

ASYMPTOTIC DISTRIBUTION OF TRACES OF SINGULAR MODULI

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ABSTRACT. We determine the asymptotic behavior of twisted traces of singular moduli with a power-saving error term in both the discriminant and the order of the pole at $i\infty$. Using this asymptotic formula, we obtain an exact formula for these traces involving the class number and a finite sum involving the exponential function evaluated at CM points.

1. INTRODUCTION

The modular j -function

$$\begin{aligned} j(z) &= q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24} \left(1 + 240 \sum_{n=1}^{\infty} \sum_{m|n} m^3 q^n \right)^3 \\ &= q^{-1} + 744 + 196884q + \dots = \sum_{n=-1}^{\infty} c(n)q^n, \quad q = e(z) = e^{2\pi iz}, \end{aligned}$$

is of fundamental importance in number theory. These Fourier coefficients $c(n)$ are integral linear combinations of the dimensions of the irreducible representations of the monster group¹ and its values at imaginary quadratic irrationalities are algebraic integers with well-known properties in class field theory.

In the 1930s, Petersson [19] and Rademacher [21] independently discovered the formula

$$c(n) = 2\pi n^{-\frac{1}{2}} \sum_{c=1}^{\infty} \frac{K(n, c)}{c} I_1\left(\frac{4\pi\sqrt{n}}{c}\right), \quad (1.1)$$

where I_ν is the I -Bessel function and $K(n, c)$ is the ordinary Kloosterman sum

$$K(n, c) = \sum_{\substack{d \bmod c \\ (c, d)=1}} e\left(\frac{\bar{d} + nd}{c}\right), \quad d\bar{d} \equiv 1 \pmod{c}.$$

In his paper, Rademacher remarked that without (1.1), the $c(n)$ “can be found... by troublesome computations, which for higher n are practically inexecutable.”² The convergence of (1.1) is quite slow. However, (in principle, at least) the formula can be used to compute $c(n)$ using a finite number of terms of the series since one knows a priori that $c(n) \in \mathbb{Z}$. The Weil bound $|K(n, c)| \leq \sigma_0(c)\sqrt{c}$, where σ_0 is the sum of divisors function, is enough to show that $C\sqrt{n}$ terms suffice as long as n is large enough compared to C .

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¹For a comprehensive survey of this and related results, see [12] and the many references therein.

²He also commented that the coefficients “do not seem to have attracted much attention before.” Certainly the state of the subject has changed somewhat in the intervening years.

Also of great interest are certain special values of j . For each negative fundamental discriminant d , let z_d be the point in the complex upper half-plane \mathbb{H} given by

$$z_d = \begin{cases} \frac{1}{2}\sqrt{d} & \text{if } d \equiv 0 \pmod{4}, \\ -\frac{1}{2} + \frac{1}{2}\sqrt{d} & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

From the classical theory of complex multiplication we know that the singular moduli $j_1(z_d)$, where $j_1 = j - 744$, are algebraic integers of degree $h(d)$, the class number of $\mathbb{Q}(\sqrt{d})$. In particular, the algebraic trace $\text{Tr } j_1(z_d)$, i.e. the sum of the Galois conjugates of $j_1(z_d)$, is an integer.

Kaneko [14, 15] expressed the coefficients $c(n)$ in terms of the traces $\text{Tr } j_1(z_d)$ using a result of Zagier [24], who showed that the traces appear as coefficients of weakly holomorphic modular forms of half-integral weight. Zagier's results sparked significant interest in traces of singular moduli which has led to numerous papers on the subject (of which we will only mention a few). Bruinier, Jenkins, and Ono [6] proved a formula like (1.1) for the traces involving half-integral weight Kloosterman sums, and computations led them to conjecture the limit

$$\lim_{d \rightarrow -\infty} \frac{1}{h(d)} \left(\text{Tr } j_1(z_d) - \sum_{z_Q \in \mathcal{R}(1)} e(-z_Q) \right) = -24, \quad (1.2)$$

where $\mathcal{R}(Y)$ is the rectangle

$$\mathcal{R}(Y) = \left\{ z = x + iy \in \mathbb{H} : -\frac{1}{2} \leq x < \frac{1}{2} \text{ and } y > Y \right\}.$$

