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Inventiones mathematicae

Bounds for automorphic *L*-functions

W. Duke[1,*], **J. Friedlander**[2,*] **and H. Iwaniec**[1]

[1] Rutgers University, New Brunswick, NJ 08903, USA

[2] University of Toronto, Toronto Ont. M5S1A1, Canada

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1 Introduction

The L-functions of Dirichlet, for primitive characters χ modulo q, satisfy the functional equation

$$(1.1)\qquad \left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)=\varepsilon_\chi i^{-a}\left(\frac{q}{\pi}\right)^{(1-s)/2}\Gamma\left(\frac{1-s+a}{2}\right)L(1-s,\bar{\chi})$$

where $a=\frac{1}{2}(\chi(1)-\chi(-1))$ and $\varepsilon_\chi=\tau(\chi)q^{-1/2}$ with $\tau(\chi)=\sum_{b\,(\mathrm{mod}\,q)}\chi(b)e\left(\frac{b}{q}\right)$ the Gauss sum, so $|\varepsilon_\chi|=1$. From this and the Phragmen-Lindelöf convexity principle, it follows that they satisfy the bound

$$(1.2)\qquad L(s,\chi)\ll q^{1/4}\log q$$

on the line $\operatorname{Re} s=\frac{1}{2}$, the implied constant depending on s. This classical estimate resisted improvement for many years until Burgess [B] reduced the exponent from $\frac{1}{4}$ to $\frac{3}{16}$, many important applications following therefrom. The proof of Burgess appeals to the Riemann Hypothesis for curves established by Weil.

Another method to break the convexity barrier was given recently in [F-I]. This method, as well as being more elementary, combines well with the methods developed in the series [D-I$_2$] to allow us here to treat the more difficult automorphic L-functions.

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* *Current address*: Mathematical Science Research Institute, Berkeley, CA 94720, USA

Although it is clear that the method extends to more general L-functions of rank one, we restrict here, for the sake of exposition, to those L-functions attached to an arbitrary holomorphic cusp form of weight k for the full modular group:

$$f(z) = \sum_{m=1}^{\infty} b_m m^{(k-1)/2} e(mz) . \tag{1.3}$$

In this Fourier expansion the coefficients have been normalized so that Deligne's bound asserts that

$$b_m \ll \tau(m) \tag{1.4}$$

where the latter is the divisor function.

With χ as before, we associate to the cusp form $f(z)$, the automorphic L-function

$$L_f(s, \chi) = \sum_{m=1}^{\infty} b_m \chi(m) m^{-s} .$$

These were studied by Hecke who proved that they are entire and satisfy the following functional equation:

$$\left(\frac{q}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L_f(s, \chi) = \varepsilon_\chi^2 i^k \left(\frac{q}{2\pi}\right)^{1-s} \Gamma\left(1 - s + \frac{k-1}{2}\right) L_f(1 - s, \bar{\chi}), \tag{1.5}$$

where ε_χ is as before, cf. [Sh, p. 93]. By the duplication formula this can also be written as

$$\left(\frac{q}{\pi}\right)^s \Gamma\left(\frac{s + a_1}{2}\right) \Gamma\left(\frac{s + a_2}{2}\right) L_f(s, \chi)$$
$$= \varepsilon_\chi^2 i^{-a_1 - a_2} \left(\frac{q}{\pi}\right)^{1-s} \Gamma\left(\frac{1 - s + a_1}{2}\right) \Gamma\left(\frac{1 - s + a_2}{2}\right) L_f(1 - s, \bar{\chi}) ,$$

where $a_1 = \frac{1}{2}(k - 1)$ and $a_2 = \frac{1}{2}(k + 1)$, displaying more clearly its relation to (1.1).

As before it follows by convexity that

$$L_f(s, \chi) \ll q^{1/2} (\log q)^2 , \tag{1.6}$$

for $\operatorname{Re} s = \frac{1}{2}$, the implied constant depending only on s. The factor $(\log q)^2$ can be deleted by refining the convexity argument.

The Burgess method is not applicable here to reduce the exponent $\frac{1}{2}$. In the case that χ is real and at the special point $s = \frac{1}{2}$, the exponent was reduced to $\frac{3}{7}$ in [I] and in [D]. This was obtained by the combination of an estimate for the Fourier coefficients of half integral weight cusp forms together with Waldspurger's theorem and therefore does not apply at other points or for non-real characters.

In this paper we prove

Theorem 1 *Let χ be a primitive character to modulus q and let $\operatorname{Re} s = \frac{1}{2}$. Then we have*

$$L_f(s, \chi) \ll |s|^2 q^{5/11} \tau(q)^2 \log q \tag{1.7}$$

where the implied constant depends only on f.

