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Bilinear forms in the Fourier coefficients of half-integral weight cusp forms and sums over primes

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1. Introduction

Among the most fascinating arithmetic functions are the Fourier coefficients of holomorphic modular forms for a congruence group. The sequence of coefficients of an integral weight form restricted to values prime to the level may be expressed as a finite linear combination of multiplicative functions, namely eigenvalues of the Hecke operators. There is no adequate theory of Hecke operators for forms of weight half an odd integer. The Fourier coefficients of such forms along square-free numbers cannot be multiplicative, unless they are zero.

In this paper we shall give quantitative evidence for this phenomenon in the case of holomorphic cusp forms of weight $k=\frac{1}{2}+\ell$ with $\ell \geqq 2, \ell \in \mathbb{Z}$ for $\Gamma_0(N)$, $N \equiv 0 (\bmod 4)$. If

$$f(z)=\sum_{n=1}^{\infty} \hat{f}_n e(nz) \tag{1}$$

is such a form and $\hat{f}_n = n^{(k-1)/2} f_n$ then it will be shown that

$$\sum_{n \leqq X} \sum_{m \leqq Y} a_m b_n f_{mn} \ll (X^{1/2}+X^{1/4}Y)(XY)^{\varepsilon} \|a\| \, \|b\| \tag{2}$$

where (a_m) is a sequence of complex numbers supported on square-free integers and (b_n) is any sequence, the constant implied in the symbol $\ll$ depending on ε and f only. This result follows, by Cauchy's inequality, from

$$\sum_{n \leqq X} \left| \sum_{m \leqq Y} a_m f_{mn} \right|^2 \ll (X+X^{1/2}Y^2)(XY)^{\varepsilon} \|a\|^2 \tag{3}$$

and the latter will follow from the estimate

$$\sum_n f_{nr} \bar{f}_{ns} e^{-n/X} \ll \delta_{rs} X + (rsX)^{1/2+\varepsilon} \tag{4}$$

for r and s square-free integers congruent mod 4 and prime to a number depending

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on f only. In both estimates the constant implied in $\ll$ depends on ε and f only. We expect that for the sum (3) restricted to square-free n the bound should be $c(\varepsilon)(XY)^{\varepsilon}(X+Y)\|a\|^2$.

It may be assumed that $f(z)$ is a Poincaré series and so f_{ns} may be evaluated as a sum of Kloosterman sums. The sum over n is then transformed using Poisson's summation for Fourier coefficients. The resulting sums are evaluated in terms of Gauss–Ramanujan sums and finally estimated to give (4). An alternative approach to (4) would be through the Rankin–Selberg method. This leads, however, to certain technical difficulties which are avoided by the above exponential sums method. The exponential sums method seems to better reveal the nature of the Fourier coefficients of cusp forms and it allows transformations that are familiar in the theory of Hecke operators.

We shall also estimate special bilinear forms (2) in which the coefficients b_n are Dirichlet characters. In fact we shall deal with

$$b_n = \left(1 - \frac{n}{X}\right)\hat{\psi}(n), \tag{5}$$

say, where $\hat{\psi}$ is the Gauss sum of a Dirichlet character ψ to modulus $c \equiv 0 (\bmod N)$ and $X \geqq 2$. The smoothing factor $1 - nX^{-1}$ is introduced to claim nicer results. We shall prove that

$$\sum_{n \leqq X, n \equiv 0 (\bmod r)} b_n \hat{f}_n \ll c^2 r^{1/2+\varepsilon} X^{(k-1)/2} \tag{6}$$

for $(r,c)=1$ and b_n given by (5), where the constant implied in $\ll$ depends on ε and f only. Another bound will be given on average with respect to r, namely

$$\sum_{\substack{n \leqq X \\ n \equiv 0 (\bmod r)}} \sum_{\substack{r \leqq R \\ (r,c)=1}} a_r b_n \hat{f}_n \ll c^2 \|a\| R^{1/2} X^{k/2 - 1/4 + \varepsilon} \tag{7}$$

where b_n is given by (5) and a_r is any sequence of complex numbers supported on square-free numbers, the constant implied in $\ll$ depending on ε and f only.

A general bilinear inequality of type (2) and the special one of type (7) are crucial ingredients in Vinogradov's combinatorial method of treating the sum of a non-multiplicative function over primes. The method was first applied by him in 1937 to estimate

$$\sum_{p \leqq X} e(\alpha p). \tag{8}$$

Vinogradov's method was remarkably simplified by Vaughan in 1976. We shall apply one of Vaughan's identities to show that for the Fourier coefficients above and a primitive or principal character $\psi (\bmod c)$, $c \equiv 0 (\bmod N)$,

$$\sum_{p \leqq X} \psi(p) f_p \ll X^{155/156+\varepsilon}, \tag{9}$$

the implied constant depending only on c, ε and f. Such an estimate for the Fourier coefficients of an integral weight eigenform would be equivalent to the non-vanishing of the associated L-function in a strip to the left of its line of absolute convergence and is thus far beyond current methods.

Apparently, there is a similarity between our estimate (9) and that of Heath–Brown and Patterson for sums of cubic Gauss sums to prime moduli [2]. Nevertheless, their technique is quite different since for estimating the relevant bilinear forms they use a kind of twisted multiplicativity for the cubic Gauss sums. The authors thank the referee for comments which led to improvements in the exposition of this work.

2. Preliminaries

The group $SL_2(\mathbb{R})$ acts on the upper-half plane H by

$$\gamma z = \frac{az+b}{cz+d},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z = x + iy \in H$. We set $j_\gamma(z) = cz + d$, so

$$j_{\gamma_1\gamma_2}(z) = j_{\gamma_1}(\gamma_2 z) j_{\gamma_2}(z).$$

For $z \in \mathbb{C} \backslash \{0\}$ we define

$$z^k = \exp(k \log|z| + ik \arg z),$$

where $\log|z| \in \mathbb{R}$ and $\arg z \in (-\pi, \pi]$. Since the function

$$\arg j_{\gamma_1\gamma_2}(z) - \arg j_{\gamma_1}(\gamma_2 z) - \arg j_{\gamma_2}(z)$$

is constant in H we may put for $k > 0$

$$j_k(\gamma_1, \gamma_2) = j_{\gamma_1\gamma_2}(z)^{-k} j_{\gamma_1}(\gamma_2 z)^k j_{\gamma_2}(z)^k.$$

Let Γ be a discontinuous and co-finite subgroup of $SL_2(\mathbb{R})$ with respect to the invariant measure $d\mu z = y^{-2} dx dy$, and suppose Γ contains $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. A function $v: \Gamma \to \mathbb{C}$ is called a multiplier system of weight k for Γ if it satisfies the consistency conditions

$$|v(\gamma)| = 1, \quad v\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right) = e^{-\pi i k},$$

$$v(\gamma_1 \gamma_2) = v(\gamma_1) v(\gamma_2) j_k(\gamma_1, \gamma_2). \tag{10}$$

A holomorphic function $f: H \to \mathbb{C}$ is called a modular form for Γ with respect to the multiplier system v of weight $k > 0$ if

$$f(\gamma z) = v(\gamma) j_\gamma(z)^k f(z) \tag{11}$$

for all $\gamma \in \Gamma$ and $z \in H$, and f is holomorphic at each cusp. In particular, the function

$$|f(z)| y^{k/2} \tag{12}$$

is Γ-invariant. If this function is bounded in H then $f(z)$ is called a cusp form. The space $S_k(\Gamma, v)$ of holomorphic cusp forms for Γ with respect to the multiplier system v of weight k is a finite dimensional Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\Gamma\backslash H} f(z)\bar{g}(z)y^k\,d\mu z.$$

Notice that the function $f(z)\bar{g}(z)y^k$ is Γ-invariant and bounded in H.

