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Erratum

Bounds for automorphic L–functions. II

W. Duke[1], **J. Friedlander**[2], **H. Iwaniec**[1,2]

[1] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
[2] Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1

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Our paper has been recently a source for the work "Mollification of the fourth moment of automorphic L–functions and arithmetic applications" by E. Kowalski, P. Michel and J. VanderKam, in which they extended and improved significantly our results giving new important applications for the simultaneous non-vanishing of central values of twisted automorphic L–functions. In the process of adopting some of our arguments they found that we made a mistake when changing the variable of integration in (44). Consequently $\Lambda_h(z, y)$ should be s times larger in the next line. Unaware of this missing factor s we were able to apply relatively simple arguments in the following three sections and, not surprisingly, we produced some estimates which appeared to be stronger than they truly are.

In this errata, rather than modifying particular parts of the original paper we propose to replace entirely Sects. 7, 9, 10, 11, 12 by the new ones below. These corrections do not cause any change in the rest of the paper. In particular all of the theorems and corollaries remain as stated in the paper.

We are grateful to the authors of "Mollification of ..." for pointing out the error and we apologize for any trouble which they may have experienced when correcting our arguments themselves. Yet, we wish to say that our corrections were worked out completely independently of theirs, and later, after seeing their manuscript we found substantial differences. Actually, these authors went much further by establishing very strong asymptotic formulas for the relevant sums while we give here only upper bounds of true order of magnitude.

Finally we express special thanks to Philippe Michel for encouraging correspondence.

7 Estimates for Bessel functions

Recall that $k \geq 2$. Throughout we shall make frequent appeal to the following estimates

$$z^\ell J_{k-1}^{(\ell)}(z) \ll z(1+z)^{-\frac{3}{2}}, \tag{7.1}$$

$$z^\ell Y_0^{(\ell)}(z) \ll (1+|\log z|)(1+z)^{-\frac{1}{2}}, \tag{7.2}$$

$$z^\ell K_0^{(\ell)}(z) \ll (1+|\log z|)(1+z)^{-\frac{1}{2}}, \tag{7.3}$$

for all $z > 0$ and any $\ell \geq 0$, where the implied constant depends on k, ℓ. These estimates can be deduced from the corresponding power series if $0 < z \leq 2$, and the asymptotic expansions if $z \geq 2$ (see Sects. 8.44 and 8.45 of [GR]).

9 Evaluation of $T_h^-(c)$ and $T_h^+(c)$

First we consider $h = 0$. Notice that

$$T_0^+(c) = 0 \tag{9.1}$$

because the equation $m + sn = 0$ has no solution in positive integers. However the sum $T_0^-(c)$ is not void; it runs over positive integers m, n satisfying the equation $m - sn = 0$ and counted with multiplicities being the divisor numbers. In view of [DFI2] this should be regarded as a singular determinant equation. Therefore we have

$$T_0^-(c) = -2\pi \sum_n \tau(sn)\tau(n) \int_0^\infty Y_0\left(\frac{4\pi}{c}\sqrt{snx}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{snx}\right) F(x,n)dx\,. \tag{9.2}$$

Here we could further execute the summation with a good error term, but we do not need to do so.

Suppose $h \neq 0$. The asymptotic evaluation of $T_h^-(c)$ and $T_h^+(c)$ are the problems which have been solved in [DFI2]. In the case of $T_h^-(c)$ we shall apply Theorem 1 of [DFI2] for the test function

$$f(z,y) = -2\pi \int_0^\infty Y_0\left(\frac{4\pi}{c}\sqrt{zx}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{xy}\right) F\left(x, \frac{y}{s}\right) dx \tag{9.3}$$

with $z > 0$ and $y > 0$. Actually we have $Y \leq y \leq 2Y$ with $Y = sN$, because $F(x, y)$ is supported on $[M, 2M] \times [N, 2N]$. Put $P = 1 + \sqrt{sMN}/c$ and $Z = c^2P^2M^{-1}$ so $Z > Y$. If $z > Z$ we integrate by parts in x several times while we do not integrate by parts if $z \le Z$. We then differentiate i times in z and j times in y. Finally we use the estimates (7.1), (7.2) showing that