Their conjecture was confirmed by the second author in [9]. The limit (1.2) converges very slowly. We will see in Corollary 2.3 below that by modifying and lengthening the exponential sum we can obtain a more quickly convergent limit; in particular for any $\varepsilon > 0$, the trace $\text{Tr } j_1(z_d)$ is the nearest integer to the quantity

$$-24h(d) + \sum_{z_Q \in \mathcal{R}(|d|^{-1+\varepsilon})} (e(-z_Q) - e(-\bar{z}_Q)) \quad (1.3)$$

as long as $|d|$ is sufficiently large with respect to ε . For example, when $d = -303$, we have $h(d) = 10$ and

$$\text{Tr } j_1(z_{-303}) = -561\,766\,949\,784\,377\,042\,888\,940,$$

while (1.3) with $\varepsilon = \frac{1}{100}$ equals

$$-561\,766\,949\,784\,377\,042\,888\,939.643\dots$$

In our main theorems we treat traces of any weakly holomorphic modular form with integer coefficients and give estimates that are uniform in the order of its pole at $i\infty$. We also allow twists by genus characters. The main series from which these results originate ((3.3) below) resembles (1.1), except that it involves half-integral weight Kloosterman sums. However, for our results it is not enough to use the Weil bound for Kloosterman sums to prove either (1.2) or our improved results. The proof of (1.2) in [9] uses the uniform distribution of CM points [8], but to obtain stronger estimates it is essential to measure the cancellation among the Kloosterman sums directly.

2. STATEMENT OF RESULTS

Our results improve and generalize the limit formula (1.2). We begin by fixing notation and explaining the more general setting. Each conjugate of $j_1(z_d)$ is of the form $j_1(z_Q)$, where $Q = [a, b, c]$ is a positive definite integral binary quadratic form of discriminant $d = b^2 - 4ac$, and

$$z_Q = \frac{-b + \sqrt{d}}{2a}.$$

In fact, we can choose $z_Q \in \mathcal{F}$, where

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \operatorname{Re}(z) \leq 0 \text{ and } |z| \geq 1 \right\} \cup \left\{ z \in \mathbb{H} : 0 < \operatorname{Re}(z) < \frac{1}{2} \text{ and } |z| > 1 \right\}$$

is the usual fundamental domain for the action of $\operatorname{PSL}_2(\mathbb{Z})$ on \mathbb{H} . This point of view leads to a straightforward generalization to non-fundamental discriminants $d \equiv 0, 1 \pmod{4}$ by defining

$$\operatorname{Tr} j_1(z_d) = \sum'_{z_Q \in \mathcal{F}} j_1(z_Q),$$

where Q runs over all positive definite integral binary quadratic forms of discriminant d with $z_Q \in \mathcal{F}$. The primed sum indicates that terms are weighted by $1/\omega_Q$, where $\omega_Q = 1$ unless Q is $\operatorname{PSL}_2(\mathbb{Z})$ -equivalent to $[a, 0, a]$ or $[a, a, a]$, in which case it equals 2 or 3, respectively.