Suppose that $\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\in\Gamma$, so a cusp form $f(z)$ has the Fourier expansion

$$f(z)=\sum_{n=1}^{\infty}\hat{f}_n e(nz). \tag{13}$$

By the boundedness of $|f(z)|y^{k/2}$ we infer

$$\sum_n |\hat{f}_n|^2 e^{-4\pi ny}=\int_0^1 |f(z)|^2\,dx \ll y^{-k}$$

for any $y>0$. Hence on taking $y=X^{-1}$ and discarding some positive terms we conclude that

$$\sum_{n\leqq X}|\hat{f}_n|^2 \ll X^k, \tag{14}$$

where the constant implied in the symbol $\ll$ depends on f only. This is a Hardy and Hecke type extimate. By a similar argument it follows that

$$\sum_{n\leqq X}\hat{f}_n e(\alpha n) \ll X^{k/2}\log X, \tag{15}$$

where the constant in $\ll$ depends on f only.

From now on we assume that $\Gamma=\Gamma_0(N)$ is the Hecke congruence group of level $N\equiv 0(\operatorname{mod} 4)$, that the weight k is half an odd positive integer and that the multiplier system is that of a theta-series, i.e.

$$v(\gamma)=\chi(d)\bar{\varepsilon}_d\left(\frac{c}{d}\right)$$

for $\gamma\in\Gamma_0(N)$, where χ is a Dirichlet character to the modulus N, $\left(\frac{c}{d}\right)$ is the extended quadratic residue symbol (see [5]) and

$$\varepsilon_d=\begin{cases}1 & \text{if } d\equiv 1(\operatorname{mod} 4),\\ i^{2k} & \text{if } d\equiv -1(\operatorname{mod} 4).\end{cases}$$

In particular we have $\left(\frac{c}{-d}\right)=\left(\frac{c}{d}\right)$ if $c,d>0$. Notice that

$$\varepsilon_{d_1d_2}=(-1)^{(1/4)(d_1-1)(d_2-1)}\varepsilon_{d_1}\varepsilon_{d_2}.$$

For any $c\equiv 0(\operatorname{mod} N)$ and $(d,c)=1$ we shall use the notation $v(c,d)=v(\gamma)$, where $\gamma=\begin{pmatrix}* & *\\ c & d\end{pmatrix}\in\Gamma_0(N)$, and $v_r(c,d)=\varepsilon_d\bar{\varepsilon}_{dr}v(c,d)$ if $2\nmid r$.

3. A functional equation

If ψ is a periodic function (mod c) we define its Fourier transform by

$$\hat{\psi}(n) = \sum_{d \pmod c} \psi(d) e\left(\frac{dn}{c}\right). \tag{16}$$

Suppose that ψ is a multiplicative character and that $c \equiv 0 \pmod N$. We shall establish a functional equation for the twisted form

$$f(z, \hat{\psi}) = \sum_n \hat{\psi}(n) \hat{f}_n e(nz). \tag{17}$$

We have

$$f(z, \hat{\psi}) = \sum_{\substack{d \pmod c \\ (c,d)=1}} \psi(-d) f\left(z - \frac{d}{c}\right).$$

Since there exists $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(N)$ by (11) we obtain

$$f\left(z - \frac{d}{c}\right) = \bar{v}(\gamma)(cz)^{-k} f\left(\frac{\bar{d}}{c} - \frac{1}{c^2 z}\right).$$

Hence changing d into $\bar{d}$ we obtain

$$\begin{aligned} f(z, \hat{\psi}) &= (cz)^{-k} \sum_{d \pmod c} \bar{\psi}(-d) \bar{v}(c, \bar{d}) f\left(\frac{d}{c} - \frac{1}{c^2 z}\right) \\ &= (cz)^{-k} \sum_n \left(\sum_{d \pmod c} \bar{\psi}(-d) \chi(d) \varepsilon_d \left(\frac{c}{d}\right) e\left(\frac{dn}{c}\right)\right) \hat{f}_n e\left(\frac{-n}{c^2 z}\right). \end{aligned}$$

Letting

$$\rho_c(d) = \bar{\psi}(-d) \chi(d) \varepsilon_d \left(\frac{c}{d}\right) \tag{18}$$

we conclude the following functional equation

$$f(z, \hat{\psi}) = (cz)^{-k} f\left(\frac{-1}{c^2 z}, \hat{\rho}_c\right). \tag{19}$$

Now suppose r is a positive integer with $(c, r) = 1$. We shall derive a functional equation for

$$f_r(z, \hat{\psi}) = \sum_{n \equiv 0 \pmod r} \hat{\psi}(n) \hat{f}_n e(nz). \tag{20}$$

Let ψ_s be the principal character to the modulus s, so its Fourier transform

$$\hat{\psi}_s(n) = \sum_{\substack{d \pmod s \\ (d,s)=1}} e\left(\frac{dn}{s}\right)$$

is the Ramanujan sum. We detect the divisibility $n \equiv 0 \pmod r$ in (20) by means of

$\hat{\psi}_s$ as follows

$$\sum_{s|r} \hat{\psi}_s(n) = \sum_{d(\bmod r)} e\left(\frac{dn}{r}\right) = \begin{cases} r, & \text{if} \quad n \equiv 0(\bmod r) \\ 0, & \text{if} \quad n \not\equiv 0(\bmod r). \end{cases}$$

We also have $\hat{\psi}_s \hat{\psi} = \bar{\psi}(s)(\psi_s \psi)^\wedge$, so by (19) we conclude that

$$f_r(z, \hat{\psi}) = \frac{1}{r} \sum_{s|r} \bar{\psi}(s) f(z, \psi_s \psi)^\wedge = \frac{1}{r} \sum_{s|r} \bar{\psi}(s)(csz)^{-k} f\left(\frac{-1}{c^2 s^2 z}, \hat{\rho}_{cs}\right).$$

The Fourier transform of ρ_{cs} factors. Indeed we have

$$\begin{aligned} \hat{\rho}_{cs}(n) &= \sum_{d(\bmod cs)} \bar{\psi}(-d)\chi(d)\varepsilon_d \left(\frac{cs}{d}\right) e\left(\frac{dn}{cs}\right) \\ &= \sum_{\substack{d_1(\bmod c) \\ d_2(\bmod s)}}\sum \bar{\psi}(-d_1 s)\chi(d_1 s)\varepsilon_{d_1 s}\left(\frac{cs}{d_1 s + d_2 c}\right) e\left(\frac{d_1 n}{c} + \frac{d_2 n}{s}\right), \end{aligned}$$

and by the quadratic reciprocity law we have

$$\begin{aligned} \left(\frac{sc}{d_1 s + d_2 c}\right) &= \left(\frac{c}{d_1 s}\right)\left(\frac{d_2 c}{s}\right)(-1)^{((s-1)/2)((d_1 s - 1)/2)} \\ &= \left(\frac{c}{d_1}\right)\left(\frac{d_2}{s}\right)(-1)^{((s-1)/2)((d_1-1)/2)+(s-1)/2}, \end{aligned}$$

and

$$\varepsilon_{d_1 s}(-1)^{((s-1)/2)((d_1-1)/2)+(s-1)/2} = \varepsilon_{d_1}\varepsilon_s(-1)^{(s-1)/2} = \varepsilon_{d_1}\bar{\varepsilon}_s.$$