(9.4)
$$z^i y^j f^{(ij)}(z, y) \ll \left(1 + \frac{z}{Z}\right)^{-A} \left(1 + \frac{y}{Y}\right)^{-A} c^{-1}\sqrt{sMN}\, MP^{i+j-2} \log c$$

for any $i, j, A > 0$, the implied constant depending on i, j, A. This verifies the condition (2) of [DFI2]; therefore by (5) of [DFI2] we obtain

$$T_h^-(c) = \int_0^\infty g(h + sy, y)dy + O\left(P^{\frac{1}{4}}(sN)^{\frac{3}{4}}Mc^{\varepsilon}\right) \tag{9.5}$$

where $g(z, y) = G^-(z, y)\Lambda_h(z, y)$ and $\Lambda_h(z, y)$ is given by the rapidly convergent series of Ramanujan sums

$$\Lambda_h(z, y) = \sum_{w=1}^{\infty} \frac{(s, w)}{w^2} S(0, h; w) \left[\log z - \lambda_w\right]\left[\log y - \lambda_{sw}\right] \tag{9.6}$$

with $\lambda_w = 2\gamma + \log(w^2)$ and $\lambda_{sw} = 2\gamma + \log(sw^2/(s, w))$.

In the next section we shall be applying Poisson's summation in the h variable. This requires small preparations of (9.5) and (9.6). First we truncate the series (9.6) to $w < q$, estimating the tail by $O\left(\tau(|h|)\tau(s)q^{-1}\log^2 q\right)$ so that the resulting change in $T_h^-(c)$ is $O\left(\tau(|h|)\tau(s)MNq^{-1}\log^2 q\right)$, which is absorbed by the error term in (9.5). Next notice that $G^-(z, y)$ is very small if z is much larger than Z (see (9.4)). Hence it follows that $T_h^-(c)$ and the integral in (9.5) are both very small if $|h|$ is much larger than Z. Precisely, we can introduce freely a factor $(1 + |h|/Z)^{-2}$ into the error term in (9.5). Therefore we have

(9.7)
$$T_h^-(c) = \sum_{1 \le w < q} \frac{(s, w)}{w^2} S(0, h; w)Y(h) + O\left((1 + |h|/Z)^{-2}\, P^{\frac{1}{4}}(sN)^{\frac{3}{4}}Mc^{\varepsilon}\right) ,$$

where

$$Y(h) = -2\pi \iint \left[\log(h + sy) - \lambda_w\right]\left[\log y - \lambda_{sw}\right] Y_0\left(\frac{4\pi}{c}\sqrt{(h + sy)x}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right) F(x, y)dx\, dy\, . \tag{9.8}$$

Here the range of integration is restricted by the support of $F(x, y)$ and by the condition $z = h + sy > 0$. Similarly we show that

(9.9)
$$T_h^+(c) = \sum_{1 \le w < q} \frac{(s, w)}{w^2} S(0, h; w)K(h) + O\left((1 + h/Z)^{-2}\, P^{\frac{1}{4}}(sN)^{\frac{3}{4}}Mc^{\varepsilon}\right) ,$$

where

$$K(h) = 4 \iint \left[\log(h - sy) - \lambda_w\right]\left[\log y - \lambda_{sw}\right] K_0\left(\frac{4\pi}{c}\sqrt{(h - sy)x}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right) F(x, y)dx\,dy\,. \tag{9.10}$$

Here the range of integration is restricted by the support of $F(x, y)$ and by the condition $z = h - sy > 0$.

10 Evaluation of $T(c)$

By (36) and (9.7) we obtain (recall that (9.7) is not valid for $h = 0$)

(10.1)

$$T^-(c) = \varphi(c)T_0^-(c) + \sum_{1\le w<q} (s, w)w^{-2} \sum_{h\neq 0} S(0, h; w)S(0, h; c)Y(h) + O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\varepsilon}\right)\,.$$

Have in mind that $Y(h)$ depends also on the summation variables c, w which we shall exploit after the summation in h is executed.

For the sum of products of Ramanujan sums we are going to establish the following general formula

Lemma 10.1 *Let $f(x)$ be a function of $\mathcal{C}^2$ class on $\mathbb{R}$ such that $(1 + x^2)$ $f^{(\ell)}(x) \ll 1$ for $\ell = 0, 1, 2$. Then we have*

$$\sum_h S(0, h; c)S(0, h; w) f(h) = \varphi((c, w)) \sum_u{}' \hat{f}(u(c, w)/cw) \tag{10.2}$$

where $\hat{f}(y)$ is the Fourier transform of f. Here and hereafter the $\sum'$ restricts the summation by the condition

$$\left(u, \frac{cw}{(c, w)^2}\right) = 1\,. \tag{10.3}$$

Proof. Splitting the summation into classes $h \equiv a(\text{mod}[c, w])$ and then applying Poisson's formula we obtain

$$\frac{1}{[c, w]} \sum_{a(\text{mod}[c,w])} S(0, a; c)S(0, a; w) \sum_u e\left(\frac{-au}{[c, w]}\right) \hat{f}\left(\frac{u}{[c, w]}\right)\,.$$

