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Bilinear forms with Kloosterman fractions

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1. Introduction

Kloosterman sums have in recent years played an increasingly important role in the study of many problems in analytic number theory. The Kloosterman sum with character χ (mod c) is

$$S_\chi(a, b; c) = \sum_{x \pmod{c}} \overline{\chi}(x) e\left(\frac{a\overline{x} + bx}{c}\right) \tag{1.1}$$

where $e(t) = e^{2\pi i t}$ and $\overline{x}x \equiv 1 \pmod{c}$. If χ is the principal character this is the classical Kloosterman sum and if $\chi(x) = \left(\frac{x}{c}\right)$ is the Jacobi symbol then this is the Salié sum. In applications we usually need estimates for incomplete sums of the form

$$\sum_{x \in I} \overline{\chi}(x) e\left(\frac{a\overline{x}}{c}\right)$$

where I is a segment of an arithmetic progression. A standard method of bounding such a sum is to express it in terms of the complete sums $S_\chi(a, b; c)$ by a Fourier technique and then apply Weil's bound for the latter; see Lemma 8. When I is too small this method gives nothing, and we are fortunate that for many applications it suffices to bound these sums on average over the modulus c.

In this paper we consider general bilinear forms of the type

$$\mathscr{B}(M, N) = \sum\sum_{(m,n)=1} \alpha_m \beta_n e\left(a\,\frac{\overline{m}}{n}\right) \tag{1.2}$$

where a is a fixed (but possibly large) positive integer and α_m, β_n are arbitrary complex numbers for $M < m \le 2M$, $N < n \le 2N$, respectively and $\overline{m}m \equiv 1$ (mod n). A trivial bound for this is

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$$|\mathscr{B}(M,N)| \leq 2\|\alpha\| \; \|\beta\|(MN)^{\frac{1}{2}} \tag{1.3}$$

where $\| \cdot \|$ denotes the ℓ_2-norm. Our aim is to obtain non-trivial improvements which are needed for a number of applications; some of these are mentioned below.

Our first result is

Theorem 1. *For any positive integer a we have*

$$\mathscr{B}(M,N) \ll \|\alpha\| \; \|\beta\| \left\{ (M+N)^{\frac{1}{2}} + \left(1 + \frac{a}{MN}\right)^{\frac{1}{2}} \min(M,N) \right\} (MN)^{\varepsilon} \tag{1.4}$$

where the implied constant depends only on ε.

We prove (1.4) in Sect. 2 in the slightly stronger form (2.17) and (2.18). The argument is elementary and has some features in common with the work [FV] of Forti and Viola (on the large sieve) about which we learned years ago in conversations with Bombieri. A sharp bound expected for $\mathscr{B}(M,N)$ is conjectured in (2.20).

Theorem 1 provides non-trivial bounds except in the case when one of the ranges M, N is much larger than the other and in the case when they are nearly equal. The former case is of less interest and in general it is not possible to obtain non-trivial bounds. By contrast the case when M and N are nearly equal is very important for applications. Our second theorem provides, by more sophisticated techniques (see the remarks following (3.4)), a non-trivial bound in this case. Specifically, we have

Theorem 2. *For any positive integer a we have*

$$\mathscr{B}(M,N) \ll \|\alpha\| \; \|\beta\|(a+MN)^{\frac{3}{8}}(M+N)^{\frac{11}{48}+\varepsilon} \tag{1.5}$$

the implied constant depending only on ε.

In case $a \ll MN$, a condition usually satisfied in applications, the above result improves the trivial bound (1.3) provided $N > M^{\frac{5}{6}+\varepsilon}$ and $M > N^{\frac{5}{6}+\varepsilon}$. Combining Theorem 1 and Theorem 2, we obtain a non-trivial estimate whenever $N > M^{\varepsilon}$ and $M > N^{\varepsilon}$. More precisely, applying (1.4) if $a + MN < (M+N)^{59/30}$ and (1.5) otherwise, we deduce

Theorem 3. *For any positive integer a and any complex numbers α_m, β_n, we have*

$$\mathscr{B}(M,N) \ll \|\alpha\| \; \|\beta\|(a+MN)^{\frac{14}{29}}(M+N)^{\frac{1}{58}+\varepsilon} \tag{1.6}$$

where the implied constant depends only on ε.

With applications in mind we establish bounds for the more general weighted sum

$$\mathscr{B}_F(M,N) = \sum\sum_{(m,n)=1} \alpha_m \beta_n e\left(a\frac{\overline{m}}{n}\right) F(m,n) \tag{1.7}$$

where $F(m,n)$ is a smooth function whose partial derivatives satisfy

$$F^{(j,k)}(m,n) \ll \eta^{j+k} m^{-j} n^{-k}, \tag{1.8}$$

for $0 \le j,k \le 2$ and some $\eta \ge 1$.

Corollary. *The bounds* (1.4), (1.5), (1.6) *for* $\mathscr{B}(M,N)$ *hold also for* $\mathscr{B}_F(M,N)$, *provided that, in each case the right hand side is multiplied by* η^2.

For applications to Salié sums it is natural to consider the hermitian sum

$$\mathscr{H}(M,N) = \sum\sum_{(m,n)=1} \alpha_m \beta_n e\left(a\frac{\overline{m}}{n} - a\frac{\overline{n}}{m}\right). \tag{1.9}$$

Theorem H. *The bounds* (1.4), (1.5), (1.6) *for* $\mathscr{B}(M,N)$ *hold as stated also for* $\mathscr{H}(M,N)$.

Since for the Salié sum we have (see Lemma 5 of [I])

$$S_\chi(a,a;c) = \varepsilon_c c^{\frac{1}{2}}\left(\frac{a}{c}\right) \sum_{\substack{mn=c \\ (m,n)=1}} e\left(2a\frac{\overline{m}}{n} - 2a\frac{\overline{n}}{m}\right) \tag{1.10}$$

if $(c,2a)=1$ where $\varepsilon_c = 1$ or i according to $c \equiv 1$ or $3 \pmod 4$ we deduce

Theorem 4. *For positive integers* a, r *with* $8|r$ *and* b *with* $(b,r)=1$ *we have*

$$\sum_{\substack{c \le x, (c,a)=1 \\ c \equiv b \pmod r}} c^{-\frac{1}{2}} S_\chi(a,a;c) = 24\pi^{-2}\varepsilon_b \delta(a)\psi(ar) a E(x/2a) + O\left(r^{\frac{1}{5}}(a+x)^{\frac{47}{118}} x^{\frac{35}{59}+\varepsilon}\right) \tag{1.11}$$

where the implied constant depends only on ε *and we define* $\delta(a)=1$ *if* a *is a square and zero otherwise,*

$$\psi(q) = \prod_{p|q}\left(1+\frac{1}{p}\right)^{-1}, \quad E(y) = \int_0^y e(-1/t)\,dt.$$

Note that $E(y) = y + O(\log(y+1))$ so, if a is a square the main term in (1.11) is

$$12\pi^{-2}\varepsilon_b\psi(ar)\left\{x + O\left(a\log\left(1+\frac{x}{a}\right)\right)\right\}.$$

The estimate (1.11) is non-trivial, i.e. the error term improves upon x/r, if $x > (a^{\frac{47}{48}} r^3 + r^{142}) a^{\varepsilon}$, the main point here being that the exponent of a is less than one.