The main result of [9] determines the asymptotic behavior of $\operatorname{Tr} f(z_d)$ for any $f \in \mathbb{C}[j]$. A convenient basis for $\mathbb{C}[j]$ is given by the functions $j_m = P_m(j)$, where $P_m(x)$ are the Faber polynomials defined by the property that $j_m = q^{-m} + O(q)$. The first few basis elements are $j_0 = 1$, $j_1 = j - 744$, and

$$\begin{aligned} j_2 &= j^2 - 1488j + 159768 \\ j_3 &= j^3 - 2232j^2 + 1069956j - 36866976. \end{aligned}$$

We generalize the results of [9] by considering sums twisted by genus characters. For any factorization $D = dd'$ of the negative discriminant D , where d is a (positive or negative) fundamental discriminant and d' is a discriminant, there is an associated character

$$\chi_d(Q) = \begin{cases} \left(\frac{d}{n}\right) & \text{if } (a, b, c, d) = 1 \text{ and } Q \text{ represents } n \text{ and } (d, n) = 1, \\ 0 & \text{if } (a, b, c, d) > 1. \end{cases}$$

We say that Q represents n if $n = ax^2 + bxy + cy^2$ for some $x, y \in \mathbb{Z}$. We define the twisted traces of j_m by

$$\operatorname{Tr}_d j_m(z_D) = \sum'_{z_Q \in \mathcal{F}} \chi_d(Q) \omega_Q^{-1} j_m(z_Q).$$

Let $\delta_1 = 1$ and $\delta_d = 0$ otherwise, and let $\sigma_1(m)$ denote the sum of the divisors of m . Finally, let $\theta \in [0, \frac{7}{64}]$ be an admissible exponent toward the Ramanujan conjecture for Maass cusp forms of integral weight.

Theorem 2.1. *For each negative discriminant D , let d be any fundamental discriminant dividing D . Then for each $m \geq 1$ we have*

$$\operatorname{Tr}_d j_m(z_D) - \sum_{z_Q \in \mathcal{R}(\frac{1}{m})} \chi_d(Q) e(-mz_Q) = -24\delta_d \sigma_1(m) h(D) + O(|D|^{\frac{17}{36} + \epsilon} m^{\frac{5}{6} + \frac{\theta}{3} + \epsilon}). \quad (2.1)$$

In Theorem 2.1 we have subtracted the quantity $\sum \chi_d(Q) e(-mz_Q)$ to provide an easier comparison with (1.2), but our methods are optimized for the following modification.

Theorem 2.2. *Let D , d , and m be as in Theorem 2.1. Then for $0 < Y \ll \frac{1}{m}$ we have*

$$\begin{aligned} \mathrm{Tr}_d j_m(z_D) - \sum_{z_Q \in \mathcal{R}(Y)} \chi_d(Q) (e(-mz_Q) - e(-m\bar{z}_Q)) \\ = -24\delta_d \sigma_1(m) h(D) + O\left(m|D|^{\frac{1}{3}} Y^{\frac{1}{2}} \left(Y^{-\frac{1}{6}} + |D|^{\frac{5}{36}} m^{\frac{1}{3}(1+\theta)}\right) (m|D|/Y)^\varepsilon\right). \end{aligned}$$

Choosing Y small enough that the error term tends to zero, we obtain the following.

Corollary 2.3. *Let D , d and m be as in Theorem 2.1 and let $Y = Cm^{-A}|D|^{-B}$, where $A > 3$, $B > 1$, and $C > 0$ are constants. Then $\mathrm{Tr}_d j_m(z_D)$ is the nearest integer to*

$$-24\delta_d \sigma_1(m) h(D) + \sum_{z_Q \in \mathcal{R}(Y)} \chi_d(Q) (e(-mz_Q) - e(-m\bar{z}_Q))$$

provided $m|D|$ is sufficiently large compared to C .

We prove Theorem 2.2 by obtaining a hybrid estimate for sums of half-integral weight Kloosterman sums associated with the Kohnen plus space on $\Gamma_0(4)$. Much of the heavy lifting that leads to this estimate was done in the authors' recent work [2] studying real quadratic analogues of traces of singular moduli. In that paper, we obtained asymptotic formulas for averages of two types of real quadratic geometric invariants: integrals of j_m over modular surfaces and contour integrals of j_m along the boundaries of the surfaces. Essential here and in [2] is a variant of Kuznetsov's formula relating Kloosterman sums to products $\rho_j(d)\rho_j(d')$ of coefficients of Maass forms in Kohnen's plus space, where $D = dd'$. The main significant difference is that here d and d' have opposite sign, while in the real quadratic case d and d' have the same sign.

