree 2 and weight k.

(2) For every ϕ of the same parity as k, the function

$$\Lambda(s,\phi) = (2\pi)^{-2s}\Gamma(s + k/2 - 3/4 + ir/2)\Gamma(s + k/2 - 3/4 - ir/2)\Psi(s,\phi)$$

is entire and bounded in every vertical strip in s and satisfies the functional equation

$$\Lambda(1-s,\phi) = (-1)^k\Lambda(s,\phi).$$

□

In order to apply this to produce lifts, we need the following lemma, whose easy proof is left to the reader.

Lemma 3. For $n \in \mathbf{N}$, let $c(n) \in \mathbf{C}$ be given and define

$$A(T) = A\begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} = \sum_{d|(n,r,m)} d^{k-1} c\left(\frac{|2T|}{d^2}\right).$$

Then

$$\Psi(s,\phi) = 2^{2s+k-1}\zeta(2s)\sum_{n=1}^{\infty} c(n)b(-n)n^{-s-k/2+1}$$

where $b(-n) = n^{-1/2}\sum^*_{\substack{T>0\\|2T|=n}} \phi(z_T)$. $\square$

4 Fourier coefficients of weight-1/2 forms

We next show that $b(-n)$ are Fourier coefficients of a function which has suitable properties for applying the Rankin-Selberg method.

Suppose g is a smooth function on $\mathcal{H}$ which satisfies, for all $\gamma \in \Gamma_0(4)$, the transformation rule

$$g(\gamma z) = J(\gamma, z)g(z),$$

and has a polynomial growth in the cusps of $\Gamma_0(4)\backslash\mathcal{H}$, where $J(\gamma, z) = \vartheta(\gamma z)/\vartheta(z)$ with $\vartheta(z) = y^{1/4}\sum e(n^2 z)$. Such g has a Fourier expansion of the form

$$g(z) = \sum_{n\in\mathbf{Z}} B(n,y)e(nx).$$

Denote by T_r the space of all such g for which $B(n, y)$ for $n < 0$ can be written

$$B(n,y) = b(n)(4\pi y|n|)^{-1/4}W_{-1/4,(ir)/2}(4\pi y|n|)$$

for some number $b(n)$, the nth Fourier coefficient. Here $W_{\alpha,\beta}(y)$ is the usual Whittaker function [MO] which decays exponentially for large y. Let T_r^+ be the subspace of T_r consisting of those functions g whose Fourier coefficients satisfy $B(n, y) = 0$ if $n \equiv 2, 3 \bmod 4$.

Theorem 4. Let ϕ be an even spectral eigenfunction of Δ for the modular group with eigenvalue $1/4 + r^2$. There exists a $g \in T_r^+$ whose Fourier coefficients $b(-n)$ for $n > 0$ are given by

$$b(-n) = n^{-1/2}\sum^*_{\substack{T>0\\|2T|=n}} \phi(z_T). \qquad \square$$

Proof. If ϕ is a cusp form, this follows from [KS, pp. 197, 207] where g is computed as a theta lift of ϕ and is in fact a Maass cusp form of weight 1/2, so

$$\Delta_{1/2}\, g + (1/4 + (r/2)^2)g = 0$$

where

$$\Delta_{1/2} = y^2(\partial_x^2 + \partial_y^2) - (1/2)iy\partial_x.$$

If $\phi = E(z, 1/2 + ir)$, the theorem follows from the next lemma, part of which is a consequence of the work of Goldfeld and Hoffstein [GH], where they explicitly calculated the Fourier coefficients of Eisenstein series of weight 1/2 for $\Gamma_0(4)$ (see Propositions 1.2, 1.4, 1.5 in [GH]). We require those at cusps ∞ and 0, which are given by

$$E_\infty^{1/2}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s J(\gamma, z)^{-1}$$

and

$$E_0^{1/2}(z, s) = e^{i\pi/4} \left(\frac{z}{|z|}\right)^{-1/2} E_\infty^{1/2}(-1/(4z), s),$$