Bilinear forms of type $\mathscr{B}(M,N)$ with special coefficients arise in many problems, and our results, especially Theorem 2, have quite a number of applications. One such is to the estimation of the Fourier coefficients of forms automorphic with respect to a theta multiplier. Thus Theorem 4 gives a new and simpler derivation of a sufficiently strong bound for these coefficients. See [I], [D], and [DFI1-2] for the earlier ones. Our original motivation for this work, however, comes from the application of these theorems to the problem of improving the convexity bounds for the *L*-functions attached to the class group characters of an imaginary quadratic field, and more generally to Artin *L*-functions of degree two. In [DFI4] we apply our results to considerably strengthen the results given in [DFI3]. Some further applications, given in [DFI5], deal with the number of solutions of determinant equations and new mean value theorems for Dirichlet *L*-functions and Dirichlet polynomials. There are also applications to sums of the type occurring in the works [BFI] on the distribution of divisor functions and of primes in arithmetic progressions.

It is a simple matter to reduce the proof of Theorem 2 to estimates for incomplete Kloosterman sums; however, these are over intervals too short to be estimated by any known method. There are conjectured bounds for such short sums (see C. Hooley [H]) which would more than suffice. Of course we prefer to avoid any unproven hypothesis in this work. We find we are able to adapt the amplification method developed in the earlier works in this series [FI], [DFI1,2,3] although its use in the current paper is considerably different.

The proof of Theorem 2 applies almost without change to the twisted sum

$$\mathscr{A}(M,N) = \sum_{(m,n)=1}\sum \alpha_m \beta_n \left(\frac{m}{n}\right) e\left(a\,\frac{\overline{m}}{n}\right), \tag{1.12}$$

where $\left(\frac{m}{n}\right)$ is the Jacobi symbol. We discuss this in Sect. 4.

Our methods begin by applying Cauchy's inequality

$$|\mathscr{B}(M,N)| \leq \|\alpha\| \mathscr{C}(M,N;\beta)^{\frac{1}{2}} \tag{1.13}$$

where

$$\mathscr{C}(M,N;\beta) = \sum_{M<m\leq 2M} \Big| \sum_{\substack{N<n\leq 2N \\ (n,m)=1}} \beta_n e\left(a\,\frac{\overline{m}}{n}\right) \Big|^2, \tag{1.14}$$

and we actually establish non-trivial estimates for $\mathscr{C}(M,N;\beta)$; see Theorems 5 and 6.

In several places, in order to simplify both statements and proofs, we have refrained from trying to obtain the sharpest exponents within the reach of the methods.

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2. Proof of Theorem 1

First we establish the following

Theorem 5. *For any positive integer a and complex numbers β_n we have*

$$\mathscr{C}(M,N;\beta) \ll \|\beta\|^2\tau(a)(M+N^2)N^{\varepsilon} \tag{2.1}$$

where the implied constant depends only on ε.

This bound can be obtained, rather more quickly than we do here, by squaring out, changing the order of summation, and then applying the well-known estimate for incomplete Kloosterman sums derived (see Lemma 8) from the Weil bound [W] for complete sums. We take the opportunity, however, to show how these results follow, with not too much extra effort, from an elementary argument which avoids the appeal to Weil's bound.

We begin by assuming that the β_n are supported on integers n prime to a. At the end of the proof we remove this restriction. We write for every m

$$\sum_{(n,m)=1} \beta_n e\left(a\frac{\overline{m}}{n}\right) = \sum_{(n,qm)=1} \beta_n e\left(aq\frac{\overline{qm}}{n}\right) + \sum_{(n,m)=1} \beta_n' e\left(a\frac{\overline{m}}{n}\right)$$

where q is a prime number, to be selected later, from the interval $P < q \le 2P$, and

$$\beta_n' = \begin{cases} \beta_n & \text{if } q|n \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Hence

$$\Big|\sum_{(n,m)=1} \beta_n e\left(a\frac{\overline{m}}{n}\right)\Big|^2 \le 2\sum_p\Big|\sum_{(n,qpm)=1} \beta_n e\left(aq\frac{\overline{pm}}{n}\right)\Big|^2 + 2\Big|\sum_{(n,m)=1} \beta_n' e\left(a\frac{\overline{m}}{n}\right)\Big|^2$$

where p runs over all primes in the interval $P < p \le 2P$. Summing over m in the interval $M < m \le 2M$ we obtain

$$\mathscr{C}(M,N;\beta) \le 2\mathscr{C}(P,M,N;\beta) + 2\mathscr{C}(M,N;\beta') \tag{2.3}$$

where

$$\mathscr{C}(P,M,N;\beta) = \sum_p\sum_m\Big|\sum_{(n,qpm)=1} \beta_n e\left(aq\frac{\overline{pm}}{n}\right)\Big|^2. \tag{2.4}$$

Regarding $\ell = pm$ as a single variable which occurs with multiplicity $\le \log \ell$ in the interval $L < \ell \le 4L$ for $L = PM$, we find that

$$\mathscr{C}(P,M,N;\beta) \le \sum_{\ell} f(\ell)\Big|\sum_{(n,q\ell)=1} \beta_n e\left(aq\frac{\overline{\ell}}{n}\right)\Big|^2 \tag{2.5}$$

where $f(\ell)$ is a smooth function supported on $\frac{1}{2}L < \ell \le 8L$ such that $f^{(j)}(\ell) \ll L^{-j}\log L$ for all $j \ge 0$. Squaring out and changing the order of summation, we obtain

$$\sum\sum_{\substack{n_1 \neq n_2 \\ (n_1 n_2, q)=1}} \beta_{n_1}\overline{\beta}_{n_2} \sum_{(\ell, n_1 n_2)=1} f(\ell) e\left(aq(n_2 - n_1)\frac{\overline{\ell}}{n_1 n_2}\right) + O\left(\|\beta\|^2 L\right). \tag{2.6}$$

We shall choose P sufficiently large so that the sum over $\ell \asymp L = PM$ is essentially a Ramanujan sum. More precisely, by Poisson summation we have

$$\sum_{\ell} = \frac{1}{n_1 n_2}\sum_h \hat{f}\left(\frac{h}{n_1 n_2}\right) S(aq(n_2 - n_1), h; n_1 n_2) \tag{2.7}$$

where $S(b, h; c)$ is the Kloosterman sum. For $h = 0$ this is the Ramanujan sum which is bounded by

$$|S(b, 0; c)| \le (b, c).$$

For $h \neq 0$ we can afford to use the trivial bound

$$|S(b, h; c)| \le c$$

because the Fourier transform is very small; namely, by partial integration

$$\hat{f}\left(\frac{h}{n_1 n_2}\right) \ll \left(1 + \frac{|h|L}{N^2}\right)^{-A} L \log L \tag{2.8}$$

for any $A > 1$. Hence

$$\sum_{h \neq 0} \left|\hat{f}\left(\frac{h}{n_1 n_2}\right)\right| \ll \left(1 + \frac{L}{N^2}\right)^{-A} N^2 \log L.$$

Assuming $L > N^{2+\varepsilon}$, we choose $A = 3\varepsilon^{-1}$ and infer that

$$\sum_{\ell} \ll (n_2 - n_1, n_1 n_2) L N^{-2}.$$

Hence

$$\mathscr{C}(P, M, N; \beta) \ll LN^{-2} \sum\sum_{n_1 \neq n_2} |\beta_{n_1}\beta_{n_2}|((n_2 - n_1), n_1 n_2) + \|\beta\|^2 L.$$

Since, for given n_1,

$$\sum_{n_2 \neq n_1} (n_2 - n_1, n_1 n_2) = \sum_{n_2 \neq n_1} (n_2 - n_1, n_1^2) \le 2 \sum_{0 < n < N} (n, n_1^2) \le 2N\tau(n_1^2),$$

this gives

$$\mathscr{C}(P, M, N; \beta) \ll \|\beta\|^2 L.$$

Hence we conclude by (2.3) that

(2.9) $$\mathscr{C}(M,N;\beta)\le 2\mathscr{C}(M,N;\beta')+O(\|\beta\|^2L),$$

where β' was defined in (2.2).