Therefore

$$\hat{\rho}_{cs} = \bar{\psi}(-s)\chi(s)\hat{\rho}_c\hat{\lambda}_s$$

where

$$\lambda_s(d) = \bar{\varepsilon}_s\left(\frac{d}{s}\right).$$

Notice that $\hat{\lambda}_s$ is multiplicative in s. We conclude

Theorem 1. *Let* $c \equiv 0(\bmod N), \psi$ *be a character to the modulus* c *and* $(c, r) = 1$. *We then have*

$$f_r(z, \hat{\psi}) = \frac{\psi(-1)}{r} \sum_{s|r} \chi(s)\bar{\psi}^2(s)(csz)^{-k} f\left(\frac{-1}{c^2 s^2 z}, \hat{\rho}_c \hat{\lambda}_s\right). \tag{21}$$

Corollary. *We have*

$$\sum_{\substack{n<X \\ n \equiv 0(\bmod r)}} (X-n)\hat{\psi}(n)\hat{f}_n = \psi(-1)\frac{c}{2\pi i^k r}\sum_{s|r} s\chi(s)\bar{\psi}^2(s)$$

$$\cdot \sum_m \hat{\rho}_c(m)\hat{\lambda}_s(m)\hat{f}_m\left(\frac{X}{m}\right)^{(k+1)/2} J_{k+1}\left(\frac{4\pi}{cs}\sqrt{mX}\right).$$

Proof. For $\operatorname{Re} z > 0$ we have

$$\sum_{n \equiv 0 (\bmod r)} \hat{\psi}(n) \hat{f}_n \exp(-2\pi n z) = \frac{\psi(-1)}{r} \sum_{s \mid r} \chi(s) \bar{\psi}^2(s)$$

$$\cdot \sum_m \hat{\rho}_c(m) \hat{\lambda}_s(m) \hat{f}_m (csiz)^{-k} \exp\left(\frac{-2\pi m}{c^2 s^2 z}\right).$$

Moreover, for $\sigma > 0, \alpha > 0, \beta > 0, \gamma \in \mathbb{R}$ we have

$$\frac{1}{2\pi i} \int_{(\sigma)} e^{\gamma z} z^{-2} dz = \max\{\gamma, 0\},$$

and

$$\frac{1}{2\pi i} \int_{(\sigma)} e^{\alpha z - \beta z^{-1}} z^{-\nu-1} dz = \left(\frac{\alpha}{\beta}\right)^{\nu/2} J_\nu(2\sqrt{\alpha\beta}).$$

These formulas for $\gamma = 2\pi(X-n)$, $\alpha = 2\pi X, \beta = 2\pi X, \beta = 2\pi m(cs)^{-2}$ and $\nu = k+1$ yield the assertion of the corollary.

4. Estimates for the mean-values

Theorem 2. *Let $c \equiv 0 (\bmod N)$, ψ be a character to the modulus c and $(r, c) = 1$. Then for $X \geqq 1$ we have*

$$\sum_{\substack{n \leqq X \\ n \equiv 0 (\bmod r)}} (X-n) \hat{\psi}(n) \hat{f}_n \ll c^2 (r\tau(r))^{1/2} X^{(k+1)/2},$$

where the constant implied in $\ll$ depends on f only.

Proof. It follows from the Corollary to Theorem 1 by the following estimates

$$|\hat{\rho}_c(m)| \leqq c,$$
$$|\hat{\lambda}_s(m)| \leqq s^{1/2}(s,m)^{1/2},$$
$$J_\nu(y) \ll \min\{y^\nu, y^{-1/2}\},$$

and by (14) with Cauchy's inequality.

Obviously Theorem 2 yields (6). In order to prove (7) we first establish the following

Lemma 1. *For any complex numbers α_m, β_s we have*

$$\sum_{m \leqq M} \sum_{s \leqq S}^{*} \alpha_m \beta_s \hat{\lambda}_s(m) \ll \|\alpha\| \, \|\beta\| \{(MS)^{1/2} + M^{1/4} S^{1+\varepsilon}\},$$

where $\sum^*$ means that s ranges over square-free odd numbers, the constant implied in $\ll$ depending on ε only.

Proof. We have

$$\hat{\lambda}_s(m) = \left(\frac{m}{s}\right) s^{1/2}.$$

Splitting into progressions (mod 4) we can assume that any two values of s are congruent (mod 4). Then by Cauchy's inequality the sum is bounded by

$$\begin{aligned}&\|\alpha\|\Bigg(\sum_{m\leq M}\Bigg|\sum_{s\leq S}^{*}\beta_s\left(\frac{m}{s}\right)s^{1/2}\Bigg|^2\Bigg)^{1/2}\\ &\leq\|\alpha\|\,\|\beta\|(SM)^{1/2}+\|\alpha\|\Bigg(S\sum_{s_1\neq s_2\leq S}^{*}\sum^{*}|\beta_{s_1}\beta_{s_2}|\Bigg|\sum_{m\leq M}\left(\frac{m}{s_1s_2}\right)\Bigg|\Bigg)^{1/2}\\ &\leq\|\alpha\|\,\|\beta\|\Bigg\{(SM)^{1/2}+S^{1/2}\Bigg(\sum_{n\leq S^2}{}'\,\tau(n)\Bigg|\sum_{m\leq M}\left(\frac{m}{n}\right)\Bigg|^2\Bigg)^{1/4}\Bigg\}.\end{aligned}$$

where $\sum'$ means that n ranges over integers $\equiv 1 \pmod 4$ different from a square. For this sum we apply a bound of Jutila [3], giving

$$O(MS^{2+\varepsilon}),$$

from which the assertion of Lemma 1 follows.

By the Corollary to Theorem 1 and by Lemma 1 we shall infer

Theorem 3. *For any sequence a_r of complex numbers supported on square-free integers we have*

$$\sum_{n\leq X}(X-n)\hat{\psi}(n)\hat{f}_n\sigma_n(R)\ll\|a\|c^2R^{1/2}X^{k/2+3/4+\varepsilon},$$

where

$$\sigma_n(R)=\sum_{\substack{r\leq R\\ r|n,\,(r,c)=1}}a_r$$

and the constant implied in $\ll$ depends on ε and f only.

Proof. Clearly we may assume that $R\leq X$ for the proof. The left-hand side is bounded by

$$\frac{c^2}{2\pi}\sum_{t\leq R}t^{-1}\sum_m\left(\frac{X}{m}\right)^{(k+1)/2}|\hat{f}_m|\Bigg|\sum_{s\leq R/t}a_{st}\chi(s)\bar{\psi}^2(s)\lambda_s(m)J_{k+1}\left(\frac{4\pi\sqrt{mX}}{cs}\right)\Bigg|.$$

Given t we break the summation over m and s into subintervals of type

$$m>(cX)^2,\quad s\leq R/t \tag{22}$$

and

$$M<m\leq 2M,\quad S<s\leq 2S \tag{23}$$

with $1<2M\leq(cX)^2$ and $1<2S\leq R/t$.

In the range (22) we estimate trivially by means of (14) and $J_{k+1}(y)\ll y^{-1/2}$ getting

$$\begin{aligned}&c^2\sum_{t\leq R}t^{-1}\sum_{m>(cX)^2}\left(\frac{X}{m}\right)^{(k+1)/2}|\hat{f}_m|\sum_{s\leq R/t}|a_{st}|c^{1/2}s(mX)^{-1/4}\\ &\ll c^{5/2}RX^{k/2+1/4}\Bigg(\sum_{r\leq R}|a_r|\Bigg)\sum_{m>(cX)^2}|\hat{f}_m|m^{-k/2-3/4}\end{aligned}$$

$$\ll c^2 R\left(\sum_{r \leqq R} |a_r|\right) X^{k/2-1/4} \ll \|a\| c^2 R^{3/2} X^{k/2-1/4},$$

which is satisfactory because $R \leqq X$.