3. FROM TRACES TO QUADRATIC WEYL SUMS

We begin by relating the twisted traces of singular moduli to the quadratic Weyl sums

$$T_m(d, d'; c) = \sum_{\substack{b \pmod{c} \\ b^2 \equiv D(c)}} \chi_d\left(\left[\frac{c}{4}, b, \frac{b^2 - D}{c}\right]\right) e\left(\frac{2mb}{c}\right), \quad (3.1)$$

where, as in the introduction, $D = dd'$. In [11, (4.10)–(4.11)] the function $j_m(z)$ is expressed in terms of (the analytic continuation of) a Poincaré series $G_{-m}(z, s)$ evaluated at $s = 1$:

$$j_m(z) = G_{-m}(z, 1) - 24\sigma_1(m).$$

Thus by Proposition 4 of [11] we have³

$$\mathrm{Tr}_d j_m(z_D) = -24\delta_d \sigma_1(m) h(D) + \pi(2m)^{\frac{1}{2}} |D|^{\frac{1}{4}} \lim_{s \rightarrow 1^+} \sum_{4|c} \frac{T_m(d, d'; c)}{\sqrt{c}} I_{s-\frac{1}{2}}\left(\frac{4\pi m}{c} |D|^{\frac{1}{2}}\right), \quad (3.2)$$

where I_ν is the I -Bessel function. In Section 4 we will prove the following estimate for averages of the Weyl sums $T_m(d, d'; c)$.

³In [11] the Weyl sums are denoted $S_m(d, d'; c)$ but we have used the notation $T_m(d, d'; c)$ to avoid confusion with the Kloosterman sums in the next section.

Theorem 3.1. *Suppose that $D = dd'$ is a negative discriminant and that d is a fundamental discriminant. Then for any $m \geq 1$ we have*

$$\sum_{4|c \leq x} \frac{T_m(d, d'; c)}{\sqrt{c}} \ll \left(x^{\frac{1}{6}} + |D|^{\frac{2}{9}} m^{\frac{1}{3}(1+\theta)} \right) (m|D|x)^\varepsilon.$$

Theorem 3.1 and [7, (10.29.1) and (10.30.1)] together justify exchanging the sum and the limit in (3.2), so we conclude that

$$\mathrm{Tr}_d j_m(z_D) = -24\delta_d \sigma_1(m) h(D) + \sum_{4|c} T_m(d, d'; c) \sinh\left(\frac{4\pi m}{c} |D|^{\frac{1}{2}}\right). \quad (3.3)$$

This formula generalizes [9, (12)]. Suppose that $x \gg m|D|^{1/2}$. For the tail of the c -sum in (3.3) we have, by partial summation,

$$\sum_{4|c \geq x} T_m(d, d'; c) \sinh\left(\frac{4\pi m}{c} |D|^{\frac{1}{2}}\right) \ll m|D|^{\frac{1}{2}} x^{-\frac{1}{2}} \left(x^{\frac{1}{6}} + |D|^{\frac{2}{9}} m^{\frac{1}{3}(1+\theta)} \right) (m|D|x)^\varepsilon. \quad (3.4)$$

Setting $x = \frac{2}{Y}|D|^{1/2}$, Theorem 2.2 will follow after we relate the terms $c < x$ in (3.3) to the CM points $z_Q \in \mathcal{R}(Y)$.

Lemma 3.2. *For any $Y > 0$ and $\xi \in \{-1, 1\}$ we have*

$$\frac{1}{2} \sum_{4|c < \frac{2}{Y}|D|^{1/2}} T_m(d, d'; c) \exp\left(\xi \frac{4\pi m}{c} |D|^{\frac{1}{2}}\right) = \sum_{z_Q \in \mathcal{R}(Y)} \chi_d(Q) e(-mz_{Q,\xi}), \quad (3.5)$$

where $z_{Q,\xi} = z_Q$ when $\xi = 1$ and $z_{Q,\xi} = \bar{z}_Q$ when $\xi = -1$.