For any $u \in \mathbb{Z}$ the resulting complete sum over the classes $a(\text{mod}[c, w])$ normalized by $[c, w]^{-1}$ is equal to the number of solutions to the congruence

$$\frac{dw}{(c, w)} + \frac{\delta c}{(c, w)} \equiv u \left(\text{mod}\, \frac{cw}{(c, w)^2}\right)$$

in $d(\operatorname{mod} c)$, $(d,c)=1$ and $\delta(\operatorname{mod} w)$, $(\delta,w)=1$. There are no solutions unless u satisfies (10.3) in which case the number of solutions equals $\varphi((c,w))$. This completes the proof.

Remarks. The $u=0$ term on the right-hand side of (10.2) contributes $\delta_{cw}\varphi(w)\hat{f}(0)$ where δ_{cw} is the diagonal symbol of Kronecker. Therefore Lemma 10.1 shows an orthogonality property of Ramanujan sums which is particularly strong if $\hat{f}(y)$ has small support. In our applications this diagonal term is outside the range of summation, because $w<q\le c$.

Applying Lemma 10.1 for $Y(h)$ we get

$$\text{(10.4)}\quad \sum_{h\neq 0} S(0,h;c)S(0,h;w)Y(h) = -\varphi(c)\varphi(w)Y(0)+\varphi((c,w))\sum_u{}' \hat{Y}(u(c,w)/cw)$$

where $Y(0)$ is given by (see (9.8))

$$\text{(10.5)}\quad Y(0) = -2\pi\iint \left[\log sy-\lambda_w\right]\left[\log y-\lambda_{sw}\right] Y_0\left(\frac{4\pi}{c}\sqrt{sxy}\right)J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)F(x,y)dx\,dy$$

and

(10.6)

$$\hat{Y}(v) = -2\pi\iint\left(\int_0^\infty \left[\log z-\lambda_w\right]Y_0\left(\frac{4\pi}{c}\sqrt{zx}\right)\cos 2\pi v(z-sy)dz\right) \left[\log y-\lambda_{sw}\right]J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)F(x,y)dx\,dy$$

by combining the terms for u and $-u$ (recall that $u\neq 0$ by the last remarks). Inserting (10.4) into (10.1) we get

$$\text{(10.7)}\quad T^-(c) = \varphi(c)T_0^-(c)-\varphi(c)\sum_{1\le w<q}\varphi(w)(s,w)w^{-2}Y(0) + \sum_{1\le w<q}\varphi((c,w))(s,w)w^{-2}\sum_u{}'\hat{Y}(u(c,w)/cw) + O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\varepsilon}\right).$$

Next we evaluate $T^+(c)$. The arguments are similar to those used for $T^-(c)$. By (36) and (9.9) we obtain (recall that $T_0^+(c) = 0$)

$$T^+(c) = \sum_{1\le w<q} (s,w)w^{-2} \sum_h S(0,h;c)S(0,h;w)K(h) + O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\varepsilon}\right) . \tag{10.8}$$

Applying Lemma 10.1 for $K(h)$ we get

$$\sum_h S(0,h;c)S(0,h;w)K(h) \;=\; \varphi((c,w)) \sum_u{}' \hat{K}(u(c,w)/cw) \tag{10.9}$$

where

$$\hat{K}(v) = 4 \iint \left(\int_0^\infty \left[\log z - \lambda_w\right] K_0\left(\frac{4\pi}{c}\sqrt{zx}\right) \cos 2\pi v(z+sy)dz \right) \left[\log y - \lambda_{sw}\right] J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right) F(x,y)dx\,dy \tag{10.10}$$

by combining the terms for u and $-u$. Inserting (10.9) into (10.8) we get

$$T^+(c) = \sum_{1\le w<q} \varphi((c,w))(s,w)w^{-2} \sum_u{}' \hat{K}(u(c,w)/cw) + O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\varepsilon}\right) . \tag{10.11}$$

For computational reasons (and for aesthetics) it is natural to consider the following linear combinations of the special functions Y_0, K_0 (see also Theorem 4.10 of the Iwaniec Rutgers Lecture Notes, Fall 1997):

$$C(z) = 4K_0(2z) - 2\pi Y_0(2z) , \tag{10.12}$$

$$S(z) = 4K_0(2z) + 2\pi Y_0(2z) . \tag{10.13}$$

These are given by the integral convolutions (see (3.864.1) and (3.864.2) of [GR])

$$C(z) = 4\int_0^\infty \cos(zt)\cos\left(\frac{z}{t}\right)\frac{dt}{t} , \tag{10.14}$$

$$S(z) = 4\int_0^\infty \sin(zt)\sin\left(\frac{z}{t}\right)\frac{dt}{t} . \tag{10.15}$$