In the finite subintervals we shall apply Lemma 1. This requires the variables m, s to be independent. In order to separate m from s in the argument of the Bessel function we use the integral representation

$$J_{k+1}(2y) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma\left(\frac{k+1-v}{2}\right)}{\Gamma\left(\frac{k+3+v}{2}\right)} y^v dv \tag{24}$$

valid for $y > 0$ with $0 < \sigma < k+1$. In the range (23) with $(cS)^2 X^{-1} < M < (cX)^2$ we use (24) with $\sigma = \varepsilon$, say, getting the following bound

$$\left(\frac{\sqrt{MX}}{cS}\right)^{\varepsilon} c^2 t^{-1} \left(\frac{X}{M}\right)^{(k+1)/2} \sum_{M < m \leqq 2M} |\hat{f}_m| \left|\sum_{S < s \leqq 2S} \beta_s \hat{\lambda}_s(m)\right|$$

with some β_s such that $|\beta_s| \leqq |a_{st}|$. Then by Lemma 1 and (14) we get

$$\left(\frac{\sqrt{MX}}{cS}\right)^{\varepsilon} c^2 t^{-1} \left(\frac{X}{M}\right)^{(k+1)/2} M^{k/2} \left(\sum_{S < s \leqq 2S} |a_{st}|^2\right)^{1/2} \{(MS)^{1/2} + M^{1/4} S^{1+\varepsilon}\}.$$

Similarly in the range (23) with $M < (cS)^2 X^{-1}$ we use (14) with $\sigma = k+1-\varepsilon$ getting the following bound

$$\left(\frac{\sqrt{MX}}{cS}\right)^{-\varepsilon} c^2 t^{-1} \left(\frac{X}{cS}\right)^{k+1} \sum_{M < m \leqq 2M} |\hat{f}_m| \left|\sum_{S < s \leqq 2S} \beta_s \hat{\lambda}_s(m)\right|$$

with some β_s such that $|\beta_s| \leqq |a_{st}|$. Then by Lemma 1 and (14) we get

$$\left(\frac{\sqrt{MX}}{cS}\right)^{-\varepsilon} c^2 t^{-1} \left(\frac{X}{cS}\right)^{k+1} M^{k/2} \left(\sum_{S < s \leqq 2S} |a_{st}|^2\right)^{1/2} \{(MS)^{1/2} + M^{1/4} S^{1+\varepsilon}\}.$$

Summing over M of type $M = 2^a$ from both estimates we get

$$c^2 t^{-1} \left(\sum_{S < s \leqq 2S} |a_{st}|^2\right)^{1/2} S^{1/2} X^{k/2+3/4+2\varepsilon}.$$

Next we sum over S of type $S = 2^b \leqq R/t$ and over $t \leqq R$ giving the desired bound. This completes the proof of Theorem 3.

Obviously (7) follows from Theorem 3.

5. Poisson's summation formulas

The modular relation (11) when expressed in terms of the Fourier coefficients becomes a Poisson summation type formula. Indeed by (11) for $\gamma = \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma$

and $z = -\frac{d}{c} + i(c\zeta)^{-1}$ with $\operatorname{Re}\zeta > 0$ we obtain

$$\sum_n \hat{f}_n e\left(\frac{a}{c}n\right)\exp\left(-2\pi\zeta\frac{n}{c}\right) = v(\gamma)\left(\frac{i}{\zeta}\right)^k \sum_n \hat{f}_n e\left(-\frac{d}{c}n\right)\exp\left(-\frac{2\pi n}{\zeta c}\right).$$

An extension of this formula will result from application of the Laplace inversion formula

$$p(x) = \frac{1}{2\pi i}\int_{(\sigma)} \exp(-x\zeta)q(\zeta)d\zeta,$$

$$q(\zeta) = \int_0^\infty \exp(x\zeta)p(x)dx$$

valid for any $p(x)$ of class C^2 and compactly supported in $\mathbb{R}^+$. We obtain

$$\sum_n \hat{f}_n e\left(\frac{a}{c}n\right)p\left(\frac{2\pi n}{c}\right) = i^k v(\gamma)\sum_n \hat{f}_n e\left(-\frac{d}{c}n\right)r\left(\frac{2\pi n}{c}\right),$$

where

$$\begin{aligned} r(y) &= \frac{1}{2\pi i}\int_{(\sigma)} \exp(-y\zeta^{-1})\zeta^{-k}q(\zeta)d\zeta \\ &= \int_0^\infty p(x)\frac{1}{2\pi i}\int_{(\sigma)} \exp(x\zeta - y\zeta^{-1})\zeta^{-k}d\zeta \\ &= \int_0^\infty p(x)\left(\frac{x}{y}\right)^{(k-1)/2} J_{k-1}(2\sqrt{xy})dx. \end{aligned}$$

Setting $G(x) = p(x)x^{(k-1)/2}$ we obtain

$$\sum_n f_n e\left(\frac{a}{c}n\right)G\left(\frac{2\pi n}{c}\right) = i^k v(\gamma)\sum_n f_n e\left(-\frac{d}{c}n\right)H\left(\frac{2\pi n}{c}\right), \tag{25}$$

where

$$H(y) = \int_0^\infty G(x)J_{k-1}(2\sqrt{xy})dx. \tag{26}$$

Here the assumption that $G(x)$ is compactly supported can be replaced by a weaker one that $G(x)$ is of C^2 class in $\mathbb{R}^+$ such that

$$|G^{(j)}(x)| \leqq (1 + x^2)^{-1} \quad \text{for} \quad j = 0, 1, 2 \tag{27}$$

by a suitable approximation and by the estimate

$$\sum_{n \leqq X} |f_n|^2 \ll X, \tag{28}$$

which follows from (14) by partial summation.

The formula (25) holds true for any co-finite discontinuous group with $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ in it. If $\Gamma = \Gamma_0(N)$ we shall generalize the result as follows.

Theorem 4. *Let $c \equiv 0 \pmod N$, $(c,d)=1$ and r be square-free with $(r,N)=1$. We then have*

$$\sum_n f_{rn} e\left(\frac{d}{c}n\right) G\left(\frac{2\pi n}{c}\right)$$
$$= i^k v_r(c,\bar{d}) \sum_{r=qvw,\, q|c} \bar{\tilde{\varepsilon}}_v \varepsilon_{qv} \chi(w) \left(\frac{dw}{q}\right)\left(\frac{c}{vw}\right) v^{-1/2} \sum_n f_{qn} \left(\frac{n}{v}\right) e\left(\frac{-\overline{dvwn}}{c}\right) H\left(\frac{2\pi nw}{cv}\right), \tag{29}$$

where $\tilde{\varepsilon}_v$ is the sign of the Gauss sum, i.e.

$$\tilde{\varepsilon}_v = \begin{cases} 1 & \text{if } v = 1 \pmod 4 \\ i & \text{if } v \equiv -1 \pmod 4. \end{cases} \tag{30}$$

Proof. The sum on the left-hand side is equal to

$$\mathscr{L} = \frac{1}{r} \sum_{\substack{\alpha (\mathrm{mod}\, cr) \\ \alpha \equiv d (\mathrm{mod}\, c)}} \sum_n f_n e\left(\frac{\alpha n}{cr}\right) G\left(\frac{2\pi n}{cr}\right).$$

The fraction α/cr reduces to β/cv with $(\beta, cv)=1$, $r=vw$, $(c,w)=1$, $\beta w \equiv d \pmod c$ and the innermost sum transforms by (25) into

$$w i^k v(cv, \bar{\beta}) \sum_n f_n e\left(\frac{-\bar{\beta}n}{cv}\right) H\left(\frac{2\pi n w}{cv}\right).$$