We remark that here, and throughout the paper, we have not made any effort to obtain the best exponent in $|s|$, in $\tau(q)$, or in $\log q$.

Combining the above estimate with the Waldspurger theorem (see [K]) we get the estimate

$$c(q) \ll q^{5/22}\tau(q)\log q \tag{1.8}$$

for the Fourier coefficient of half integral weight forms on $\Gamma_0(4)$. The exponent $\frac{5}{22}$, although larger than the exponent $\frac{3}{14}$ given in [I], is still strong enough to provide a simplification of the solution of the Linnik problem for the sphere given in [D] (see also [Sa]). The technique given here applies also to those L-functions which are similarly related to the more general ternary Linnik problems treated in [D] and [D-SP].

2 The delta-symbol

Define

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 . \end{cases} \tag{2.1}$$

Let $\omega(t)$ be an even function on $\mathbb{R}$ with $\omega(0) = 0$ and compactly supported such that

$$\sum_{k=1}^{\infty} \omega(k) = 1 .$$

Put

$$\delta_k(n) = \omega(k) - \omega\left(\frac{n}{k}\right) . \tag{2.2}$$

We then have

$$\delta(n) = \sum_{k \mid n} \delta_k(n) . \tag{2.3}$$

Using additive characters this yields

$$\delta(n) = \sum_{k} k^{-1} \sum_{h (\mathrm{mod}\ k)} e\left(\frac{hn}{k}\right) \delta_k(n) .$$

Putting

$$\Delta_c(n) = \sum_{r} r^{-1} \delta_{cr}(n) \tag{2.4}$$

we get, with $r = (h, k)$, $a = h/r$, $c = k/r$,

$$\delta(n) = \sum_{c} c^{-1} \sum_{a (\mathrm{mod}\ c)}^{*} e\left(\frac{an}{c}\right) \Delta_c(n) . \tag{2.5}$$

In practice we apply the above identity to integers $|n| < \frac{N}{2}$, say, with test function $\omega(t)$ supported on $\frac{K}{2} < |t| < K$, and whose derivatives satisfy $\omega^{(j)}(t) \ll K^{-j-1}$. Then $\delta_k(n)$ vanishes save for $1 \leqq k < \max(K, \frac{N}{K}) = K$ by choosing $K = N^{1/2}$. Hence $\Delta_c(n)$ vanishes save for $1 \leqq c < K$ and $\Delta_c(n) \ll K^{-1}$.

3 A mean-value theorem

Let a_n be complex numbers for $1 \leqq n \leqq \frac{N}{2}$. In this section our aim is to give an estimate for

$$S = \sum_{\chi (\mathrm{mod}\, q)}^{*} \left| \sum_n a_n \chi(n) \right|^2 ,$$

where star restricts the summation to primitive characters. Expanding each primitive χ using Gauss sums and then extending the resulting summation to all characters mod q, we get by orthogonality,

$$S \leqq \frac{\varphi(q)}{q} \sum_{a (\mathrm{mod}\, q)}^{*} \left| \sum_n a_n e\left(\frac{an}{q}\right) \right|^2 .$$

Next, extending to all residue classes we get

$$S \leqq \varphi(q) \sum_{h \equiv 0 (\mathrm{mod}\, q)} S_h ,$$

where

$$S_h = \sum_{n_1 - n_2 = h} a_{n_1} \bar{a}_{n_2} .$$

For $h = 0$ we have the diagonal contribution $\varphi(q) S_0$, where

$$S_0 = \sum |a_n|^2 . \tag{3.1}$$

We denote the remaining contribution from $h \neq 0$ by S_*. Given $h \neq 0$ we apply (2.5) to split

$$S_h = \sum_c c^{-1} S_{hc} ,$$

where

$$S_{hc} = \sum_{a (\mathrm{mod}\, c)}^{*} \sum_{n_1, n_2} a_{n_1} \bar{a}_{n_2} e\left(\frac{a}{c}(n_1 - n_2 - h)\right) \Delta_c(n_1 - n_2 - h) .$$