Proof. Let $x = \frac{2}{Y}|D|^{1/2}$. Replacing c by $4c$ and using the definition (3.1) we have

$$\frac{1}{2} \sum_{c < \frac{x}{4}} T_m(d, d'; 4c) \exp\left(\xi \frac{\pi m}{c} |D|^{\frac{1}{2}}\right) = \sum_{c < \frac{x}{4}} \sum_{\substack{b \bmod 2c \\ b^2 \equiv D(4c)}} \chi_d\left(\left[c, b, \frac{b^2 - D}{4c}\right]\right) e\left(\frac{-m(-b + i\xi\sqrt{|D|})}{2c}\right),$$

where we have used that the b -summands are unchanged by replacing b by $b+2c$. The quadratic forms Q of discriminant D with $-\frac{1}{2} \leq \mathrm{Re}(z_Q) < \frac{1}{2}$ are in one-to-one correspondence with pairs of integers (b, c) for which $-c < b \leq c$ and $\frac{b^2 - D}{4c} \in \mathbb{Z}$. Furthermore, $\mathrm{Im}(z_Q) = \frac{1}{2c}|D|^{1/2}$, so the condition $c < \frac{x}{4}$ is equivalent to $\mathrm{Im}(z_Q) > \frac{2}{x}|D|^{1/2}$. Thus the left-hand side of (3.5) equals

$$\sum_{z_Q \in \mathcal{R}(\frac{2}{x}|D|^{1/2})} \chi_d(Q) e(-mz_{Q,\xi}),$$

from which the lemma follows after replacing x by $\frac{2}{Y}|D|^{1/2}$. \square

Proof of Theorem 2.2. By (3.3), Lemma 3.2, and (3.4) with $x = \frac{2}{Y}|D|^{1/2}$, we find that

$$\begin{aligned} \mathrm{Tr}_d j_m(z_D) &= -24\delta_d \sigma_1(m) h(D) + \sum_{z_Q \in \mathcal{R}(Y)} \chi_d(Q) (e(-mz_Q) - e(-m\bar{z}_Q)) \\ &\quad + O\left(m|D|^{\frac{1}{3}} Y^{\frac{1}{2}} \left(Y^{-\frac{1}{6}} + |D|^{\frac{5}{36}} m^{\frac{1}{3}(1+\theta)}\right) (m|D|/Y)^\varepsilon\right), \end{aligned}$$

as desired. \square

Proof of Theorem 2.1. Setting $Y = 1/m$ in Theorem 2.2, we obtain (2.1) with $e(-mz_Q)$ replaced by $e(-mz_Q) - e(-m\bar{z}_Q)$. By Lemma 3.2 it remains to estimate the sum

$$\sum_{c \leq 2m|D|^{1/2}} T_m(d, d'; c) \exp\left(-\frac{4\pi m}{c}|D|^{1/2}\right). \quad (3.6)$$

A straightforward argument involving Theorem 3.1 and partial summation shows that the sum in (3.6) is $\ll |D|^{\frac{17}{36} + \varepsilon} m^{\frac{5}{6} + \frac{\theta}{3} + \varepsilon}$. \square