Let $V(M,N)$ denote the norm of the linear operator given by the matrix $(e(a\frac{\overline{m}}{n}))$ with $M<m\le 2M$, $N<n\le 2N$, $(n,am)=1$ so $\mathscr{C}(M,N;\beta)\le\|\beta\|^2V(M,N)$ for any complex numbers β_n. Our inequality (2.9) asserts that

$$\mathscr{C}(M,N;\beta)\le 2\|\beta'\|^2V(M,N)+O(\|\beta\|^2L).$$

Now we choose q in $P<q\le 2P$ such that

$$\|\beta'\|^2\ll\|\beta\|^2P^{-1}\log N$$

which we can do for every interval $(P,2P]$ since this holds on average for primes in such an interval. Therefore we have established that

$$\mathscr{C}(M,N;\beta)\ll\|\beta\|^2(V(M,N)P^{-1}\log N+L)$$

for all complex numbers β_n with $(n,a)=1$. In other words,

$$V(M,N)\ll V(M,N)P^{-1}\log N+L.$$

Choosing $P=(1+N^2M^{-1})N^{\varepsilon}$, we obtain

$$V(M,N)\ll L=PM=(M+N^2)N^{\varepsilon}$$

which gives

(2.10) $$\mathscr{C}(M,N)\ll\|\beta\|^2(M+N^2)N^{\varepsilon}.$$

This is (2.1) without the factor $\tau(a)$. Now, at the cost of re-inserting this factor, we remove the restriction on n prime to a. To this end we note

$$\Big|\sum_n\beta_n e\left(a\frac{\overline{m}}{n}\right)\Big|^2\le\tau(a)\sum_{cd=a}\Big|\sum_{(n,d)=1}\beta_{cn}e\left(d\frac{\overline{m}}{n}\right)\Big|^2$$

and for each d apply (2.10) getting Theorem 5.

The same method works for slightly perturbed sums of the type

(2.11) $$\mathscr{C}_g(M,N;\beta)=\sum_{M<m\le 2M}\Big|\sum_{(n,m)=1}\beta_n e\left(a\frac{\overline{m}}{n}+g_n(m)\right)\Big|^2$$

where g_n is a smooth function with small derivatives, precisely

(2.12) $$g_n^{(j)}\ll\theta M^{-j}$$

for all $j\ge 0$ with some $\theta\ge 1$ and the implied constant depending only on j. This perturbation term appears in (2.4) as $g_n(pm/q)$ and in (2.5) in the form $g_n(\ell/q)$. Then we have in (2.6) the factor $e\left(g_{n_1}(\ell/q)-g_{n_2}(\ell/q)\right)$. The formula (2.7) remains valid but with $\hat{f}(h/n_1n_2)$ replaced by

$$\int f(t) e\left(\frac{ht}{n_1 n_2} + g_{n_1}\left(\frac{t}{q}\right) - g_{n_2}\left(\frac{t}{q}\right)\right) dt. \tag{2.13}$$

This integral satisfies the same bound (2.8) provided $L = PM > \theta N^{2+\varepsilon}$. Therefore the remaining arguments are exactly the same except that we choose P somewhat larger, namely $P = (1 + \theta N^2 M^{-1}) N^\varepsilon$ so $L = PM = (M + \theta N^2) N^\varepsilon$. This yields

Theorem 5′. *For any positive integer a, any complex numbers β_n and any functions g_n satisfying (2.12) with $\theta \geq 1$ we have*

$$\mathscr{C}_g(M, N; \beta) \ll \|\beta\|^2 \tau(a)(M + \theta N^2) N^\varepsilon \tag{2.14}$$

where the implied constant depends only on ε.

If we apply (2.14) for $g_n(m) = -a/mn$, so (2.12) holds with $\theta = 1 + a/MN$, and apply the reciprocity formula

$$\frac{\overline{m}}{n} + \frac{\overline{n}}{m} \equiv \frac{1}{mn} \pmod 1 \tag{2.15}$$

we obtain

$$\sum_{M < m \leq 2M} \Big| \sum_{(n,m)=1} \beta_n e\left(a \frac{\overline{n}}{m}\right) \Big|^2 \ll \|\beta\|^2 \tau(a)(M + N^2 + aNM^{-1}) N^\varepsilon. \tag{2.16}$$

Now we are ready to prove Theorem 1. Using (1.13), we derive by (2.1) and (2.16) two estimates

$$\mathscr{B}(M, N) \ll \|\alpha\| \, \|\beta\| \left(M^{\frac{1}{2}} + N\right)(aN)^\varepsilon \tag{2.17}$$

$$\mathscr{B}(M, N) \ll \|\alpha\| \, \|\beta\| \left(N^{\frac{1}{2}} + M + a^{\frac{1}{2}} M^{\frac{1}{2}} N^{-\frac{1}{2}}\right)(aM)^\varepsilon . \tag{2.18}$$

Applying (2.17) if $M \geq N$ and (2.18) if $M < N$, we get (1.4). Note that the factor $\tau(a)$ is not needed because $\tau(a) \ll a^\varepsilon$ and the estimate (1.4) is trivial if $a > (MN)^2$.

Remarks. We expect, but cannot prove, that the norm $V(M, N)$ of the linear operator given by the matrix $\left(e\left(a \frac{\overline{m}}{n}\right)\right)$ with $(m, n) = 1$, $M < m \leq 2M$, $N < n \leq 2N$ is bounded by $(M + N)(aMN)^\varepsilon$, that is

$$\mathscr{C}(M, N; \beta) \ll \|\beta\|^2 (M + N)(aMN)^\varepsilon \tag{2.19}$$

for any complex numbers β_n. Hence, by (1.14), one could get

$$\mathscr{B}(M, N) \ll \|\alpha\| \, \|\beta\| (M + N)^{\frac{1}{2}} (aMN)^\varepsilon. \tag{2.20}$$

This estimate seems to be out of reach of present methods.

3. Proofs of Theorems 2 and H

By virtue of (1.13), to prove Theorem 2, it suffices to prove

Theorem 6. *For any positive integer a and any complex numbers β_n, we have*

$$\mathscr{C}(M, N; \beta) \ll \|\beta\|^2 (a + MN)^{\frac{3}{4}} (M + N)^{\frac{11}{24}} (MN)^{\varepsilon} \tag{3.1}$$

where the implied constant depends only on ε.

For this purpose we consider a slightly more general sum

$$\mathscr{C}^*(M, N; \beta) = \sum_{(m,b)=1} \Big| \sum_{(n,m)=1} \beta_n e\left(\frac{a\overline{m}}{bn}\right) \Big|^2 \tag{3.2}$$

for which we prove:

Theorem 6$^\flat$. *For any positive co-prime integers a, b and any complex β_n supported on squarefree numbers, we have*

$$\mathscr{C}^*(M, N; \beta) \ll \|\beta\|^2 (a + bMN)^{\frac{1}{2}} (M + N)^{\frac{19}{24}} N^{\frac{1}{8}} (bMN)^{\varepsilon} \tag{3.3}$$

where the implied constant depends only on ε.