Now we specialize the sequence (a_n) to be the convolution

$$a_n = \sum_{\ell m = n} \lambda_\ell b_m g(m) ,$$

where λ_ℓ are arbitrary complex numbers for $1 \leqq \ell \leqq L$ and g is a $\mathscr{C}^2$ function, supported in [M, 2M] and satisfying

$$|g^{(j)}(m)| \leqq M^{-j} \quad \text{for } j = 0, 1, 2 .$$

We then have

$$S_{hc} = \sum_{\ell_1, \ell_2} \lambda_{\ell_1} \bar{\lambda}_{\ell_2} T_{\ell_1 \ell_2}(c) ,$$

where

$$T_{l_1 l_2}(c) = \sum_{a (\mathrm{mod}\, c)}^{*} e\left(-\frac{ah}{c}\right) \sum_{m_1, m_2} b_{m_1} \bar{b}_{m_2} e\left(\frac{a}{c}(\ell_1 m_1 - \ell_2 m_2)\right) F(m_1, m_2)$$

and $F(m_1, m_2) = g(m_1)\bar{g}(m_2)\Delta_c(\ell_1 m_1 - \ell_2 m_2 - h)$. Note that (with $K = N^{1/2} = 2(LM)^{1/2}$)

$$F^{(i,j)} \ll K^{-1}(cM/K)^{-i-j} \quad \text{for } 0 \leqq i, j \leqq 2\,. \tag{3.2}$$

Next we shall transform the sum of the Fourier coefficients $b_{m_1}\bar{b}_{m_2}$ by the following Poisson-type formula (cf. [D-I$_1$, p. 792]).

Lemma 1 *Let F be a smooth and compactly supported function on $\mathbb{R}^+$. For any integers $c \geqq 1$ and $(a, c) = 1$ we have*

$$\sum_m b_m e\left(\frac{am}{c}\right) F(m) = \sum_r b_r e\left(-\frac{\bar{a}r}{c}\right)\check{F}(r)\,, \tag{3.3}$$

where $a\bar{a} \equiv 1 (\operatorname{mod} c)$ and $\check{F}(r)$ is the Hankel-type transform

$$\check{F}(y) = 2\pi i^k c^{-1} \int_0^\infty F(x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{xy}\right) dx\,,$$

where J is the usual Bessel function.

Applying Lemma 1 in each variable m_1, m_2 we obtain

$$T_{\ell_1\ell_2}(c) = \sum_{a(\operatorname{mod} c)}^* e\left(\frac{ah}{c}\right) \sum_{r_1, r_2} b_{r_1}\bar{b}_{r_2} e\left(\frac{\overline{a\ell'}}{c_1} r_1 - \frac{\overline{a\ell''}}{c_2} r_2\right)\check{F}(r_1, r_2) \tag{3.4}$$

where $\ell' = \ell_1/(\ell_1, c)$, $\ell'' = \ell_2/(\ell_2, c)$, $c_1 = c/(\ell_1, c)$, $c_2 = c/(\ell_2, c)$ and

$$\check{F}(r_1, r_2) = \frac{4\pi^2}{c_1 c_2} \int_0^\infty \int_0^\infty F(x_1, x_2) J_{k-1}\left(\frac{4\pi}{c_1}\sqrt{x_1 r_1}\right) J_{k-1}\left(\frac{4\pi}{c_2}\sqrt{x_2 r_2}\right) dx_1 dx_2\,.$$

From the recurrence formula

$$\frac{d}{dz}(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z)$$

and the bound

$$J_\nu(z) \ll (1 + z)^{-1/2}$$

we get, on integrating by parts and combining with (3.2),

$$\check{F}(r_1, r_2) \ll \frac{M^2}{c_1 c_2 K}\left(1 + \frac{cMr_1}{c_1^2 K}\right)^{-5/4}\left(1 + \frac{cMr_2}{c_2^2 K}\right)^{-5/4}.$$

Hence, using the bound $\sum_{r \leqq x} |b_r|^2 \ll x$ (which follows from Parseval's equation), we infer

$$\sum_{r_1, r_2} b_{r_1}\bar{b}_{r_2}|\check{F}(r_1, r_2)| \ll K\,. \tag{3.5}$$

The sum over a in (3.4) is a Kloosterman sum $S(h, *; c)$ to which we apply Weil's bound. Together with (3.5) this gives

$$\begin{aligned} T_{\ell_1\ell_2}(c) &= \sum_{r_1, r_2} b_{r_1}\bar{b}_{r_2}\check{F}(r_1, r_2) S(h, *; c) \\ &\ll (h, c)^{1/2} c^{1/2} \tau(c)(LM)^{1/2}. \end{aligned}$$

Hence

$$S_{hc} \ll (h,c)^{1/2} c^{1/2} \tau(c)(LM)^{1/2} (\sum |\lambda_\ell|)^2 .$$

Next summing over $c < 2(LM)^{1/2}$ we obtain

$$S_h \ll \tau(h)(LM)^{3/4}(\log LM)(\sum |\lambda_\ell|)^2 .$$

Summing over $h \equiv 0 (\mathrm{mod}\, q)$, $0 < |h| \leqq LM$ we obtain

$$S_* \ll \tau(q)(LM)^{7/4}(\log LM)^2 (\sum |\lambda_\ell|)^2 . \tag{3.6}$$

For our choice of a_n (3.1) gives

$$S_0 \ll M(\log M) \sum_\ell |\lambda_\ell|^2 \tau(\ell) . \tag{3.7}$$

By combining (3.6) and (3.7) we complete the proof of the following mean-value theorem.