Introducing (10.7) and (10.11) into (33) we derive that

$$T(c) = T^*(c) + \varphi(c)T_0(c) - \varphi(c)\sum_{1\le w<q}\varphi(w)(s,w)w^{-2}Y(0)$$

$$(10.16)\qquad + \sum_{1\le w<q}\varphi((c,w))(s,w)w^{-2}\sum_u{}'(\hat{C}-\hat{S})(u(c,w)/cw)$$

$$+ O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\varepsilon}\right) ,$$

where

$$(10.17)\quad \hat{C}(v) = \iint \left(\int_0^\infty \left[\log z - \lambda_w\right] C\left(\frac{2\pi}{c}\sqrt{zx}\right)\cos(2\pi zv)dz\right) \left[\log y - \lambda_{sw}\right] J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)\cos(2\pi syv)F(x,y)dx\,dy ,$$

$$(10.18)\quad \hat{S}(v) = \iint \left(\int_0^\infty \left[\log z - \lambda_w\right] S\left(\frac{2\pi}{c}\sqrt{zx}\right)\sin(2\pi zv)dz\right) \left[\log y - \lambda_{sw}\right] J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)\sin(2\pi syv)F(x,y)dx\,dy .$$

The innermost integrals in the z–variable are evaluated in the Sect. 12. Changing the variable z into $z/2\pi v$ and then using the formulas (12.3)–(12.6) we obtain

$$(10.19)\quad \hat{C}(v) = \frac{1}{v}\iint \left[\log\frac{x}{c^2v^2} - \lambda_w\right]\left[\log y - \lambda_{sw}\right] F(x,y) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)\cos(2\pi syv)\cos\left(\frac{2\pi x}{c^2v}\right)dx\,dy ,$$

$$(10.20)\quad \hat{S}(v) = \frac{1}{v}\iint \left[\log\frac{x}{c^2v^2} - \lambda_w\right]\left[\log y - \lambda_{sw}\right] F(x,y) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)\sin(2\pi syv)\sin\left(\frac{2\pi x}{c^2v}\right)dx\,dy .$$

Subtracting we arrive at

$$(10.21)\quad \hat{C}(v) - \hat{S}(v) = \frac{1}{v}\iint \left[\log\frac{x}{c^2v^2} - \lambda_w\right]\left[\log y - \lambda_{sw}\right] F(x,y) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)\cos 2\pi\left(syv + \frac{x}{c^2v}\right)dx\,dy .$$

We conclude this section with the crude estimate

$$\hat{C}(v) - \hat{S}(v) \ll v^{-1}\left(1 + vsNP^{-1}\right)^{-1} MN(\log MN)^2 \tag{10.22}$$

which follows from (10.21) by partial integration. This is good enough to truncate the series in (10.16) to $u < q$, the tail being absorbed by the error term already present in (10.16).

11 Estimation of $\mathcal{B}(s)$

We need to sum $T(c)$ over all $c \equiv 0 (\operatorname{mod} q)$ in (28). To this end we use (10.16) for $c \leq 2C$ and (27) for $c \geq C$ with a smooth partition of the transition range, where C is at our disposal subject to $C \geq \sqrt{sMN}$. We obtain

$$\begin{aligned} \mathcal{B}(s) \ll &(MN/s)^{\frac{1}{2}} q^{\varepsilon+1} + MNq^{\varepsilon-1} \\ &+ q \sum_{N<n<2N} \tau(sn)\tau(n)|Q(n)| \\ &+ q \sum_{1\leq w<q} \frac{(s,w)}{w}|R(w)| \\ &+ q \sum\sum_{1\leq u,w<q} \frac{(s,w)}{uw}|R(u,w)| \\ &+ (sMN)^{\frac{1}{2}} MNC^{\varepsilon-1} \\ &+ \left(C + q^{-\frac{5}{4}}(sMN)^{\frac{9}{8}}\right)(sN)^{\frac{3}{4}} C^{\varepsilon} \\ &+ q(MN)^{\frac{3}{4}} C^{\varepsilon-\frac{1}{2}} . \end{aligned} \tag{11.1}$$