First notice we can assume that β_n is supported on numbers co-prime with ab by pulling out the highest common factor (ab, n) and using Cauchy's inequality.

We introduce to $\mathscr{C}^*(M, N; \beta)$ characters to modulus m as follows:

(3.4)
$$\mathscr{D}(M, L, N) = \sum_{(m,b)=1} \frac{1}{\varphi(m)} \sum_{\chi \pmod m} \Big| \sum_{\ell} \chi(\ell)\lambda_\ell \Big|^2 \Big| \sum_n \chi(n)\beta_n e\left(\frac{a\overline{m}}{bn}\right) \Big|^2$$

where the range of summation is $M < m \le 2M$, $L < \ell \le 2L$, $N < n \le 2N$, where $L \le 3MN$, and λ_ℓ are complex numbers to be chosen suitably to amplify the contribution of the principal character $\chi_0 \pmod m$. This operation is somewhat wasteful and it looks artificial, nevertheless it is vital. The losses will be recovered completely due to the orthogonality of characters and we shall gain some flexibility in the resulting sums.

We choose

$$\lambda_\ell = \begin{cases} 1 & \text{if } \ell \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

because it yields a good bound

(3.5)
$$\sum_{\ell} \chi_0(\ell)\lambda_\ell \ge \frac{1}{\log 2L} \sum_{L<\ell\le 2L} \chi_0(\ell) \log \ell \ge \frac{1}{\log 2L} \Big(\sum_{L<\ell\le 2L} \log \ell - \log M \Big) \gg \frac{L}{\log L}$$

(provided $L > 2 \log M$) while simultaneously the requirement that ℓ be prime substantially simplifies the technical details. This choice is assumed from now on. By (3.5) we get

(3.6) $$\mathscr{C}^*(M,N;\beta) \ll ML^{-2}(\log L)^2 \mathscr{D}(M,L,N).$$

By the orthogonality of characters we obtain

(3.7) $$\mathscr{D}(M,L,N) = \sum\sum\sum\sum\sum_{\substack{(m,b\ell_1\ell_2n_1n_2)=1 \\ \ell_1n_1\equiv\ell_2n_2 \pmod m}} \beta_{n_1}\overline{\beta}_{n_2} e\left(\frac{a\overline{m}}{bn_1} - \frac{a\overline{m}}{bn_2}\right)$$

where the range of summation is as before and ℓ_1, ℓ_2 run over primes.

Let $\mathscr{D}'$ denote the partial sum of $\mathscr{D}(M,L,N)$ restricted by either of the following two conditions: $\ell_1 \mid b\ell_2n_1n_2$ or $\ell_2 \mid b\ell_1n_1n_2$. Estimating trivially, we get

(3.8) $$\mathscr{D}' \ll \|\beta\|^2(M+N)L\log bN.$$

The remaining terms yield

(3.9) $$\mathscr{D}^* = \sum\sum_{\ell_1\neq\ell_2} \sum\sum\sum_{\substack{(bn_1n_2,m\ell_1\ell_2)=1 \\ \ell_1n_1\equiv\ell_2n_2 \pmod m}} \beta_{n_1}\overline{\beta}_{n_2} e\left(\frac{a\overline{m}}{bn_1} - \frac{a\overline{m}}{bn_2}\right).$$

Note that the summation conditions imply $(\ell_1\ell_2, m) = 1$ and $\ell_1n_1 \neq \ell_2n_2$, the latter saying $\mathscr{D}^*$ has no diagonal terms. First we estimate the partial sum of $\mathscr{D}^*$ restricted by $(n_1, n_2) = 1$ which is

(3.10) $$\mathscr{D}_1^* = \sum\sum_{(n_1,n_2)=1} \beta_{n_1}\overline{\beta}_{n_2} \sum_{(m,b)=1} \sum\sum_{\substack{\ell_1\neq\ell_2 \\ (\ell_1\ell_2,bn_1n_2)=1 \\ \ell_1n_1\equiv\ell_2n_2 \pmod m}} e\left(\frac{a\overline{m}}{bn_1} - \frac{a\overline{m}}{bn_2}\right).$$

Note that the summation conditions in $\mathscr{D}_1^*$ imply $(m, n_1n_2) = 1$.

Next we split the summation over m into primitive residue classes to modulus b, say

(3.11) $$m \equiv c \pmod b \quad \text{with} \quad (c,b) = 1.$$

Then we replace m by the complementary divisor d of $\ell_1n_1 - \ell_2n_2$ where

(3.12) $$\ell_1n_1 - \ell_2n_2 = dm.$$

Note that the summation conditions in $\mathscr{D}_1^*$ imply $(d, \ell_1\ell_2n_1n_2) = 1$. If m is large $|d|$ is small. Precisely, the inequalities $M < m \leq 2M$ translate into

(3.13) $$M < (\ell_1n_1 - \ell_2n_2)d^{-1} \leq 2M$$

whence $0 < |d| < D$ for $D = 3LNM^{-1}$. Moreover, (3.11) together with (3.12) can be interpreted as one congruence

(3.14) $$\ell_1n_1 - \ell_2n_2 \equiv cd \pmod{b|d|}$$

and $m \pmod{bn_1n_2}$ is determined as the solution to the following system:

$$
\begin{aligned}
m &\equiv c \pmod{b}\\
m &\equiv -\overline{d}\ell_2 n_2 \pmod{n_1} \qquad (3.15)\\
m &\equiv \overline{d}\ell_1 n_1 \pmod{n_2}.
\end{aligned}
$$

Since b, n_1, n_2 are pairwise co-prime, we deduce from the system of congruences (3.15) that

$$e\left(\frac{a\overline{m}}{bn_1}-\frac{a\overline{m}}{bn_2}\right)=e\left(a\overline{c}\left(\frac{\overline{n_1}}{b}-\frac{\overline{n_2}}{b}\right)-ad\left(\frac{\overline{b\ell_1 n_1}}{n_2}+\frac{\overline{b\ell_2 n_2}}{n_1}\right)\right).$$

Having interpreted each occurrence of m in terms of d, we are ready to estimate $\mathscr{D}_1^*$ as follows:

$$|\mathscr{D}_1^*|\le \sideset{}{^*}\sum_{c \pmod{b}}\ \sum_{0<|d|<D}\ \sum\sum_{(n_1,n_2)=1}|\beta_{n_1}\beta_{n_2}|\ \Big|\sum\sum_{\ell_1\neq\ell_2}\Big|$$

where

$$\sum\sum_{\ell_1\neq\ell_2}=\sum\sum_{\substack{\ell_1\neq\ell_2\\(\dots)}}e\left(ad\left(\frac{\overline{b\ell_1 n_1}}{n_2}+\frac{\overline{b\ell_2 n_2}}{n_1}\right)\right). \qquad (3.16)$$

Here the three dots remind us that the summation is restricted to prime numbers ℓ_1, ℓ_2 in the interval $L<\ell_1,\ell_2\le 2L$ which satisfy the inequalities (3.13) and the congruence (3.14) in addition to $(\ell_1\ell_2, bdn_1n_2)=1$. One could put $|d|$ into the inner summation to obtain a stronger result (due to a longer diagonal) but we have chosen not to do so for simplicity. By Cauchy's inequality

$$|\mathscr{D}_1^*|^2\le\|\beta\|^4 2bD\sum_{0<|d|<D}\mathscr{E}(d) \qquad (3.17)$$

where

$$\mathscr{E}(d)=\sum\sum_{\substack{(n_1,n_2)=1\\(n_1n_2,abd)=1}}\ \sum_{c \pmod{b}}\Big|\sum\sum_{\ell_1\neq\ell_2}\Big|^2.$$