Hence

$$\mathscr{L} = i^k \sum_{\substack{r=vw \\ (c,w)=1}} v^{-1} \sum_{\substack{\beta (\mathrm{mod}\, cv) \\ \beta w \equiv d (\mathrm{mod}\, c)}} v(cv, \bar{\beta}) \sum_n f_n e\left(\frac{-\bar{\beta}n}{cv}\right) H\left(\frac{2\pi nw}{cv}\right).$$

Here the multiplier is equal to

$$v(cv,\bar{\beta}) = \chi(\bar{\beta})\bar{\varepsilon}_\beta \left(\frac{cv}{\beta}\right) = \chi(\bar{d})\chi(w)\bar{\varepsilon}_{dw}\left(\frac{c}{dw}\right)\left(\frac{v}{\beta}\right)$$
$$= v_r(c,\bar{d})\chi(w)\left(\frac{c}{w}\right)\left(\frac{\beta}{v}\right)\varepsilon_{dr}\bar{\varepsilon}_{dw}(-1)^{((dw-1)/2)(v-1)/2}$$

by the quadratic reciprocity law. Furthermore we have

$$\varepsilon_{dr}\bar{\varepsilon}_{dw}(-1)^{((dw-1)/2)(v-1)/2} = \varepsilon_v.$$

Hence we obtain

$$\mathscr{L} = i^k v_r(c,\bar{d}) \sum_{r=vw} v^{-1}\varepsilon_v \chi(w)\left(\frac{c}{w}\right)\sum_n f_n H\left(\frac{2\pi nw}{cv}\right) g_n(c,d;v,w)$$

with

$$g_n(c,d;v,w) = \sum_{\substack{\beta (\mathrm{mod}\, cv) \\ \beta d \equiv w (\mathrm{mod}\, c)}} \left(\frac{\beta}{v}\right) e\left(\frac{-\bar{\beta}n}{cv}\right).$$

The latter sum vanishes unless $(c,v)|n$. Letting $v=qv_1$, with $q|c$, $(v_1,c)=1$ and $n=qn_1$ we obtain

$$g_n(c,d;v,w)=q\left(\frac{dw}{q}\right)\sum_{\substack{\beta(\bmod cv_1)\\ \beta d\equiv w(\bmod c)}}\left(\frac{\beta}{v_1}\right)e\left(-\frac{\beta n_1}{cv_1}\right)$$
$$=q\left(\frac{dw}{q}\right)\left(\frac{cn_1}{v_1}\right)\bar{\varepsilon}_{v_1}v_1^{1/2}e\left(-\frac{\overline{dv_1}wn_1}{c}\right)$$

from which (29) follows.

6. Poincaré series and Kloosterman sums

From now on let $k>2$. The Poincaré series defined by

$$P_m(z;\Gamma,\nu)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\bar{\nu}(\gamma)j_\gamma(z)^{-k}e(m\gamma z),$$

where m is a positive integer, is a cusp form of weight k, and it has the Fourier expansion

$$P_m(z;\Gamma,\nu)=\sum_{n=1}^{\infty}(n/m)^{(k-1)/2}p_n(m)e(nz).$$

say, with the coefficients $p_n(m)$ given by (see [4]).

$$p_n(m)=\delta_{mn}+2\pi i^k\sum_{c>0}c^{-1}S(m,n;c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),\tag{31}$$

where $S(m,n;c)$ is the Kloosterman sum to the modulus c defined by

$$S(m,n;c)=\sum_{\left(\begin{smallmatrix}a & *\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty}\bar{\nu}(\gamma)e\left(\frac{ma+nd}{c}\right).\tag{32}$$

In the forthcoming arguments we shall also use the following sums

$$S_\kappa(m,n;c)=\sum_{d(\bmod c)}\bar{\nu}_\kappa(c,d)e\left(\frac{m\bar{d}+nd}{c}\right).\tag{33}$$

Lemma 2. *If $p\nmid mN$, $p|n$ and $p^2|c$ then $S(m,n;c)=0$.*

Proof. We have

$$S(m,n;c)=\sum_{d(\bmod c)}\bar{\nu}(c,d)e\left(\frac{m\bar{d}+nd}{c}\right)$$
$$=\sum_{\substack{d_1(\bmod c/p)\\ d_2(\bmod p)}}\sum\bar{\nu}\left(c,d_1+\frac{c}{p}d_2\right)e\left(\frac{m\bar{d}_1(1-c\bar{d}_1d_2/p)+n(d_1+cd_2/p)}{c}\right)$$
$$=\sum_{d_1(\bmod c/p)}\bar{\nu}(c,d_1)e\left(\frac{m\bar{d}_1+nd_1}{c}\right)\sum_{d_2(\bmod p)}e\left(\frac{-m\bar{d}_1^2d_2}{p}\right)=0.$$

Lemma 3. *If* $(s,mN)=1$ *then* $S(m,ns;c)=0$ *unless*

$$s=tu,\ t\quad \textit{square-free} \tag{34}$$

and

$$c=tc_1,\quad (s,c_1)=1 \tag{35}$$

in which case we have

$$S(m,ns;c)=\tilde{\varepsilon}_t\bar{\varepsilon}_t t^{1/2}\chi(u)\left(\frac{m}{t}\right)\left(\frac{c_1}{s}\right)S_s(mu\bar{t},n;c_1). \tag{36}$$

Proof. If c is not of type (35) then $S(m,ns;c)$ vanishes by Lemma 2. Thus we assume that (34) and (35) hold. We write

$$d\equiv d_1\bar{t}t+d_2\bar{c}_1c_1(\operatorname{mod} c_1t)$$

with d_1,d_2 ranging over the reduced residue classes modulo c_1,t respectively. We obtain

$$\bar{d}\equiv\overline{d_1t}t+\overline{d_2c_1}c_1(\operatorname{mod} c_1t),$$

$$\nu(c,d)=\chi(d_1)\bar{\varepsilon}_{d_1}\left(\frac{c_1}{d_1}\right)\left(\frac{t}{d}\right),$$

$$\left(\frac{t}{d}\right)=(-1)^{((t-1)/2)((d-1)/2)}\left(\frac{d}{t}\right)=(-1)^{((t-1)/2)((d_1-1)/2)}\left(\frac{d_2}{t}\right),$$

$$e\left(\frac{m\bar{d}+nsd}{c}\right)=e\left(\frac{m\overline{d_1t}+nud_1}{c_1}\right)e\left(m\frac{\overline{d_2c_1}}{t}\right),$$

$$\sum_{d_2(\operatorname{mod} t)}\left(\frac{d_2}{t}\right)e\left(m\frac{\overline{d_2c_1}}{t}\right)=\left(\frac{mc_1}{t}\right)\tilde{\varepsilon}_t t^{1/2},$$

$$S(m,ns;c)=\left(\frac{m}{t}\right)\tilde{\varepsilon}_t t^{1/2}\sum_{d_1(\operatorname{mod} c_1)}\bar{\chi}(d_1)\varepsilon_{d_1}(-1)^{((t-1)/2)((d_1-1)/2)}\left(\frac{c_1}{td_1}\right)e\left(\frac{m\overline{d_1t}+nud_1}{c_1}\right).$$

Change $d_1\to\bar{u}d_1$ and use

$$\varepsilon_{ud_1}(-1)^{((t-1)/2)((ud_1-1)/2)}=\varepsilon_{sd_1}\bar{\varepsilon}_t,$$

completing the proof.

7. Convolution series

Our objective is to evaluate the series

$$\mathscr{L}_{rs}(f\otimes g;\omega)=\sum_1^{\infty}f_{rn}\bar{g}_{sn}\omega(n), \tag{37}$$

where $f,g\in S_k(\Gamma,\nu),\omega(x)$ is a suitable test function and r,s are positive integers. Traditionally the above series is investigated by the Rankin–Selberg method. In this paper we give another approach based on the formulas (29), (31) and (36).