Here the first term comes from the estimation of $T(0)$ given by (31), the second term comes from the estimation of $T^*(c)$ given by (43), and the third term comes from $T_0^-(c)$ given by (9.2) where we have

$$Q(n) = \sum_{c\equiv 0(q)} \varphi(c)c^{-2} \int_0^\infty Y_0\left(\frac{4\pi}{c}\sqrt{snx}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{snx}\right) F(x,n)dx. \tag{11.2}$$

The fourth term comes by estimating $\varphi(w) \leq w$ in (10.16) where

$$\begin{aligned} R(w) = \sum_{c\equiv 0(q)} \varphi(c)c^{-2} \iint &\left[\log sy - \lambda_w\right]\left[\log y - \lambda_{sw}\right] F(x,y) \\ &Y_0\left(\frac{4\pi}{c}\sqrt{sxy}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right) dx\, dy \end{aligned} \tag{11.3}$$

by (10.5). In the fifth term we have

(11.4)
$$R(u,w)=\sideset{}{'}\sum_{c\equiv 0(q)}\frac{\varphi((c,w))}{c(c,w)}\iint\left[\log\frac{xw^2}{u^2(c,w)^2}-\lambda_w\right]\left[\log y-\lambda_{sw}\right]$$
$$F(x,y)\,J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right)\cos\frac{2\pi}{c}\left(syu\frac{(c,w)}{w}+\frac{xw}{u(c,w)}\right)dx\,dy\ ,$$

where $\sum'$ restricts the variable c by the condition (10.3) (see (10.16) and see (10.21) with $v=u(c,w)/cw$). Notice that we have no restriction $c\le 2C$ in the series (11.2), (11.3), (11.4), although we were cutting in the range $C<c<2C$. The tails of these series over $c>C$ are estimated trivially by $(sMN)^{\frac{1}{2}}MNC^{\varepsilon-1}$ which comprises the sixth term on the right-hand side of (11.1). The seventh term comes from the relevant sum of the error term in (10.16) with $c<2C$. Finally the eighth term comes from the application of (27) to the part of the series (28) over $c>C$ (recall that $k\ge 2$ and $M,N\ll q^{1+\varepsilon}$).

We choose $C=s\sqrt{MN}$ making the sixth and the eighth terms negligible by comparison to the first term. The second term is also negligible.

It remains to estimate $Q(n)$, $R(w)$ and $R(u,w)$. Changing the variable x into c^2t^2 we write

(11.5)
$$Q(n)=\int_0^\infty Y_0\left(4\pi t\sqrt{sn}\right)J_{k-1}\left(4\pi t\sqrt{sn}\right)\left(\sum_{c\equiv 0(q)}\varphi(c)F(c^2t^2,n)\right)dt^2\ .$$

In this way we eliminated c in the oscillating Bessel functions. For the last sum we use the following

Lemma 11.1 *Let f be a smooth function compactly supported on $\mathbb{R}^+$. Then*

$$\sum_{c\equiv 0(q)}\frac{\varphi(c)}{c}f(c)=\frac{1}{\zeta(2)\nu(q)}\int f(x)dx+O\left(\frac{\varphi(q)}{q}\int|f'(x)|\log\left(1+\frac{x}{q}\right)dx\right)\tag{11.6}$$

where

$$\nu(q)=q\prod_{p|q}\left(1+\frac{1}{p}\right)\tag{11.7}$$

and the implied constant is absolute.

Proof. Our sum is equal to

$$S = \sum_d \frac{\mu(d)}{d} \sum_n f\,(n[d,q])\ .$$

Hence we derive by the Euler-Maclaurin formula

$$\sum_\ell F(\ell) = \int \big(F(t) + \{t\}F'(t)\big)\,dt \tag{11.8}$$

(this is valid for any $F \in \mathcal{C}_0^\infty(\mathbb{R})$) that

$$S = \left(\sum_d \frac{\mu(d)}{d[d,q]}\right) \int f(x)dx + \int \xi_q(x)\, f'(x)dx\ .$$

Here the factor in front of $\hat{f}(0)$ equals $1/\zeta(2)\nu(q)$ and

$$\xi_q(x) = \sum_d \frac{\mu(d)}{d} \left\{\frac{x}{[d,q]}\right\}\ .$$

Another expression for this is

$$\xi_q(x) = \frac{\varphi(q)}{q} \sum_{(d,q)=1} \frac{\mu(d)}{d} \left\{\frac{x}{dq}\right\}\ .$$

Hence it follows that

$$\xi_q(x) \ll \frac{\varphi(q)}{q} \sum_d \frac{1}{d} \min\left(1, \frac{x}{dq}\right) \ll \frac{\varphi(q)}{q} \log\left(1 + \frac{x}{q}\right)\ .$$

This completes the proof of (11.6).

