A NOTE ON CAYLEY'S ONE SENTENCE PAPER

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ABSTRACT. The relevance of a forgotten paper of Cayley that comprises one sentence to the theory of certain partitions is explained.

Arthur Cayley (1821–1895) published some 996 papers and one book. One of his papers [1] from 1873, "An elliptic-transcendent identity", contains but one sentence:

The following is a singular identity:

$$(1+q)(1+q^3)(1+q^5)(1+q^7)^2(1+q^9)\cdots \\ -(1-q)(1-q^3)(1-q^5)(1-q^7)^2(1-q^9)\cdots \\ = 2q(1+q^2)(1+q^4)(1+q^6)(1+q^8)(1+q^{10})(1+q^{12})(1+q^{14})^2(1+q^{16})\cdots, \tag{0.1}$$

where in each of the three terms every factor has the exponent 1 or 2 according as the exponent of q is not, or is, divisible by 7.

Given the title of his paper and his expertise in elliptic functions [2, p. 192], it is highly likely that Cayley knew that this identity is equivalent to the classical "irrational" modular equation of degree 7, first found by Guetzlaff [4] in 1834:

$$\sqrt[4]{k}\sqrt[4]{l}+\sqrt[4]{k'}\sqrt[4]{l'}=1, \quad \text{where} \quad k'^2=1-k^2 \ \text{ and } \ l'^2=1-l^2. \tag{0.2}$$

This equation relates the moduli k, k' and l, l' in the elliptic integral transformation formula

$$\frac{L'}{L}=7\,\frac{K'}{K},$$

where K, K', L, L' are the respective standard complete elliptic integrals. Thus, for instance,

$$K=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Now (0.2) reduces to (0.1) after the moduli are expressed in terms of Jacobi's theta constants as functions of $q=e^{-\pi\frac{K'}{K}}$ and $q^7=e^{-\pi\frac{L'}{L}}$, giving

$$\sqrt{\vartheta_4(q)\vartheta_4(q^7)}+\sqrt{\vartheta_2(q)\vartheta_2(q^7)}=\sqrt{\vartheta_3(q)\vartheta_3(q^7)},$$

and then the product expansions of these theta constants is applied, for instance

$$\vartheta_4(q)=\prod_{n\geq 1}(1-q^{2n-1})(1-q^n).$$

A good introductory reference for these modular equations that explains how the identities were classically proven in the case of degree 3 is Chapter XII of Hardy's book [5].

In view of the way that he expressed it, it is also quite possible that Cayley was aware that (0.1) has a remarkable interpretation in terms of partitions. One of several ways to express that interpretation follows.

Theorem 1. *The number of partitions of an odd integer $m > 1$ into an odd number of odd parts that are distinct unless divisible by 7 equals the number of partitions of $\frac{m-1}{2}$ into parts that are distinct unless divisible by 7. In both counts the multiplicity of parts divisible by 7 is at most two and they are counted twice when their multiplicity is one.*

Expanding in a q-series half of either side of (0.1) gives

$$q + q^3 + q^5 + 2q^7 + 2q^9 + 3q^{11} + 4q^{13} + 6q^{15} + 7q^{17} + 9q^{19} + 12q^{21} + \cdots ,$$

with the coefficient of q^m giving the number of partitions of m of the first type and of $\frac{m-1}{2}$ of the second type. For instance this number is 7 when $m = 17$:

$$\begin{aligned} 17 = 1+3+13 &= 1+5+11 = 1+7+9 \\ &= 1+7+9 = 3+5+9 = 3+7+7 \text{ and} \\ \tfrac{17-1}{2} = 8 = 1+7 &= 1+7 = 1+2+5 = 1+3+4 = 2+6 = 3+5. \end{aligned}$$

Of the other well-known irrational modular equations there is one whose q-series equivalent is very similar in form to (0.1), namely the modular equation of degree 23, which has been known since at least 1861 [7] and reads

$$\sqrt[4]{k}\sqrt[4]{l} + \sqrt[4]{k'}\sqrt[4]{l'} + \sqrt{2}\,\sqrt[12]{4klk'l'} = 1. \tag{0.3}$$