Since the space $S_k(\Gamma,\nu)$ is spanned by a finite number of Poincaré series we can assume without loss of generality that

$$g(z)=m^{(k-1)/2}P_m(z;\Gamma,\nu) \tag{38}$$

for some $m \geqq 1$, in which case we have (by (31))

$$g_{sn} = \delta_{m,ns} + 2\pi i^k \sum_{c \equiv 0 (\mathrm{mod}\, N)} c^{-1} S(m, ns; c) J_{k-1}\left(\frac{4\pi\sqrt{mns}}{c}\right). \tag{39}$$

For technical simplicity we assume that $(rs, mN) = 1$, $r \equiv s (\mathrm{mod}\, 4)$ and r, s are square-free. We then get by Lemma 3

$$g_{sn} = \delta_{m,ns} + 2\pi i^k \sum_{s=tu} \tilde{\varepsilon}_t \bar{\varepsilon}_t \chi(u) \left(\frac{m}{t}\right) t^{-1/2} \sum_{c \equiv 0 (\mathrm{mod}\, N)} c^{-1} \left(\frac{c}{s}\right) S_s(mu\bar{t}, n; c) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mnu}{t}}\right). \tag{40}$$

Hence

$$\mathscr{L}_{rs}(f \otimes g; \omega) = \omega\left(\frac{m}{s}\right) f_{rm/s} + 2\pi i^{-k} \sum_{s=tu} \bar{\tilde{\varepsilon}}_t \varepsilon_t \bar{\chi}(u) \left(\frac{m}{t}\right) t^{-1/2} \sum_{c \equiv 0 (\mathrm{mod}\, N)} c^{-1} \left(\frac{c}{s}\right) \mathscr{P}_c(u/t), \tag{41}$$

where

$$\mathscr{P}_c(u/t) = \sum_n f_{rn} \bar{S}_s(mu\bar{t}, n; c) G\left(\frac{2\pi n}{c}\right) \tag{42}$$

with

$$G(x) = \omega\left(\frac{cx}{2\pi}\right) J_{k-1}\left(2\sqrt{\frac{2\pi mux}{ct}}\right). \tag{43}$$

Here $f_{rm/s}$ is defined to be 0 if $s \nmid rm$. Then by (29) and (33) we infer that

$$\mathscr{P}_c(u/t) = i^{-k} \chi(-1) \sum_{\substack{r = qvw \\ q \mid c}} \bar{\tilde{\varepsilon}}_v \varepsilon_{qv} \chi(w) \left(\frac{w}{q}\right) \left(\frac{c}{vw}\right) v^{-1/2} \sum_n f_{qn} \left(\frac{n}{v}\right) \mathscr{R}_c(htv, q) H\left(\frac{2\pi nw}{cv}\right), \tag{44}$$

where

$$h = muv - nwt \tag{45}$$

and $\mathscr{R}_c(htv, q)$ is a Gauss–Ramanujan sum defined by

$$\begin{aligned}
\mathscr{R}_c(htv, q) &= i^{2k} \chi(-1) \sum_{d (\mathrm{mod}\, c)} v_r(c, \bar{d}) v_s(c, -d) \left(\frac{d}{q}\right) e\left(h\frac{\overline{dtv}}{c}\right) \\
&= \left(\frac{tv}{q}\right) \sum_{\substack{d (\mathrm{mod}\, c) \\ (c,d)=1}} \left(\frac{d}{q}\right) e\left(\frac{dh}{c}\right) \\
&= \left(\frac{tv}{q}\right) \sum_{c = qk\ell} \mu(\ell) \left(\frac{\ell}{q}\right) \sum_{d (\mathrm{mod}\, kq)} \left(\frac{d}{q}\right) e\left(\frac{dh}{kq}\right) \\
&= \left(\frac{tv}{q}\right) \sum_{\substack{c = qk\ell \\ k \mid h}} k\mu(\ell) \left(\frac{\ell}{q}\right) \sum_{d (\mathrm{mod}\, q)} \left(\frac{d}{q}\right) e\left(\frac{dh/k}{q}\right) \\
&= \tilde{\varepsilon}_q \left(\frac{tv}{q}\right) q^{1/2} \sum_{\substack{c = qkl \\ k \mid h}} k\mu(\ell) \left(\frac{\ell h/k}{q}\right).
\end{aligned}$$

We set

$$r_c(h,q)=\sum_{\substack{c=k\ell\\ k|h}} k\mu(\ell)\left(\frac{\ell h/k}{q}\right). \tag{46}$$

Next we express $H\left(\frac{2\pi nw}{cv}\right)$ in (44) in terms of the original test function $\omega(x)$. We recall that $H(y)$ is the transform (26) of $G(x)$ given by (43). Thus we have

$$H\left(\frac{2\pi nw}{cv}\right)=\left(\frac{tv}{mnuw}\right)^{1/4} j_\omega\left(\frac{2\pi}{c}\sqrt{\frac{mu}{t}},\frac{2\pi}{c}\sqrt{\frac{nw}{v}}\right),$$

where

$$j_\omega(A,B)=\sqrt{AB}\int\omega(x)J_{k-1}(2A\sqrt{x})J_{k-1}(2B\sqrt{x})dx. \tag{47}$$

Gathering together the above evaluations we conclude

Theorem 5. *Let* $N\equiv 0 \pmod 4$, $m\geqq 1$, $(rs,mN)=1, r\equiv s \pmod 4$, and r,s be square-free. Let $\omega(x)$ be a function of C^2 class on $\mathbb{R}^+$ satisfying (27). We then have

$$\mathscr{L}_{rs}(f\otimes g,\omega)=\omega\left(\frac{m}{s}\right)f_{rm/s}+2\pi i^{-2k}\sum_{\substack{s=tu\\ r=vwq}}\bar{\chi}(u)\chi(-w)\left(\frac{m}{t}\right)\left(\frac{u}{q}\right)\varepsilon(t,v,q)$$

$$\sum_{c\equiv 0(\mathrm{mod}\,N)} c^{-1}\left(\frac{c}{sr/q}\right)\sum_n f_{qn}\left(\frac{n}{v}\right)r_c(h,q)j_\omega\left(\frac{2\pi}{cq}\sqrt{\frac{mu}{t}},\frac{2\pi}{cq}\sqrt{\frac{nw}{v}}\right)(mnrsq)^{-1/4}, \tag{48}$$

where

$$\varepsilon(t,v,q)=\bar{\bar{\varepsilon}}_t\bar{\bar{\varepsilon}}_v\bar{\bar{\varepsilon}}_q\varepsilon_t\varepsilon_{qv}=\pm 1,$$

h is given by (45), $r_c(h,q)$ is given by (46) and $j_\omega(A,B)$ is the transform of $\omega(x)$ given by (47).