4. PLUS-SPACE KUZNETSOV TRACE FORMULA WITH OPPOSITE SIGNS

The estimate in Theorem 3.1 will follow from an estimate for sums of Kloosterman sums of half-integral weight associated to Kohnen's plus space on $\Gamma_0(4)$. For additional background see Section 3 of [2]. Let $k = \pm\frac{1}{2} = \lambda + \frac{1}{2}$ and suppose that $(-1)^\lambda m, (-1)^\lambda n \equiv 0, 1 \pmod{4}$. The plus-space Kloosterman sums are

$$S_k^+(m, n; c) = e\left(-\frac{k}{4}\right) \sum_{d \pmod{c}} \left(\frac{c}{d}\right) \varepsilon_d^{2k} e\left(\frac{m\bar{d} + nd}{c}\right) \times \begin{cases} 1 & \text{if } 8 \mid c, \\ 2 & \text{if } 4 \parallel c, \end{cases}$$

where $d\bar{d} \equiv 1 \pmod{c}$ and $\varepsilon_d = 1$ or i according to $d \equiv 1$ or $3 \pmod{4}$, respectively. These are related to the quadratic Weyl sums via Kohnen's identity (Lemma 8 of [16])

$$T_m(d, d'; c) = \sum_{n \mid (m, \frac{c}{4})} \left(\frac{d}{c}\right) \sqrt{\frac{2n}{c}} S_{\frac{1}{2}}^+\left(d', \frac{m^2}{n^2}d; \frac{c}{n}\right). \quad (4.1)$$

Furthermore, we have the relation

$$S_k^+(m, n; c) = S_k^+(n, m; c) = S_{-k}^+(-m, -n; c). \quad (4.2)$$

Theorem 3.1 follows in a straightforward way from (4.1), (4.2), and the following theorem.

Theorem 4.1. *Let k, λ, m, n be as above, with the additional assumption that $m > 0, n < 0$, and $(-1)^\lambda m = dv^2, (-1)^\lambda n = d'w^2$, with d, d' fundamental discriminants. Then*

$$\sum_{4|c \leq x} \frac{S_k^+(m, n; c)}{c} \ll \left(x^{\frac{1}{6}} + |dd'|^{\frac{2}{9}}(vw)^{\frac{1}{3}(1+\theta)}\right) (|mn|x)^\varepsilon.$$

Individually, the Kloosterman sums satisfy the Weil bound

$$|S_k^+(m, n, c)| \leq 2\sigma_0(c) \gcd(m, n, c)^{\frac{1}{2}} \sqrt{c}, \quad (4.3)$$

see Lemma 6.1 of [10]. Thus the sum in Theorem 4.1 is trivially bounded above by $x^{1/2}|mnx|^\varepsilon$.

To prove Theorem 4.1 we will use a version of Kuznetsov's formula relating the Kloosterman sums $S_k^+(m, n; c)$ to Fourier coefficients of Maass cusp forms residing in the Kohnen plus-space. Let $\vartheta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z)$ denote the usual Jacobi theta function, and let ν_ϑ denote the associated multiplier system. Let $\Gamma = \Gamma_0(4)$ and $(k, \nu) = (\frac{1}{2}, \nu_\vartheta)$ or $(-\frac{1}{2}, \bar{\nu}_\vartheta)$. The plus-space \mathcal{V}_k^+ of Maass cusp forms of weight k for Γ is spanned by functions $u : \mathbb{H} \rightarrow \mathbb{C}$ satisfying

- $(\Delta_k + \frac{1}{4} + r^2)u = 0$, where $\Delta_k = y^2(\partial_x^2 + \partial_y^2) - ik y \partial_x$ is the hyperbolic Laplacian,
- $u(\gamma z) = \nu(\gamma) j(\gamma, z)^k u(z)$ for all $\gamma \in \Gamma$, where $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = \frac{cz+d}{|cz+d|}$,
- the Fourier coefficients of $u(z)$ are supported on exponents $\equiv 0, (-1)^\lambda \pmod{4}$,
- $\|u\| < \infty$, where $\|\cdot\|$ is the Petersson norm.

The quantity r is called the spectral parameter.