Note that we have dropped, by positivity, the condition $(c,b)=1$. Squaring out and changing the order of summation, we arrange that

$$\mathscr{E}(d)=\sum\sum\sum\sum_{\substack{\ell_1\neq\ell_2,\ell_1'\neq\ell_2'\\(\ell_1\ell_2\ell_1'\ell_2',bd)=1}}\mathscr{E}_d(\ell_1,\ell_2;\ell_1',\ell_2') \qquad (3.18)$$

with

$$\mathscr{E}_d(\ell_1,\ell_2;\ell_1',\ell_2')=\sum\sum_{\substack{(n_1,n_2)=1\\(***)}}e\left(ad(\ell_1'-\ell_1)\frac{\overline{b\ell_1'\ell_1 n_1}}{n_2}+ad(\ell_2'-\ell_2)\frac{\overline{b\ell_2'\ell_2 n_2}}{n_1}\right) \qquad (3.19)$$

where the three stars denote the six conditions:

$$
\begin{aligned}
N &< n_1, n_2 \leq 2N \\
M &< (\ell_1 n_1 - \ell_2 n_2) d^{-1} \leq 2M \\
M &< (\ell_1' n_1 - \ell_2' n_2) d^{-1} \leq 2M
\end{aligned} \tag{3.20}
$$

and

$$
\begin{gathered}
(n_1 n_2, abd\ell_1\ell_2\ell_1'\ell_2') = 1 \\
\ell_1 n_1 \equiv \ell_2 n_2 \pmod{|d|} \\
(\ell_1' - \ell_1) n_1 \equiv (\ell_2' - \ell_2) n_2 \pmod{b|d|}.
\end{gathered} \tag{3.21}
$$

Observe that (3.21) implies

$$
\ell_2'\ell_1 \equiv \ell_2\ell_1' \pmod{|d|}. \tag{3.22}
$$

Put

$$
\Delta = \ell_1'\ell_1(\ell_2' - \ell_2) - \ell_2'\ell_2(\ell_1' - \ell_1) \tag{3.23}
$$

and note that $\Delta = 0$ if and only if $\ell_1 = \ell_1'$ and $\ell_2 = \ell_2'$.

If $\Delta = 0$ we use the trivial bound

(3.24)

$$
\mathscr{E}_d(\ell_1, \ell_2; \ell_1', \ell_2') \ll \#\{(n_1, n_2) : (n_1 n_2, \ell_1\ell_2) = 1,\ \ell_1 n_1 \equiv \ell_2 n_2 \pmod{|d|}\}.
$$

Summing this over $\ell_1 = \ell_1' \neq \ell_2 = \ell_2'$ we deduce that the terms with $\Delta = 0$ contribute to $\mathscr{E}(d)$ at most

$$
\mathscr{E}_0(d) \ll |d|^{-1} L^2 N^2 \log N. \tag{3.25}
$$

If $\Delta \neq 0$ we reduce the problem to the estimation of incomplete Kloosterman sums. First we write

$$
\frac{\overline{b\ell_1'\ell_1 n_1}}{n_2} \equiv -\frac{\overline{bn_2}}{\ell_1'\ell_1 n_1} - \frac{\overline{\ell_1'\ell_1 n_1 n_2}}{b} + \frac{1}{b\ell_1'\ell_1 n_1 n_2} \pmod{1}
$$

whence the exponential in (3.19) is equal to

$$
e\left(ad\Delta\,\frac{\overline{b\ell_2'\ell_2 n_2}}{\ell_1'\ell_1 n_1} - ad(\ell_1' - \ell_1)\,\frac{\overline{\ell_1'\ell_1 n_1 n_2}}{b} + \frac{ad(\ell_1' - \ell_1)}{b\ell_1'\ell_1 n_1 n_2}\right). \tag{3.26}
$$

Here the middle term can be transformed using (3.21) into

$$
e\left(ad(\ell_2' - \ell_2)\,\frac{\overline{\ell_1'\ell_1 n_1^2}}{b}\right)
$$

which renders visible its independence of n_2. By the above transformations we obtain

$$
|\mathscr{E}_d(\ell_1, \ell_2; \ell_1', \ell_2')| \leq \sum_{n_1} \left| \sum_{\substack{(n_2, n_1) = 1 \\ (***)}} e\left(ad\Delta\,\frac{\overline{b\ell_2'\ell_2 n_2}}{\ell_1'\ell_1 n_1} - \frac{ad(\ell_1' - \ell_1)}{b\ell_1'\ell_1 n_1 n_2}\right)\right|. \tag{3.27}
$$

The double sum on the right-hand side of (3.27) is a bilinear form of type (1.2) however with special coefficients. One could attempt to apply results [DI] from the spectral theory of automorphic forms but this would be very complicated and it is not clear how good the estimate would be, given the extremely large level of the relevant group $\Gamma_0(b\ell_1\ell_2\ell_1'\ell_2')$.

Therefore we choose the simpler route of bounding the sum over the single variable n_2. This is an incomplete Kloosterman sum with congruence conditions (3.21) and small perturbation,

$$\frac{ad(\ell_1'-\ell_1)}{b\ell_1'\ell_1 n_1 n_2} \ll \frac{aD}{bLN^2} \ll \frac{a}{bMN}\ . \tag{3.28}$$

From Lemma 8 we deduce that

$$\sum_{n_2} \ll \left(1+\frac{a}{bMN}\right)(ad\Delta,\ell_1'\ell_1 n_1)^{\frac12} LN^{\frac12+\varepsilon}. \tag{3.29}$$

Summing this over n_1 we get

$$\mathscr{E}_d(\ell_1,\ell_2;\ell_1',\ell_2') \ll \left(1+\frac{a}{bMN}\right)(a,\ell_1'\ell_1)^{\frac12} LN^{\frac32+\varepsilon}. \tag{3.30}$$

Next summing this over $\ell_1,\ell_2,\ell_1',\ell_2'$ subject to (3.22) we deduce that the terms with $\Delta\neq 0$ contribute to $\mathscr{E}(d)$ at most

$$\mathscr{E}_1(d) \ll |d|^{-1}\left(1+\frac{a}{bMN}\right) L^5 N^{\frac32+\varepsilon} \tag{3.31}$$

using the fact that $\ell_2'\ell_1 \equiv \ell_1'\ell_2 \pmod{|d|}$ but $\ell_2'\ell_1 \neq \ell_1'\ell_2$. Adding (3.31) to (3.25) we get

$$\mathscr{E}(d) \ll |d|^{-1}\left(1+\frac{a}{bMN}\right)\left(L^2N^2+L^5N^{\frac32}\right)N^{\varepsilon}. \tag{3.32}$$

Hence, by (3.17)

$$\mathscr{D}_1^* \ll \|\beta\|^2\left(b+\frac{a}{MN}\right)^{\frac12}\left(\frac{LN}{M}\right)^{\frac12}\left(LN+L^{\frac52}N^{\frac34}\right)N^{\varepsilon}. \tag{3.33}$$

The same bound (3.33) holds true for $\mathscr{D}^*$; to see this pull out the greatest common divisor $\nu=(n_1,n_2)$, apply (3.33) with $b\nu$, $N\nu^{-1}$ in place of b, N and then sum over ν.