This equation is equivalent to the following identity, written like (0.1):

$$\begin{aligned} &(1+q)(1+q^3)(1+q^5)\cdots(1+q^{22})(1+q^{23})^2(1+q^{24})\cdots \\ &-(1-q)(1-q^3)(1-q^5)\cdots(1-q^{22})(1-q^{23})^2(1-q^{24})\cdots \\ &= 2q + 2q^3(1+q^2)(1+q^4)(1+q^6)\cdots(1+q^{44})(1+q^{46})^2(1+q^{48})\cdots , \end{aligned}$$

where in each of the three terms every factor has the exponent 1 or 2 according as the exponent of q is not, or is, divisible by 23.

The associated partition identity is a very close analogue of that in Theorem 1. It seems that there are no others of this exact type and so (0.1) is not quite, but nearly, "singular".

Theorem 2. *The number of partitions of an odd $m > 3$ into an odd number of odd parts that are distinct unless divisible by 23 equals the number of partitions of $\frac{m-3}{2}$ into parts that are distinct unless divisible by 23, where in both counts the multiplicity of parts divisible by 23 is at most two and they are counted twice when their multiplicity is one.*

The relevant q-series begins

$$q + q^3 + q^5 + q^7 + 2q^9 + 2q^{11} + 3q^{13} + 4q^{15} + 5q^{17} + 6q^{19} + 8q^{21} + 10q^{23} + \cdots .$$

For example, the coefficient of q^{23} is 10 and

$$\begin{aligned} 23 &= 23 = 1+3+19 = 1+5+17 = 1+7+15 = 1+9+13 \\ &= 3+5+15 = 3+7+13 = 3+9+11 = 5+7+11 \text{ while} \\ \tfrac{23-3}{2} &= 10 = 1+9 = 2+8 = 3+7 = 4+6 = 1+2+7 = 1+3+6 \\ &= 1+4+5 = 2+3+5 = 1+2+3+4. \end{aligned}$$

Other modular equations, better known these days, relate the values of the classical modular function

$$j(q) = \frac{\left(1 + 240\sum_{n\ge1}\left(\sum_{d|n} d^3\right) q^n\right)^3}{q\prod_{n\ge1}(1-q^n)^{24}} = q^{-1} + 744 + 19688q + 21493760q^2 + \cdots$$

at q and q^n, for a positive integer n. These equations are much more complicated to write down and have large coefficients. Around the time of Cayley's paper, Smith [8] computed that for $x = 2^{-8}j(q)$ and $x' = 2^{-8}j(q^3)$

$$\begin{aligned} x(x+48000)^3 + {}& x'(x'+48000)^3 - 2^{16}x^3x'^3 + 571392\,x^2x'^2(x+x') \\ & - 1069956\,xx'(x^2+x'^2) + 13408902\,x^2x'^2 \\ & + 34338816000\,xx'(x+x') - 11762176000000\,xx' = 0. \end{aligned}$$

This equation can be interpreted as representing the modular curve $X_0(3)$, which is associated to the subgroup $\Gamma_0(3)$ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Modular equations like the rational versions of (0.2) and (0.3), where also k' and l' are eliminated, are related to subgroups of $\Gamma(2)$ and are much simpler, which is due in large part to the fact that $\Gamma(2)$ contains no elliptic elements. This simplicity is responsible for the elegant interpretations of their irrational versions as partition identities.

It is unfortunate, but understandable, that the enigmatic paper [1] has remained virtually unnoticed for more than 150 years. Nevertheless, an equivalent form of Theorem 1, among other results, was independently discovered by Farkas and Kra in [3]. A number of papers after [3] have appeared with the same theme (see e.g. [6], which may be consulted for further references). Had the relevance of Cayley's one sentence paper not been overlooked, the development of this pretty chapter of partition theory surely would have started earlier.

References

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