The result simplifies a bit if $r=s=1$. We obtain

Corollary. *If $r=s=1$ then we have*

$$\mathscr{L}(f\otimes g,\omega)=\omega(m)f_m+2\pi i^{-2k}\sum_{k\ell\equiv 0(\mathrm{mod}\,N)}\mu(\ell)\ell^{-1}\sum_{n\equiv m(\mathrm{mod}\,k)} f_n j_\omega\left(\frac{2\pi\sqrt{m}}{k\ell},\frac{2\pi\sqrt{n}}{k\ell}\right)(mn)^{-1/4}. \tag{49}$$

6. Estimation of $\mathscr{L}^0_{rs}(f\otimes g,\omega)$

Let $\mathscr{L}^0_{rs}(f\otimes g,\omega)$ stand for the partial sum of the right hand side of (48) restricted by

$$h=muv-nwt=0. \tag{50}$$

We have $r_c(0,q)=0$ unless $q=1$ in which case $r_c(0,1)=\varphi(c)$. By (50) we get $t=v$ and $w|u$, so $r|s$. Assuming that $s\leqq r$ (which can be done without loss of generality) we infer from (50) that $r=s$ and $m=n$ giving

$$\mathscr{L}^0_{rr}(f\otimes g;\omega)=2\pi i^{-2k}\chi(-1)\frac{f_m}{\sqrt{mr}}\sum_{v|r}\sum_{\substack{c\equiv 0(\bmod N)\\(c,r)=1}}\varphi(c)c^{-1}j_\omega\left(\frac{2\pi}{cv}\sqrt{mr}\right),$$

where

$$j_\omega(A)=j_\omega(A,A)=A^{-1}\int\omega(xA^{-2})J^2_{k-1}(2\sqrt{x})dx.$$

Thus

$$\mathscr{L}^0_{rr}(f\otimes g,\omega)=i^{-2k}\chi(-1)\frac{f_m}{mr}\int J^2_{k-1}(2\sqrt{x})K(x)dx,$$

where

$$K(x)=\sum_{v|r}v\sum_{\substack{c\equiv 0(\bmod N)\\(c,r)=1}}\varphi(c)\omega\left(\frac{c^2v^2x}{4\pi^2mr}\right).$$

Let $\hat{\omega}(z)$ be the Mellin transform of $\omega(x)$, so

$$\omega(x)=\frac{1}{2\pi i}\int_{(\sigma)}\hat{\omega}(z)x^{-z}dz,\quad \sigma>1$$

and $K(x)$ is the Mellin inverse transform of

$$\begin{aligned}\hat{K}(z)&=\sum_{v|r}v\sum_{\substack{c\equiv 0(\bmod N)\\(c,r)\equiv 1}}\varphi(c)\left(\frac{c^2v^2}{4\pi^2mr}\right)^{-z}\hat{\omega}(z)\\&=\varphi(N)\left(\frac{4\pi^2mr}{N^2}\right)^z\sum_{v|r}v^{1-2z}\sum_{(c,r)=1}\frac{\varphi(cN)}{\varphi(N)}c^{-2z}\hat{\omega}(z)\\&=\varphi(N)\left(\frac{2\pi}{N}\right)^{2z}(mr)^z\prod_{p|r}(1-p^{2-4z})\prod_{p|rN}(1-p^{-2z})^{-1}\frac{\zeta(2z-1)}{\zeta(2z)}\hat{\omega}(z).\end{aligned}$$

From the above expression it is plainly seen that $\hat{K}(z)$ is meromorphic in $\operatorname{Re}z\geqq\frac{1}{2}$ with only a simple pole at $z=1$ with residue

$$\operatorname*{res}_{z=1}\hat{K}(z)=12\left(N\prod_{p|r}\left(1+\frac{1}{p}\right)\right)^{-1}\hat{\omega}(1)mr=Kmr,\tag{51}$$

say. The point $z=\frac{1}{2}$ is a zero of $\hat{K}(z)$. On the line $\operatorname{Re}z=\frac{1}{2}$ we have

$$\hat{K}(z)\ll(mr)^{1/2}\tau(r)\sum_{p|r}\left(1+\frac{1}{p}\right)|z-\tfrac{1}{2}||z|^{-1/2}|\hat{\omega}(z)|,\tag{52}$$

where the constant implied in $\ll$ is absolute.

From the above evaluations we obtain

$$\begin{aligned}\mathscr{L}^0_{rr}(f\otimes g,\omega)&=i^{-2k}\chi(-1)\frac{f_m}{mr}\frac{1}{2\pi i}\int_{(2)}\hat{J}(1-z)\hat{K}(z)dz\\&=i^{-2k}\chi(-1)\frac{f_m}{mr}\frac{1}{2\pi i}\int_{(1/2)}\hat{J}(1-z)\hat{K}(z)dz+i^{-2k}\chi(-1)f_mK\hat{J}(0),\end{aligned}$$

where $\hat{J}(z)$ is the Mellin transform of $J_{k-1}^2(2\sqrt{x})$,

$$\hat{J}(z) = \Gamma(1-2z)\Gamma(k-1+z)/\Gamma(k-z)\Gamma^2(1-z).$$

Hence we conclude that

$$\mathscr{L}_{rr}^0(f\otimes g,\omega) = \frac{12}{k-1} i^{-2k}\chi(-1) f_m \hat{\omega}(1)\left(N\prod_{p|N}\left(1+\frac{1}{p}\right)\right)^{-1}$$
$$+ i^{-2k}\chi(-1)\frac{f_m}{mr}\frac{1}{2\pi i}\int_{(1/2)}\frac{\Gamma(2z-1)\Gamma(k-z)}{\Gamma^2(z)\Gamma(k+z-1)}\hat{K}(z)dz.$$

Now we specify the test function $\omega(x)$ to be

$$\omega(x) = \exp(-xX^{-1}) \tag{54}$$

with $X \geqq 2$. Then

$$\hat{\omega}(z) = \Gamma(z)X^z,$$

so $\hat{\omega}(1) = X$. By (52) and (53) we finally get

Lemma 4. *We have*

$$\mathscr{L}_{rr}^0(f\otimes g,\omega) = \frac{12X(k-1)^{-1}}{[\Gamma_0(1){:}\Gamma_0(N)]} i^{-2k}\chi(-1)f_m + O\left(|f_m|\left(\frac{X}{mr}\right)^{1/2}\tau(r)\prod_{p|r}\left(1+\frac{1}{p}\right)\right), \tag{55}$$

where the constant implied in O *is absolute.*

7. Estimation of $\mathscr{L}_{rs}^*(f\otimes g,\omega)$

Let $\mathscr{L}_{rs}^*(f\otimes g,\omega)$ stand for the partial sum of the right-hand side of (48) restricted by $h = muv - nwt \neq 0$ and let $\omega(x)$ be given by (54). We then have

$$j_\omega(A,B) = \sqrt{AB}XI_{k-1}(2ABX)\exp(-(A^2+B^2)X),$$

where $I_\nu(z)$ is the Bessel function (see [1, p. 51]). Since

$$I_\nu(z) \ll \min\{z^\nu, z^{-1/2}\}e^z$$

we obtain for $k \geqq \frac{5}{2}$ that

$$j_\omega(A,B) \ll X^{1/2}\min\{1,(ABX)^2\}\exp(-(A-B)^2X)$$
$$\ll X^{1/2}\min\left\{\left(\frac{A}{B}\right)^2,\left(\frac{B}{A}\right)^2,(ABX)^2\right\}.$$

In particular for $A = \frac{2\pi}{c}\sqrt{\frac{mu}{t}}$ and $B = \frac{2\pi}{c}\sqrt{\frac{nw}{v}}$ we get

$$j_\omega(A,B) \ll X^{1/2}\min\left\{\frac{muv}{nwt},\ \frac{nwt}{muv},\ \frac{mnuw}{c^4tv}X^2\right\},$$

and

$$|\mathscr{L}_{rs}^*(f\otimes g,\omega)| \leqq 2\pi \sum_{\substack{s=tu\\ r=vwq}} q^{-1/4} \sum_{\substack{c=k\ell\equiv 0(\mathrm{mod}\,N)\\ k|h,h\neq 0}} \ell^{-1}\sum |f_{qn}|(mnrs)^{-1/4}|j_\omega|$$

$$\ll X^{1/2}(mrs)^{-1/4} \sum_{\substack{s=tu\\ r=vwq}} q^{-1/4} \sum_{\substack{n\\ h\neq 0}} |f_{qn}| n^{-1/4} \tau(|h|) \min\left\{\frac{muv}{nwt}, \frac{nwt}{muv}\right\}$$

$$\cdot \log(mnrsX)$$

$$\ll X^{1/2}(mrs)^{-1/4} \sum_{\substack{s=tu\\ r=vwq}} q^{-1/4} \sum_{n} |f_{qn}| n^{-1/4} \min\left\{\frac{muv}{nwt}, \frac{nwt}{muv}\right\} (mnrsX)^{\varepsilon}$$

$$\ll (mrsX)^{\varepsilon}(mrs)^{-1/4} X^{1/2} \sum_{\substack{s=tu\\ r=vwq}} q^{1/4}\left(\frac{muv}{wt}\right)^{3/4},$$

by (28) and Cauchy's inequality. Hence we conclude

Lemma 5. *We have*

$$\mathscr{L}^*_{rs}(f \otimes g, \omega) \ll (mrsX)^{1/2+\varepsilon}, \tag{56}$$

where the constant implied in $\ll$ depends on ε and f only.