respectively. ■

Lemma 5. Let

$$g(z, s) = \Lambda(s) \left(E_0^{1/2}(z, s/2 + 1/4) + 2^s E_\infty^{1/2}(z, s/2 + 1/4)\right)$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $g(z, 1/2 + ir) \in T_r^+$ with Fourier coefficients

$$b(-n) = n^{-1/2} \sum_{\substack{T>0 \\ |2T|=n}}^{*} E(z_T, 1/2 + ir)$$

for $n > 0$. □

To deal with the constant function $\phi = \sqrt{3/\pi}$, we use the fact that $E(z, s)$ has a simple pole at $s = 1$ with residue $3/\pi$ and the fact, revealed by a calculation (see [Ku]), that $\Lambda^{-1}(s)g(z, s)$ has a simple pole at $s = 1$ with residue $c\vartheta(z)$ for some $c \neq 0$. Next we define

$$g(z) = \sqrt{\pi/3} \lim_{s\to 1} \left(\Lambda^{-1}(s)g(z, s) - c\frac{\vartheta(z)}{s - 1}\right).$$

It can be shown that

$$\Delta_{1/2}\, g + (3/16)g = c_1\vartheta(z)$$

for $c_1 = (c/4)\sqrt{\pi/3}$. It follows that $g \in T_{i/2}^+$ by separation of variables and by using the fact that the Fourier coefficients of $\vartheta(z)$ are supported on the positive squares.

5 Rankin-Selberg convolution

We complete the proof of Theorem 1 by establishing the required properties of $\Lambda(s,\phi)$ in Theorem 2. In view of Lemma 3 we must consider a Rankin–Selberg convolution.

Theorem 6. Suppose $f(z) \in S^+_{k-1/2}$ has Fourier coefficients $c(n)$, and $g(z) \in T^+_r$ has Fourier coefficients $b(-n)$, for $n > 0$. Define

$$\Lambda_\infty(s; f, g) = \pi^{-2s}\Gamma(s + k/2 - 3/4 + ir/2)\Gamma(s + k/2 - 3/4 - ir/2)\zeta(2s)\Phi_\infty(s)$$

where

$$\Phi_\infty(s) = \sum_{n=1}^{\infty} c(n)b(-n)n^{-s-k/2+1}.$$

Then $\Lambda_\infty(s; f, g)$ is entire, bounded in vertical strips, and satisfies the functional equation

$$\Lambda_\infty(1-s; f, g) = \Lambda_\infty(s; f, g). \qquad \square$$

Proof. A standard application of the Rankin-Selberg method gives the integral representation

$$\Lambda_\infty(s; f, g) = 2^{2s}\int_{\Gamma_0(4)\backslash\mathcal{H}} y^{k/2-1/4} f(z)g(z)\tilde{E}_\infty(z, s)\frac{dx\,dy}{y^2},$$

where

$$\tilde{E}_\infty(z, s) = (4\pi)^{k/2-1}\frac{\Gamma(s + k/2)}{\Gamma(s)}\Lambda(2s)E^{-k}_\infty(z, s)$$

with

$$E^{-k}_\infty(z, s) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} ((cz + d)/|cz + d|)^k \operatorname{Im}(\gamma z)^s.$$

Here, as before, $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

To proceed, we need the functional equation of $\tilde{E}_\infty(z, s)$. Let

$$\tilde{E}_0(z, s) = (-z/|z|)^k\tilde{E}_\infty(-1/4z, s)$$

and

$$\tilde{E}_{1/2}(z, s) = ((-2z + 1)/|-2z + 1|)^k\tilde{E}_\infty(-1/(4z - 2), s).$$

Lemma 7. For even k, we have the functional equation

$$\tilde{E}_\infty(z, 1-s) = \frac{2^{4s-3}}{1 - 2^{2s-2}}\tilde{E}_\infty(z, s) + \frac{2^{2s-2}(1 - 2^{2s-1})}{1 - 2^{2s-2}}(\tilde{E}_0(z, s) + \tilde{E}_{1/2}(z, s)). \qquad \square$$