Theorem 2 *Let λ_ℓ, $1 \leqq \ell \leqq L$ be complex numbers. Let $g(m)$ be a function of C^2 class supported in $[M, 2M]$ and such that*

$$|g^{(j)}(m)| \leqq M^{-j}, \quad j = 0, 1, 2 .$$

Let $b_m m^{(k-1)/2}$ be the Fourier coefficients of a cusp form f of weight k for the modular group. Put

$$S = \sum_{\chi (\mathrm{mod}\, q)}^{*} \left| \sum_m b_m \chi(m) g(m) \right|^2 \left| \sum_\ell \lambda_\ell \chi(\ell) \right|^2 .$$

Then we have

$$S \ll \varphi(q) M(\log M) \sum_\ell |\lambda_\ell|^2 \tau(\ell) + \tau(q)(LM)^{7/4}(\log LM)^2 \left(\sum_\ell |\lambda_\ell| \right)^2 , \tag{3.8}$$

where the implied constant depends only on f.

We shall apply Theorem 2 via the following

Corollary. *Let χ be a primitive character to modulus q and $g(m)$ as above. We have*

$$B_\chi = \sum_m b_m \chi(m) g(m) \ll (q^{7/22} M^{7/11} + M^{7/8}) \tau(q)^2 \log M . \tag{3.9}$$

Proof. Choose $\lambda_l = \bar{\chi}(\ell)$. The contribution to S from χ is equal to

$$|\{\ell \leqq L : (\ell, q) = 1\}|^2 |B_\chi|^2 .$$

The contribution to S from each other character $\psi (\mathrm{mod}\, q)$ is non-negative and so we can discard it. Choosing

$$L = q^{4/11} M^{-3/11} + 2q\tau(q)\varphi(q)^{-1}$$

the corollary follows from (3.8).

Remark. Our bound (3.9) is trivial for $M < q^{7/8}$.

4 Proof of Theorem 1

We shall require the following multiplicative version of the Poisson formula (3.3).

Lemma 2 *Let F be as in Lemma 1. For any primitive character χ to modulus q we have*

$$\sum_m b_m \chi(m) F(m) = \varepsilon_\chi^2 \sum_r b_r \bar{\chi}(r) \check{F}(r) , \tag{4.1}$$

where $\varepsilon_\chi = \tau(\chi) q^{-1/2}$ as before.

Proof. We have

$$\chi(m) = \frac{1}{\tau(\bar{\chi})} \sum_{a (\mathrm{mod}\, q)} \bar{\chi}(a) e\left(\frac{am}{q}\right)$$

and

$$\frac{1}{\tau(\chi)} \sum_{a (\mathrm{mod}\, q)} \chi(a) e\left(-\frac{\bar{a}r}{q}\right) = \bar{\chi}(-r) .$$

Thus (4.1) follows from (3.3).

Now we are ready to prove Theorem 1. By using a smooth dyadic partition of unity it suffices to estimate sums of type

$$H = \sum_m b_m \chi(m) m^{-s} h(m)$$

for h a smooth function supported on [M, 2M] such that $h^{(j)}(m) \ll M^{-j}$. If $M \ll q$ we apply (3.9) with $g(m) = m^{-s} h(m)$ and get

$$H \ll |s|^2 (q^{7/22} M^{3/22} + M^{3/8}) \tau(q)^2 \log q .$$

Hence the contribution of partial sums with $M \ll q$ is

$$\mathcal{O}(|s|^2 q^{5/11} \tau(q)^2 \log q) . \tag{4.2}$$

If $M \gg q$ we first apply Lemma 2 to replace g by $\check{g}$. By partial integration $\check{g}$ satisfies the bound

$$\check{g}(r) \ll |s|^2 M^{1/2} q^{-1} (1 + q^{-2} M r)^{-5/4} .$$

Making a dyadic subdivision in r we obtain from (3.9)

$$H \ll |s|^2 (q^{7/22} R^{7/11} + R^{7/8}) M^{1/2} q^{-1} (1 + q^{-2} M R)^{-5/4} \tau(q)^2 \log q .$$

for some $R \geqq 1$. This is greatest for $R = q^2 M^{-1}$ and the resulting bound is greatest for $M = q$. Hence the contribution of partial sums with $M \gg q$ satisfies the same bound (4.2). This completes the proof of Theorem 1.

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