Combining Lemmas 4 and 5 one obtains (4). Clearly (4) implies (3) for any sequence (a_m) supported on square-free integers prime to the level and of given residue class mod 4. The last two conditions can be released by standard arguments.

8. Sums over primes

Now we are ready to prove (9). We do not attempt to get the strongest result but rather to give the simplest argument within the available estimates (2) and (7). In fact it will be convenient to modify (2) as follows.

$$\sum_{\substack{mn \leq X\\ M<m\leq 2M}}\sum a_m b_n \hat{f}_{mn} \ll \left(\sum_{mn\leq 2X} |a_m b_n|^2\right)^{1/2} (X^{1/2}M^{-1/2} + X^{1/4}M^{3/4}) X^{(k-1)/2+\varepsilon}, \tag{57}$$

where (a_m) is a sequence of complex numbers supported on square-free integers, (b_n) is any sequence of complex numbers and the constant implied in $\ll$ depends on ε and f only. The proof of (57) follows easily from (3) by Perron's formula.

As regards (7) the result has convenient shape for applications. However, a new sum needs to be estimated in which the coefficient b_n is replaced by $b_n \log n$. Instead of generalizing (7) to cover this case we state rather an independent inequality (which follows immediately from (15)).

$$\sum_{n\leq X, n\equiv 0 (\mathrm{mod}\, r)} \hat{\psi}(n)\hat{f}_n \ll c X^{k/2} \log X, \tag{58}$$

where the constant implied in $\ll$ depends on f only. This result is weaker than (6) but it holds for all r and is free of the smooth factor $1-nX^{-1}$. The latter enables one to use partial summation in X or the formula

$$\log\frac{n}{r} = \int_r^n t^{-1}\,dt$$

giving

$$\sum_{\substack{n\leq X\\ n\equiv 0 (\mathrm{mod}\, r)}} \hat{\psi}(n)\hat{f}_n \log\frac{n}{r} \ll c X^{k/2} (\log X)^2. \tag{59}$$

Now let us consider the sum

$$P(X)=\sum_{n\leqq X} b_n\hat{f}_n\Lambda(n), \tag{60}$$

where b_n is given by (5) and $\Lambda(n)$ is the Mangoldt function. We appeal to a combinatorial partition of $\Lambda(n)$ à la Vaughan [6],

$$\Lambda(n)=\sum_{\substack{r\mid n\\ r\leqq R}}\mu(r)\log\frac{n}{r}-\sum_{\substack{\ell m\mid n\\ \ell\leqq Q, m\leqq R}}\mu(m)\Lambda(\ell) \tag{61}$$

valid for n with $Q<n\leqq QR=X$, say. We split the second sum into $\leqq(2\log X)^2$ sums over dyadic intervals $L<\ell\leqq 2L$, $M<m\leqq 2M$ with $2L\leqq Q$ and $2M\leqq R$, and we write accordingly

$$\Lambda(n)=\Lambda_R(n)-\sum_L\sum_M\Lambda_{LM}(n)$$

and

$$P(X)=P_R(X)-\sum_L\sum_M P_{LM}(X)+O(Q^{(k+1)/2}X^{\varepsilon})$$

where

$$P_R=\sum_{n\leqq X} b_n\hat{f}_n\Lambda_R(n),$$

$$P_{LM}(X)=\sum_{n\leqq X} b_n\hat{f}_n\Lambda_{LM}(n),$$

and the error term above takes care of terms $n\leqq Q$ on both sides which are not covered by the identity (61). This error term is obtained by an application of (14) through Cauchy's inequality.

For $P_R(X)$ we apply (59) getting

$$P_R(X)\ll RX^{k/2}(\log X)^2.$$

For $P_{LM}(X)$ we shall give two bounds. To get the first bound we intend to apply (7) with $r=lm$. This require lm to be square-free, so we split $P_{LM}(X)=P'_{LM}(X)+P''_{LM}(X)$, where $P'_{LM}(X)$ meets this requirement. We obtain

$$P'_{LM}(X)\ll LMX^{(k/2)-1/4+\varepsilon}.$$

For estimating $P''_{LM}(X)$ we use (6) giving

$$P''_{LM}(X)\ll\sum_{\substack{L<\ell\leqq 2L\\ M<m\leqq 2M\\ \mu(\ell m)=0}}\sum|\mu(m)|\Lambda(\ell)(\ell m)^{1/2}X^{(k-1)/2+\varepsilon}$$
$$\ll LM^{3/2}X^{(k-1)/2+\varepsilon}.$$

Combining both estimates we get our first bound

$$P_{LM}(X)\ll(LMX^{-3/4}+LM^{3/2}X^{-1})X^{(k+1)/2+\varepsilon}.$$

For the second bound we appeal to (57) with $a_m=\mu(m)$ giving

$$P_{LM}(X)\ll(M^{-1/2}+M^{3/4}X^{-1/4})X^{(k+1)/2+\varepsilon}.$$

From the above results we conclude that

$$P(X) \ll (RX^{-1/2} + QMX^{-3/4} + QM^{3/2}X^{-1} + M^{-1/2} + R^{3/4}X^{-1/4})X^{(k+1)/2+\varepsilon}$$

for any M, Q, R subject to $1 \leqq M \leqq R = Q^{-1}X$. We choose

$$M = X^{1/26}, \quad Q = X^{9/13}, \quad R = X^{4/13}$$

getting

$$P(X) \ll X^{(k+1)/2-(1/52)+\varepsilon}. \tag{62}$$

Finally, to prove (9) it remains to get rid of the smoothing factor $1 - nX^{-1}$. This is performed by means of a general inequality

$$|S|^3 \leqq 8FG,$$

where

$$S = \sum_{n \leqq X} a_n, \quad F = \max_{x \leqq X} \Big| \sum_{n \leqq x} (x-n) a_n \Big|, \quad G = \sum_{n \leqq X} |a_n|^2.$$

We obtain

$$\sum_{n \leqq X} \hat{\psi}(n) \hat{f}_n \Lambda(n) \ll X^{(k+1)/2-(1/156)+\varepsilon}$$

and by partial summation

$$\sum_{p \leqq X} \hat{\psi}(p) f_p \ll X^{(155/156)+\varepsilon}.$$

Here $\hat{\psi}(p) = \bar{\psi}(p)\hat{\psi}(1)$ if $p|c$, so we get (9) provided $\hat{\psi}(1) \neq 0$. The non-vanishing of the Gauss sum $\hat{\psi}(1)$ is guaranteed for primitive characters. If one wishes to remove the character $\psi(p)$ in (9) then it suffices to apply (9) for another form whose Fourier coefficients are twisted by ψ.

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