Once and for all we fix a spectrally normalized ($\|u\| = 1$) orthonormal basis $\{u_j\}_{j=0}^{\infty}$ of \mathcal{V}_k^+ consisting of eigenforms for the Hecke operators, ordered by eigenvalue $\frac{1}{4} + r_j^2$. The spectral parameter $r_0 = \frac{i}{4}$ corresponds to $u_0 = y^{1/4}\vartheta(z)$ or its conjugate. In either case, the product of the m -th and n -th Fourier coefficients of u_0 vanishes whenever $mn < 0$. Thus u_0 does not contribute to the Kuznetsov formula. For $j \geq 1$, Theorem 1.2 of [4] shows that there is a unique normalized Maass cusp form v_j of weight 0 with spectral parameter $2r_j$ which is even if $k = \frac{1}{2}$ and odd if $k = -\frac{1}{2}$, and such that the Hecke eigenvalues of u_j and v_j agree. Since each spectral parameter in weight 0 on $\mathrm{SL}_2(\mathbb{Z})$ is positive we find that $r_1 > 0$. For $j \geq 1$ we normalize the Fourier coefficients ρ_j of u_j by

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) W_{\frac{k}{2} \operatorname{sgn}(n), ir_j}(4\pi|n|y) e(nx),$$

where $W_{\mu, \nu}$ is the W -Whittaker function.

Kuznetsov proved his formula in [17] in the setting of integral weight on $\mathrm{SL}_2(\mathbb{Z})$. Later Proskurin [20] generalized Kuznetsov's formula to arbitrary weight and for Fuchsian groups of the first kind, but only when both inputs m, n are positive. Blomer [5, Proposition 2] proved a version of the Kuznetsov formula for $\nu \in \{\nu_\vartheta, \bar{\nu}_\vartheta\}$ with $mn < 0$. Using Blomer's result, and following the exact same procedure⁴ as in Section 5 of [2], we obtain the plus-space version for opposite signs, Theorem 4.2 below. We first fix some notation. Given a smooth test function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ satisfying

$$\varphi(0) = \varphi'(0) = 0 \quad \text{and} \quad \varphi^{(j)}(x) \ll x^{-2-\varepsilon} \quad \text{for } j = 0, 1, 2, 3, \quad (4.4)$$

define the integral transform

$$\check{\varphi}(r) = 2 \cosh(\pi r) \int_0^\infty K_{2ir}(x) \varphi(x) \frac{dx}{x},$$

where K_{2ir} is the K -Bessel function. If d is a fundamental discriminant, let $\chi_d = \left(\frac{d}{\cdot}\right)$ and let $L(s, \chi_d)$ denote analytic continuation of the Dirichlet L -function

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}.$$

Finally, we define

$$\mathfrak{S}_d(w, s) = \sum_{\ell|w} \mu(\ell) \frac{\chi_d(\ell)}{\sqrt{\ell}} \sum_{ab\ell=w} \left(\frac{a}{b}\right)^s.$$

Theorem 4.2. *Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a smooth test function satisfying (4.4). Let $k = \pm\frac{1}{2} = \lambda + \frac{1}{2}$. Suppose that $m > 0$, $n < 0$ with $(-1)^\lambda m, (-1)^\lambda n \equiv 0, 1 \pmod{4}$, and write*

$$(-1)^\lambda m = v^2 d', \quad (-1)^\lambda n = w^2 d, \quad \text{with } d, d' \text{ fundamental discriminants.}$$

⁴The most difficult part of the argument in [2] is the proof of Proposition 5.6, but that result already holds for all $m, n \equiv 0, 1 \pmod{4}$ with no sign restrictions.