Applying (11.6) for the sum in (11.5) we get

$$\big(2\zeta(2)\nu(q)t^2\big)^{-1} \int F(x,n)dx + O\left(\sqrt{M}\, t^{-1} \log(1 + Mt^{-1})\right)\ .$$

Hence

$$\begin{aligned} Q(n) = {} & \frac{1}{\zeta(2)\nu(q)} \left(\int_0^\infty Y_0(t)\, J_{k-1}(t) t^{-1} dt\right) \int F(x,n)dx \\ & + O\left(\sqrt{M} \int_0^\infty |Y_0(4\pi t\sqrt{sn})\, J_{k-1}(4\pi t\sqrt{sn})| \log\left(1 + \frac{M}{t}\right) dt\right)\ . \end{aligned}$$

The first integral vanishes (see (12.1)) so applying (7.1) and (7.2) with $\ell = 0$ we are left with

(11.9)
$$Q(n) \ll \sqrt{M} \int_0^\infty \frac{1 + |\log t\sqrt{sn}|}{1 + t\sqrt{sn}} \log\Big(1 + \frac{M}{t}\Big) dt \ll \left(\frac{M}{sn}\right)^{\frac{1}{2}} (\log snM)^2 .$$

Summing over n we derive that

$$\sum_{N<n<2N} \tau(sn)\tau(n)|Q(n)| \ll \tau(s) \left(\frac{MN}{s}\right)^{\frac{1}{2}} (\log sMN)^5 . \tag{11.10}$$

Similarly we show that (replace the summation in n by an integration in y)

$$R(w) \ll \left(\frac{MN}{s}\right)^{\frac{1}{2}} (\log sMN)^5 . \tag{11.11}$$

The case of $R(u, w)$ is also similar except that we shall appeal to the orthogonality formula (12.2) in place of (12.1). Moreover the estimation of the error term requires extra care. First we eliminate c from the oscillating cosine and the Bessel function by changing the variables. We write $d = (c, w)$, $c = \ell[d, q]$ so the conditions of summation are now over $d \mid w$ such that

$$\left(\frac{w}{d}, \frac{q}{(d, q)}\right) = 1 , \qquad \left(\frac{wq}{d(d, q)}, u\right) = 1, \tag{11.12}$$

and $(\ell, uw/d) = 1$. We change the variables of integration x, y into ℓxy, $\ell x/sy$ getting

$$R(u, w) = \frac{2}{s} \sum_{d|w}{}'' \frac{\varphi(d)}{d[d, q]} \int_0^\infty \int_0^\infty J_{k-1}\left(\frac{4\pi x}{[d, q]}\right) \cos\left(\frac{2\pi x}{[d, q]}\Big(\frac{du}{wy} + \frac{wy}{du}\Big)\right) \Phi_d(x, y) \frac{dx\, dy}{y} , \tag{11.13}$$

where

(11.14)
$$\Phi_d(x, y) = \sum_{(\ell, uw/d)=1} \left[\log \frac{\ell xyw^2}{u^2 d^2} - \lambda_w\right] \left[\log \frac{\ell x}{sy} - \lambda_{sw}\right] F\left(\ell xy, \frac{\ell x}{sy}\right) \ell x$$

and $\sum''$ is restricted by the conditions (11.12).

Lemma 11.2 *Let $F(t)$ be a smooth function compactly supported on $\mathbb{R}^+$. Then*

$$\sum_{(\ell,k)=1} F(\ell) = \frac{\varphi(k)}{k} \int F(t)dt + \int \eta_k(t) F'(t)dt \tag{11.15}$$

where

$$\eta_k(t) = \sum_{m|k} \mu(m) \left\{ \frac{t}{m} \right\} . \tag{11.16}$$

Proof. This follows from the Euler-Maclaurin formula (11.8).