Adding (3.33) for $\mathscr{D}^*$ to (3.8) for $\mathscr{D}'$ we obtain

$$\mathscr{D}(M,L,N) \ll \|\beta\|^2\left(b+\frac{a}{MN}\right)^{\frac12} L\left\{M+N+\left(\frac{LN}{M}\right)^{\frac12}\left(N+L^{\frac32}N^{\frac34}\right)\right\}(bN)^{\varepsilon}. \tag{3.34}$$

Inserting this in (3.6) we get

$$\mathscr{C}^*(M,N;\beta) \ll \|\beta\|^2(a+bMN)^{\frac12}\left\{(M+N)M^{\frac12}N^{-\frac12}L^{-1}+NL^{-\frac12}+N^{\frac34}L\right\}(bN)^{\varepsilon}.$$

This holds for all $L > 0$; the former assumption $L > 2\log M$ is not needed here because the result is trivial otherwise. We choose $L = (M+N)^{\frac{1}{2}} M^{\frac{1}{4}} N^{-\frac{5}{8}} + N^{\frac{1}{6}}$ getting

$$(M+N)M^{\frac{1}{2}}N^{-\frac{1}{2}}L^{-1} + NL^{-\frac{1}{2}} + N^{\frac{3}{4}}L \leq 2(M+N)^{\frac{1}{2}}M^{\frac{1}{4}}N^{\frac{1}{8}} + 2N^{\frac{11}{12}} \leq 4M^{\frac{3}{4}}N^{\frac{1}{8}} + 4N^{\frac{11}{12}}$$

which is rather stronger than (3.3). This completes the proof of Theorem $6^\flat$.

It remains to derive Theorem 6 from Theorem $6^\flat$. To this end we write $n = bn'$ where b is squarefull, n' is squarefree, and $(b, n') = 1$. Note that

$$\Big|\sum_n \beta_n \left(\frac{a\overline{m}}{n}\right)\Big|^2 \leq \Big(\sum_b b^{-1/2}\Big)\Big(\sum_b b^{1/2}\Big|\sum_{n'} \beta'_{n'} e\left(\frac{a\overline{m}}{bn'}\right)\Big|^2\Big)$$

where $\beta'_{n'} = \beta_{bn'}$ and

$$\sum_b b^{-1/2} \ll \log N\,.$$

We apply (3.3) for each $b \leq B$ and the trivial bound

$$\Big|\sum_{n'} \beta'_{n'} e\left(\frac{a\overline{m}}{bn'}\right)\Big|^2 \leq \frac{2N}{b}\sum_{n'} |\beta_{bn'}|^2$$

for each $b > B$ (the use instead of Theorem 5 would do a bit better). Hence we get

$$\begin{aligned} \mathscr{C}(M,N;\beta) &\ll (a+MN)^{\frac{1}{2}}(M+N)^{\frac{19}{24}}N^{\frac{1}{8}}\sum_{b<B} b^{\frac{1}{2}}\sum_{n'}|\beta_{bn'}|^2(bMN)^\varepsilon \\ &\quad + MN\log N\sum_{b>B} b^{-\frac{1}{2}}\sum_{n'}|\beta_{bn'}|^2 \\ &\ll (a+MN)^{\frac{1}{2}}(M+N)^{\frac{19}{24}}N^{\frac{1}{8}}(MN)^\varepsilon\|\beta\|^2B^{\frac{1}{2}} + \|\beta\|^2B^{-\frac{1}{2}}MN^{1+\varepsilon} \end{aligned}$$

for any $B > 0$. We choose

$$B = (a+MN)^{\frac{1}{2}}(M+N)^{-\frac{19}{24}}N^{-\frac{1}{8}}$$

getting

$$\mathscr{C}(M,N;\beta) \ll \|\beta\|^2(a+MN)^{\frac{3}{4}}(M+N)^{\frac{19}{48}}N^{\frac{1}{16}}(MN)^\varepsilon$$

which is slightly better than the bound (3.1) required for Theorem 2.

As indicated in the Introduction, Theorem 3 follows at once from Theorems 1 and 2. We conclude this section with the proofs of the Corollary and of Theorem H.

To obtain the Corollary we note first that the sum in (1.7) is unaltered if the function $F(m,n)$ is replaced by $G(m,n) = \psi(m,n)F(m,n)$ where $\psi(m,n)$ is a smooth function supported on $[\frac{1}{2}M, 3M] \times [\frac{1}{2}N, 3N]$ and equal to one on $[M, 2M]\times[N, 2N]$. The function ψ may be chosen so that $G^{(j,k)} \ll \eta^{j+k}M^{-j}N^{-k}$ whence its Fourier transform has an L_1 norm which is bounded by $O(\eta^2)$. The Corollary now follows from Theorems 1, 2, 3 after a Fourier inversion.

To obtain Theorem H we do not apply Theorems 1 and 2 directly but rather, in order to avoid losing a factor η^2 which would be harmful for Theorem 4, we modify their proofs to show that these bounds hold precisely as stated also for the hermitian sum (1.9). From these (1.6) follows as before.

To obtain (1.4) for $\mathscr{H}(M,N)$ we apply (2.14) with $g_n(m) = -a/2mn$ so that

$$a\,\frac{\overline{m}}{n} + g_n(m) = \frac{a}{2}\left(\frac{\overline{m}}{n} - \frac{\overline{n}}{m}\right) \tag{3.35}$$

by (2.15). This gives the bound (2.18) and also the same bound with M and N interchanged. Although the latter is slightly weaker than (2.17) it clearly suffices for the proof of (1.4).

To get (1.5) for $\mathscr{H}(M,N)$ it suffices to have the bound of Theorem 6 but for $\mathscr{C}_g(M,N;\beta)$ given by (2.11) with the choice (3.35). This version of Theorem 6 follows (and by the same argument) once we have the bound of Theorem $6^\flat$ with the sum $\mathscr{C}^*$ of (3.2) modified by the insertion of an additional factor $e(-a/2bmn)$ in the inner sum. To obtain the latter we follow the same argument as before and are led in (3.27) to the same perturbed incomplete Kloosterman sum but now with the additional perturbation

$$\frac{ad}{2b}\left(\frac{1}{n_2} - \frac{1}{n_1}\right)\left(\frac{1}{\ell_1 n_1 - \ell_2 n_2} - \frac{1}{\ell_1' n_1 - \ell_2' n_2}\right).$$

Using (3.20) we see that this perturbation also satisfies the bound (3.28) and hence we get (3.29). The rest is unchanged.

4. The twisted bilinear form

In this section we describe the (minor) modifications of the arguments in the previous section to give

Theorem 7. *The bound (1.5) of Theorem 2 holds when $\mathscr{B}(M,N)$ is replaced by*

$$\mathscr{A}(M,N) = \sum\sum_{(m,n)=1} \alpha_m \beta_n \left(\frac{m}{n}\right) e\left(a\,\frac{\overline{m}}{n}\right).$$

We first introduce in the definition of $\mathscr{C}(M,N;\beta)$ the Jacobi symbol $\left(\frac{m}{n}\right)$ inside the inner summation. It then appears in (3.4) and, in the form $\left(\frac{m}{n_1 n_2}\right)$, in (3.7), (3.9) and (3.10). Using (3.12) we rewrite this as

$$\left(\frac{m}{n_1 n_2}\right) = \left(\frac{-n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right)\left(\frac{d}{n_1 n_2}\right)\left(\frac{\ell_2}{n_1}\right)\left(\frac{\ell_1}{n_2}\right).$$

Consequently in (3.16) we get the additional factor $\left(\frac{\ell_2}{n_1}\right)\left(\frac{\ell_1}{n_2}\right)$. This appears again in (3.19). Finally in (3.27) we get the additional factor $\left(\frac{\ell_1}{n_2}\right)$. For the resulting twisted sum over n_2 we get the same bound (3.29) by virtue of Lemma 8. The remainder of the proof is verbatim.