Proof. The functional equation in the case $k = 0$ is found by computing the scattering matrix in [Iw, Theorem 6.5]. The functional equation for general even $k < 0$ is deduced from this by successively applying the Maass operator $L_k = -iy\partial_x + y\partial_y - k/2$, since $L_{-k}E^{-k} = (s + k/2)E^{-k-2}$. This is a consequence of the commutation relation $(L_k h)|_{(k-2)}\,\gamma = L_k(h|_k\gamma)$ where, as usual,

$$h|_k\gamma = \left(\frac{cz+d}{|cz+d|}\right)^{-k} h(\gamma z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is perhaps worth remarking that we make use of the fact that the scaling matrices

$$\sigma_0 = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_{1/2} = \begin{pmatrix} 1 & -1/2 \\ 2 & 0 \end{pmatrix},$$

for the cusps 0 and 1/2 normalize $\Gamma_0(4)$. ■

Now for $j = 0, 1$, we set

$$\Lambda_{j/2}(s; f, g) = \pi^{-2s}\Gamma(s + k/2 - 3/4 + ir/2)\Gamma(s + k/2 - 3/4 - ir/2)\zeta(2s)\Phi_{j/2}(s)$$

where

$$\Phi_{j/2}(s) = \sum_{n \equiv j \bmod 2} c(n)b(-n)n^{-s-k/2+1}.$$

Then it can be shown that these give the Rankin-Selberg convolutions at the cusps 0 and 1/2. Namely, the following holds.

Lemma 8. When $f \in S^+_{k-1/2}$ and $g \in T^+_r$, we have for $j = 0, 1$

$$\Lambda_{j/2}(s) = 2\int_{\Gamma_0(4)\backslash\mathcal{H}} y^{k/2-1/4} f(z)g(z)\tilde{E}_{j/2}(z, s)\frac{dx\,dy}{y^2}.$$ □

Proof. After [KS], consider the transformations

$$\sigma g(z) = \frac{\sqrt{2}}{4}\sum_{\nu \bmod 4} g\left(\frac{z+\nu}{4}\right),$$

$$\tau_2 g(z) = e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2} g\left(-\frac{1}{4z}\right).$$

We note that, as in the holomorphic case [Koh, p. 255], any function g in T^+_r satisfies $Lg = \tau_2\sigma g = g$. For $f \in S^+_{k-1/2}$, $g \in T^+_r$ and $i = 0, 1$, let

$$f_i(z) = \sum_{n \equiv i(2)} c(n)e(nz/4)$$

and

$$g_i(z) = \sum_{n \equiv i(2)} B(n, y/4) e(nx/4).$$

It follows as in [KZ, p. 190] that

$$(2z/i)^{-k+1/2} f\left(\frac{-1}{4z}\right) = \frac{i^{k^2-k}}{2^{k-1}} f_0(z),$$

$$(2z/i)^{-k+1/2} f\left(\frac{-1}{4z} + \frac{1}{2}\right) = \frac{i^{k^2-k}}{2^{k-1}} f_1(z),$$

$$e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2} g\left(-\frac{1}{4z}\right) = \sqrt{2} g_0(z),$$

and

$$e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2} g\left(\frac{-1}{4z} + \frac{1}{2}\right) = \sqrt{2} g_1(z).$$

Making the change of variables $z \to -1/4z$ in the case of $i = 0$, and $z \to -1/4z$ and $z \to z - 1/2$ in the case of $i = 1$, we deduce the identities

$$\int_{\Gamma_0(4)\backslash\mathcal{H}} y^{k/2-1/4} f_i(z) g_i(z) \tilde{E}_\infty(z, s) \frac{dx\,dy}{y^2} = \frac{2^{k-1}}{\sqrt{2}} \int_{\Gamma_0(4)\backslash\mathcal{H}} y^{k/2-1/4} f(z) g(z) \tilde{E}_{j/2}(z, s) \frac{dx\,dy}{y^2}.$$

The Rankin-Selberg method, applied to the functions f_i and g_i, now gives the result of the lemma. ■

The desired functional equation for Λ_∞ follows from Lemma 7, since clearly $\Lambda_\infty = \Lambda_0 + \Lambda_{1/2}$. ■

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