Note that $\eta_k(t) = t\varphi(k)/k$ if $0 \le t < 1$ and that $|\eta_k(t)| \le \tau(k)$. Hence

$$|\eta_k(t)| \le \tau(k) \min(1, t) . \tag{11.17}$$

In particular (11.15) for $k = uw/d$ yields

$$\Phi_d(x, y) = \frac{\varphi(k)}{k} \frac{I(y)}{x} + \int \eta_k \left(\frac{t}{x} \right) H(t, y)dt \tag{11.18}$$

where

$$I(y) = \int \Big[\log \frac{tyw^2}{u^2 d^2} - \lambda_w \Big]\Big[\log \frac{t}{sy} - \lambda_{sw} \Big] F\Big(ty, \frac{t}{sy}\Big) t\, dt \tag{11.19}$$

and

$$H(t, y) = \frac{\partial}{\partial t}\Big[\log \frac{tyw^2}{u^2 d^2} - \lambda_w \Big]\Big[\log \frac{t}{sy} - \lambda_{sw} \Big] F\Big(ty, \frac{t}{sy}\Big) t . \tag{11.20}$$

Now we introduce (11.18) into (11.13). From the leading term we obtain the following integral in the x–variable

$$\int_0^\infty J_{k-1} \left(\frac{4\pi x}{[d, q]} \right) \cos \left(\frac{2\pi x}{[d, q]} \left(\frac{du}{wy} + \frac{wy}{du} \right) \right) \frac{dx}{x} = 0$$

by virtue of (12.2), because $\dfrac{du}{wy} + \dfrac{wy}{du} \ge 2$. Therefore we are left with

$$R(u, w) = \frac{2}{s} \sum_{d|w}'' \frac{\varphi(d)}{d[d, q]} \int J_{k-1} \left(\frac{4\pi x}{[d, q]} \right) \int \eta_k \left(\frac{t}{x} \right) I(t, x)dt\, dx \tag{11.21}$$

where

$$I(t, x) = \int H(t, y) \cos \left(\frac{2\pi x}{[d, q]} \left(\frac{du}{wy} + \frac{wy}{du} \right) \right) \frac{dy}{y} . \tag{11.22}$$

We shall estimate $I(t, x)$ by means of the following

Lemma 11.3 *Suppose* $H(y)$ *is a smooth function supported on* $[Y, 2Y]$ *such that* $|H(y)| \le 1$ *and* $|H'(y)| \le Y^{-1}$*, where* $Y > 0$*. Then for any* $a, b > 0$ *we have*

$$\int H(y) \cos\left(ay + \frac{b}{y}\right) \frac{dy}{y} \ll (1 + ab)^{-\frac{1}{4}} \tag{11.23}$$

where the implied constant is absolute.

Proof. By Lemma 4.5 of Titchmarsh, Theory of the Riemann Zeta-Function applied twice, together with the trivial bound, we deduce that

$$\int_Y^y \cos\left(ax + \frac{b}{x}\right) \frac{dx}{x} \ll \min\left(1, a^{-1/2}, b^{-1/2}\right)$$

for $Y \le y \le 2Y$. Hence

$$\int H(y) \cos\left(ay + \frac{b}{y}\right) \frac{dy}{y} = -\int_Y^{2Y} \left(\int_Y^y \cos\left(ax + \frac{b}{x}\right)\frac{dx}{x}\right) H'(y)dy$$
$$\ll \min\left(1, a^{-\frac{1}{2}}, b^{-\frac{1}{2}}\right)$$

which is a slightly better bound than (11.23).

Before applying (11.23) to $H(t, y)$ we record from the support of $F(x, y)$ that

$$\sqrt{sMN} \le t \le 2\sqrt{sMN} \tag{11.24}$$

and

$$H(t, y) \ll (\log q)^2\ . \tag{11.25}$$

Therefore Lemma 11.3 yields

$$I(t, x) \ll \left(1 + \frac{x}{[d, q]}\right)^{-\frac{1}{2}} (\log q)^2\ . \tag{11.26}$$

Inserting this estimate into (11.21) and using $J_{k-1}(z) \ll (1 + z)^{-\frac{1}{2}}$ (see (7.1)) we obtain

$$R(u, w) \ll$$
$$\left(\frac{MN}{s}\right)^{\frac{1}{2}} (\log q)^2 \sum_{d|w} \tau\left(\frac{uw}{d}\right) \int_0^\infty (x + [d, q])^{-1} \min(1, \sqrt{sMN}/x)dx\ .$$

Hence we finally get

$$R(u, w) \ll \tau(u)\tau_3(w) \left(\frac{MN}{s}\right)^{\frac{1}{2}} (\log q)^3\ . \tag{11.27}$$

Now we have all the estimates ready to complete the proof of Theorem 2 in the case $r = 1$. We introduce (11.10), (11.11) and (11.27) into (11.1) getting

$$\mathcal{B}(s) \ll (MN/s)^{\frac{1}{2}} q^{1+\varepsilon} + \left(s\sqrt{MN} + q^{-\frac{5}{4}}(sMN)^{\frac{9}{8}}\right)(sN)^{\frac{3}{4}} q^{\varepsilon}$$
$$\ll q^{\varepsilon}\left(qs^{-\frac{1}{2}} + q^{\frac{3}{4}} s^{\frac{15}{8}}\right)(MN)^{\frac{1}{2}} .$$

This is slightly better than the original estimate (11) of Theorem 2 provided $s \leq q^{4/27}$. However, if $s > q^{1/9}$ then (11) follows from Corollary 1.