5. Proof of Theorem 4

By the quadratic reciprocity law $\left(\frac{a}{c}\right)\left(\frac{c}{a}\right)$ depends only on the residue class of c modulo 8 so for $c \equiv b \pmod r$ we have

$$\left(\frac{a}{c}\right)\left(\frac{c}{a}\right) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right)$$

and $\varepsilon_c = \varepsilon_b$. Therefore by (1.10) we find that the sum of Salié sums in (1.11) is equal to

$$S = \varepsilon_b \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) \sum_{\substack{mn \le x, (m,n)=1 \\ mn \equiv b \pmod r}}\sum \left(\frac{mn}{a}\right) e\left(2a\,\frac{\overline{m}}{n} - 2a\,\frac{\overline{n}}{m}\right).$$

We split the above double sum into $S_1 + \overline{S}_1 + S_2 + \overline{S}_2 + S_3 - S_4$ where

$$S_1 = \sum_{n \le y}\sum, \quad S_2 = \sum_{y<n \le z}\sum, \quad S_3 = \sum_{m,n>z}\sum$$

and

$$S_4 = \sum_{m,n \le z}\sum \ll z^2. \tag{5.1}$$

For S_3 we apply (1.5) of Theorem H getting

$$S_3 \ll (a+x)^{\frac{3}{8}}(x/z)^{\frac{11}{48}} x^{\frac{1}{2}+\varepsilon} \tag{5.2}$$

and to S_2 we apply (1.4) of Theorem H getting

$$S_2 \ll \left\{(x/y)^{\frac{1}{2}} + (1+a/x)^{\frac{1}{2}} z\right\} x^{\frac{1}{2}+\varepsilon}. \tag{5.3}$$

To be precise (5.2) and (5.3) do not follow immediately but rather after a few technical arrangements. First we split the summation into $\ll (\log x)^2$ dyadic boxes. Next, to separate the variables m and n we need to detect the constraints $mn \equiv b \pmod r$ and $mn \le x$. The first is achieved by splitting each of m, n into residue classes μ and ν modulo r with $\mu\nu \equiv b$, or alternatively by using Dirichlet characters modulo r. To achieve the latter we apply Lemma 9. Having done this we apply the bounds (1.5) and (1.4) of Theorem H to each resulting hermitian sum. Note that in our case we have $|\alpha_m| \le 1$, $|\beta_n| \le 1$ and, after insertion of these trivial bounds, the exponents of M and N in (1.4) (in addition to those of (1.5)) are non-negative. Thus the worst bounds occur for large M and N. Integrating and summing these bounds over the relevant residue classes and dyadic boxes we arrive at (5.2) and (5.3).

We choose z to balance the bounds (5.2), (5.3), namely

$$z = (a+x)^{-\frac{6}{59}} x^{\frac{35}{59}} \tag{5.4}$$

getting

$$S_3 \ll (a+x)^{\frac{47}{118}} x^{\frac{35}{59}+\varepsilon}. \tag{5.5}$$

It remains to estimate S_1 and choose y. We write

$$S_1 = \sum_{\substack{n \le y \\ (n,r)=1}} \left(\frac{n}{a}\right) S_1(n) \tag{5.6}$$

where by (2.15)

$$S_1(n) = \sum_{\substack{m \le x/n \\ mn \equiv b \pmod r}} \left(\frac{m}{a}\right) e\left(4a\,\frac{\overline{m}}{n} - \frac{2a}{mn}\right).$$

We split the summation into residue classes modulo rn getting

$$S_1(n) = \sideset{}{^*}\sum_{\beta \pmod{rn}} e\left(4a\,\frac{\overline{\beta}}{n}\right) \sum_{\substack{m \le x/n \\ m \equiv \beta \pmod{rn}}} \left(\frac{m}{a}\right) e\left(\frac{-2a}{mn}\right). \tag{5.7}$$

We write $a = a_1 a_2$ where $(a_1, r) = 1$ and $a_2 \mid r^{\infty}$. Therefore

$$\left(\frac{m}{a}\right) = \left(\frac{m}{a_1}\right)\left(\frac{\beta}{a_2}\right). \tag{5.8}$$

Now, (5.8) indicates that the case where a_1 is a square is different because there is no cancellation in the sum over m. By the Polya-Vinogradov inequality

$$\mathscr{M}(t) = \sum_{\substack{m \le t \\ m \equiv \beta \pmod{rn}}} \left(\frac{m}{a_1}\right) = \delta(a_1)\frac{\varphi(a_1)t}{a_1 rn} + O\left(a_1^{\frac{1}{2}} \log 2a_1\right) \tag{5.9}$$

since rn is co-prime to a_1. By partial summation we attach to this the factor $e(-2a/mn)$ getting for the sum over m in (5.7)

$$\begin{aligned}
\sum_m &= \sum_{m>T} + \sum_{m \le T} \\
&= \int_T^{x/n} e(-2a/tn)\, d\mathscr{M}(t) + O(1 + T/rn) \\
&= \delta(a_1)\tfrac{\varphi(a_1)}{a_1 rn} \int_T^{x/n} e(-2a/tn)\,dt \\
&\qquad + O\left(\left(1 + \int_T^{\infty} an^{-1}t^{-2}dt\right) a^{\frac{1}{2}} \log 2a + T/rn\right) \\
&= 2\delta(a_1)\tfrac{\varphi(a_1)a_2}{rn^2} E(x/2a) + O\left(a^{\frac{1}{2}} \log 2a + T^{-1}n^{-1}a^{\frac{3}{2}} \log 2a + T(rn)^{-1}\right) \\
&= 2\delta(a_1)\tfrac{\varphi(a_1)a_2}{rn^2} E(x/2a) + O\left(a^{\frac{1}{2}} \log 2a + r^{-\frac{1}{2}}n^{-1}a^{\frac{3}{4}} \log 2a\right).
\end{aligned}$$

Summing this over β we get by (5.7) and

$$\sideset{}{^*}\sum_{\beta \pmod{rn}} \left(\frac{\beta}{a_2}\right) e\left(4a\,\frac{\overline{\beta}}{n}\right) = \mu(n)\varphi(r)\delta(a_2) \tag{5.10}$$

that

(5.11) $S_1(n) = 2\delta(a)\varphi(a_1)a_2\dfrac{\varphi(r)}{r}\dfrac{\mu(n)}{n^2}E(x/2a)+O\left(rna^{\frac{1}{2}}\log 2a+r^{\frac{1}{2}}a^{\frac{3}{4}}\log 2a\right).$

Note that the main term exists only if a is a square. Summing this over n we get by (5.6) and

$$\sum_{\substack{n\leq y\\(n,r)=1}}\mu(n)\left(\frac{n}{a}\right)n^{-2}=\zeta(2)^{-1}\prod_{p|ar}(1-p^{-2})^{-1}+O(y^{-1}) \tag{5.12}$$

that

$$\begin{aligned}S_1=12\pi^{-2}\quad&\delta(a)\varphi(a_1)a_2\prod_{p|r}\left(1-\tfrac{1}{p}\right)\prod_{p|ar}(1-p^{-2})^{-1}E(x/2a)\\&+O\left(xy^{-1}+ry^2a^{\frac{1}{2}}\log 2a+r^{\frac{1}{2}}ya^{\frac{3}{4}}\log 2a\right).\end{aligned} \tag{5.13}$$

Now we balance (5.13) with (5.3) with respect to y choosing

$$y=\min\left\{r^{-\frac{2}{5}}a^{-\frac{1}{5}}x^{\frac{2}{5}},\ r^{-\frac{1}{3}}a^{-\frac{1}{2}}x^{\frac{2}{3}}\right\}. \tag{5.14}$$

By (5.13) and (5.14) the error term in (5.13) is bounded by

$$O\left(r^{\frac{1}{5}}a^{\frac{1}{10}}x^{\frac{4}{5}}\log 2a+r^{\frac{1}{6}}a^{\frac{1}{4}}x^{\frac{2}{3}}\log 2a\right). \tag{5.15}$$

Combining this with (5.1), (5.3), (5.4), (5.5), (5.14) and (5.15) we obtain that the error term in S is bounded by

$$O\left(r^{\frac{1}{5}}(a+x)^{\frac{47}{118}}x^{\frac{35}{59}+\varepsilon}\right) \tag{5.16}$$

and the main term is equal to $24\pi^{-2}\varepsilon_b\delta(a)\psi(ar)aE(x/2a)$. This completes the proof of Theorem 4.

6. Appendix 1

In this section we sketch the proof of a bound for incomplete Kloosterman sums in the form we have applied in the proofs of Theorem 2 and Theorem 7.

Lemma 8. *Let I be a segment of an arithmetic progression*

$$I=\{x:X'<x\leq X'+X,\ x\equiv\ell \pmod k\}, \tag{6.1}$$

χ a Dirichlet character to modulus c with $(c,k)=1$ and a, b integers. Then we have

$$\left|\sum_{x\in I}\overline{\chi}(x)e\left(\frac{a\overline{x}+bx}{c}\right)\right|\leq\left(\frac{X}{ck}+2\log 3c\right)(a,c)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c). \tag{6.2}$$

Proof. We begin with the well-known estimate for the complete sum in the form

$$|S_\chi(a,b;c)| \le (a,c)^{\frac12} c^{\frac12} \tau(c) \tag{6.3}$$

which for c prime is due to A. Weil [W] and in the general case can be deduced therefrom by elementary techniques, cf. T. Estermann [E]. We combine this with the Erdös-Turán inequality

(6.4)
$$\Big|\sum_{X'<x\le X'+X} f(x)\Big| \le \frac{X+1}{c}\Big|\sum_{x \pmod c} f(x)\Big| + \sum_{1\le|h|\le\frac{c}{2}} |h|^{-1}\Big|\sum_{x \pmod c} f(x) e\left(\frac{hx}{c}\right)\Big|$$

where $f(x)$ is any periodic function of period c. In this form (6.4) follows for instance from two applications of (3.14) of [I]. Taking $x = \ell + ky$ we derive from (6.4) the following formula:

$$\Big|\sum_{x\in I} f(x)\Big| \le \frac{X+k}{ck}\Big|\sum_{x \pmod c} f(x)\Big| + \sum_{1\le|h|\le\frac{c}{2}} |h|^{-1}\Big|\sum_{x \pmod c} f(x) e\left(\frac{h\overline{k}x}{c}\right)\Big|. \tag{6.5}$$

We apply this for $f(x) = \chi(x) e((a\overline{x} + bx)/c)$ showing that the sum in (6.2) is

$$\begin{aligned} &\le \tfrac{X+k}{ck} |S_\chi(a,b;c)| + \sum_{1\le|h|\le\frac{c}{2}} |h|^{-1} |S_\chi(a, b+h\overline{k}; c)| \\ &\le \Big(\tfrac{X+k}{ck} + \sum_{1\le|h|\le\frac{c}{2}} |h|^{-1}\Big)(a,c)^{\frac12} c^{\frac12} \tau(c) \end{aligned}$$

giving (6.2).

7. Appendix 2

In this section we give a result which is useful for separating integral variables m, n constrained by an inequality of type $mn \le x$. Put

$$f(u) = \min\{u, 1, [x]+1-u\}$$

on $0 \le u \le [x]+1$ and equal to zero elsewhere. Therefore, for a positive integer k we have

$$f(k) = \begin{cases} 1 & \text{if } k \le x \\ 0 & \text{otherwise}. \end{cases}$$

On the other hand $f(u)$ is given by the inverse Mellin transform

$$f(u) = \frac{1}{2\pi i}\int_{(0)} g(s) u^{-s}\, ds$$

with

$$\begin{aligned} g(s) &= \int_0^\infty f(u) u^{s-1} du \\ &= \tfrac{1}{s}\Big(\int_{[x]}^{[x+1]} u^s du - \int_0^1 u^s du\Big) \\ &= \tfrac{1}{s(s+1)}([x+1]^{s+1} - [x]^{s+1} - 1). \end{aligned}$$

These three expressions show that on Re $s = 0$,

$$|g(s)| \le \min\left\{1 + \log x,\ \frac{2}{|s|},\ \frac{2(x+1)}{|s(s+1)|}\right\}.$$

Hence

$$\frac{1}{2\pi}\int_{(0)} |g(s)ds| < \log 6x.$$

To see this integrate separately over the intervals $[0,1)$, $[1,x)$, $[x,\infty)$ getting

$$1 + \log x + 2\log x + 2(x+1)x^{-1} \le 5 + 3\log x < \pi \log 6x.$$

This yields

Lemma 9. *For $x \ge 1$ there exists a function $h(t)$ such that*

$$\int_{-\infty}^{\infty} |h(t)|dt < \log 6x \tag{7.1}$$

and for every positive integer k

$$\int_{-\infty}^{\infty} h(t)k^{it}dt = \begin{cases} 1 & \text{if } k \le x \\ 0 & \text{otherwise}. \end{cases} \tag{7.2}$$

Similar arguments apply to the separation of positive integral variables m, n constrained by an inequality of the type $\frac{m}{n} \le y$. Suppose $n \le N$ so the distinct points $\frac{m}{n}$ are spaced by $\ge N^{-2}$. Therefore the condition $\frac{m}{n} \le y$ is equivalent to $\frac{m}{n} \le u$ for any $u \in [x, x + N^{-2})$ with $x = \sup\left\{\frac{m}{n} \le y : n \le N\right\}$. Hence we deduce

Lemma 9′. *For any $y > 0$ and $N \ge y^{-1/2}$ there exists a function $h(t)$ such that*

$$\int_{-\infty}^{\infty} |h(t)|dt < \log(6yN^2) \tag{7.3}$$

and for all positive integers m, n with $n \le N$

$$\int_{-\infty}^{\infty} h(t)\left(\frac{m}{n}\right)^{it}dt = \begin{cases} 1 & \text{if } \frac{m}{n} \le y \\ 0 & \text{otherwise}. \end{cases} \tag{7.4}$$

In fact Lemma 9′ contains Lemma 9; put $N = 1$ and $y = x$. The main point to these variants of well-known lemmas such as Perron's formula is that (7.2) and (7.4) contain no error terms.

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