12 Definite integrals of special functions

Here we give the integral formulas which are needed in this work. We give proofs of those which we couldn't find in the literature.

First are the following orthogonality formulas

$$\int_0^{\infty} Y_0(z)\, J_{k-1}(z) z^{-1} dz = 0 \tag{12.1}$$

if k is even (see (6.576.6) of [GR]), and

$$\int_0^{\infty} J_{k-1}(az)\cos(bz) z^{-1} dz = 0 \tag{12.2}$$

if k is even and $b \geq a > 0$ (see (6.693.2) of [GR]).

Next we shall prove the following formulas for any $a > 0$

$$\int_0^{\infty} C(\sqrt{az}) \cos z\, dz = 2\pi \cos a\ , \tag{12.3}$$

$$\int_0^{\infty} S(\sqrt{az}) \sin z\, dz = 2\pi \sin a\ , \tag{12.4}$$

$$\int_0^{\infty} C(\sqrt{az})(\log z) \cos z\, dz = 2\pi(\log a) \cos a\ , \tag{12.5}$$

$$\int_0^{\infty} S(\sqrt{az})(\log z) \sin z\, dz = 2\pi(\log a) \sin a\ . \tag{12.6}$$

For the proof we write these formulas as Fourier transforms

$$(12.3') \qquad \int_0^\infty C(\sqrt{z})\cos(bz)dz = \frac{2\pi}{b}\cos\frac{1}{b},$$

$$(12.4') \qquad \int_0^\infty S(\sqrt{z})\sin(bz)dz = \frac{2\pi}{b}\sin\frac{1}{b},$$

$$(12.5') \qquad \int_0^\infty C(\sqrt{z})(\log z)\cos(bz)dz = -\frac{4\pi}{b}(\log b)\cos\frac{1}{b}\ ,$$

$$(12.6') \qquad \int_0^\infty S(\sqrt{z})(\log z)\sin(bz)dz = -\frac{4\pi}{b}(\log b)\cos\frac{1}{b}\ ,$$

by changing the variable z into bz with $b = a^{-1}$. Actually (12.5′), (12.6′) are obtained by this change of variables and using (12.3′), (12.4′) respectively.

Now we write

$$(12.3^*) \qquad C(\sqrt{z}) = \int_0^\infty \frac{4}{b}\left(\cos\frac{1}{b}\right)\cos(bz)db$$

by changing the variable in (10.14). This is the Fourier cosine transform of $\frac{4}{b}\cos\frac{1}{b}$. Therefore (12.3′) follows from (12.3*) by Fourier inversion. Similarly (12.4′) follows from

$$(12.4^*) \qquad S(\sqrt{z}) = \int_0^\infty \frac{4}{b}\left(\sin\frac{1}{b}\right)\sin(bz)db$$

by Fourier inversion of the Fourier sine transform.

For the proof of (12.5′) we write

$$C(\sqrt{z})\log z = \int_0^\infty \frac{4}{b}(\log bz)\left(\cos\frac{1}{b}\right)\cos(bz)db - \int_0^\infty \frac{4}{b}(\log b)\left(\cos\frac{1}{b}\right)\cos(bz)db\ .$$

Changing b into $1/bz$ in the first integral we get

$$(12.5^*) \qquad C(\sqrt{z})\log z = -\int_0^\infty \frac{8}{b}(\log b)\left(\cos\frac{1}{b}\right)\cos(bz)db\ .$$

This is the Fourier cosine transform of $-\frac{8}{b}(\log b)\cos\frac{1}{b}$. Therefore (12.5′) follows from (12.5*) by Fourier inversion. Similarly one derives (12.6′)

from

(12.6*) $$S(\sqrt{z})\log z = -\int_0^\infty \frac{8}{b}(\log b)\left(\sin\frac{1}{b}\right)\sin(bz)db\ .$$

Remarks. All the integrals considered here converge due to oscillations of trigonometric functions, but not absolutely. Nevertheless, one can justify the Fourier inversions by mollifying the integrals (12.3*) – (12.6*) with the function $\omega(b) = \exp(-\varepsilon b - \varepsilon b^{-1})$ and letting ε tend to zero.