Fix an orthonormal basis of Maass cusp forms $\{u_j\} \subset \mathcal{V}_k^+$ with associated spectral parameters r_j and coefficients $\rho_j(n)$. Then

$$\begin{aligned} \sum_{4|c>0} \frac{S_k^+(m, n, c)}{c} \varphi\left(\frac{4\pi\sqrt{|mn|}}{c}\right) &= 6\sqrt{|mn|} \sum_{j \geq 1} \frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh \pi r_j} \check{\varphi}(r_j) \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{d}{d'} \right|^{ir} \frac{L(\frac{1}{2} - 2ir, \chi_{d'})L(\frac{1}{2} + 2ir, \chi_d)\mathfrak{S}_{d'}(v, 2ir)\mathfrak{S}_d(w, 2ir)}{|\zeta(1 + 4ir)|^2 \cosh \pi r |\Gamma(\frac{1-k}{2} + ir)\Gamma(\frac{1+k}{2} - ir)|} \check{\varphi}(r) dr. \end{aligned}$$

5. PROOF OF THEOREM 4.1

Let $a = 4\pi\sqrt{|mn|}$ and $x \geq 3$ and let $1 \leq T \leq x/3$ be a free parameter to be chosen later. Following [22], we fix a test function $\varphi = \varphi_{a,x,T} : [0, \infty) \rightarrow [0, 1]$ satisfying

- (i) $\varphi(t) = 1$ for $\frac{a}{2x} \leq t \leq \frac{a}{x}$,
- (ii) $\varphi(t) = 0$ for $t \leq \frac{a}{2x+2T}$ and $t \geq \frac{a}{x-T}$,
- (iii) $\varphi'(t) \ll \left(\frac{a}{x-T} - \frac{a}{x}\right)^{-1} \ll \frac{x^2}{aT}$, and
- (iv) φ and φ' are piecewise monotonic on a fixed number of intervals (whose number is independent of a, x, T).

We apply the plus space Kuznetsov formula in Theorem 4.2 with this test function and we estimate each of the terms on the right-hand side. For this, we require an estimate for the integral transform $\check{\varphi}(r)$. All but the first estimate in the following theorem are proved in [1, Section 6]. There are some minor errors in that proof, and we have provided the corrections, along with the proof of the first estimate, in Appendix A.

Theorem 5.1. *Suppose that a, x, T , and $\varphi = \varphi_{a,x,T}$ are as above. Then*

$$\check{\varphi}(r) \ll \begin{cases} r^{-\frac{3}{2}} & \text{if } r \leq 1, \\ e^{-\frac{1}{2}r} & \text{if } 1 \leq r \leq \frac{a}{8x}, \\ r^{-1} & \text{if } \max\left(\frac{a}{8x}, 1\right) \leq r \leq \frac{a}{x}, \\ r^{-\frac{3}{2}} \min\left(1, \frac{x}{rT}\right) & \text{if } r \geq \max\left(\frac{a}{x}, 1\right). \end{cases}$$

We first give two estimates for the contribution from the Maass cusp forms

$$\mathcal{K}^m = \sqrt{|mn|} \sum_{j \geq 1} \frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh \pi r_j} \check{\varphi}(r_j).$$

The first estimate is Theorem 5.2 below, which we quote directly from [2, Section 6]. This estimate uses Young's [23] Weyl-type hybrid subconvexity estimate for central values of L -functions of Maass cusp forms of integral weight for $\mathrm{SL}_2(\mathbb{Z})$ twisted by Dirichlet characters (see also Appendix A of [2]).

Theorem 5.2. *Let $k = \pm\frac{1}{2} = \lambda + \frac{1}{2}$. Suppose that $m > 0$, $n < 0$ and write $(-1)^\lambda m = v^2 d'$ and $(-1)^\lambda n = w^2 d$ with d, d' fundamental discriminants. Then*

$$\sqrt{|mn|} \sum_{r_j \leq X} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll |dd'|^{\frac{1}{6}} (vw)^\theta X^2(|mn|X)^\epsilon.$$

Our second estimate comes from Section 4 of [2].

Theorem 5.3. *Suppose that $(k, \nu) = (\frac{1}{2}, \nu_\vartheta)$ or $(-\frac{1}{2}, \bar{\nu}_\vartheta)$. Then for all $n \neq 0$ we have*

$$n \sum_{X \leq r_j \leq 2X} |\rho_j(n)|^2 e^{-\pi r_j} \ll X^{-k \operatorname{sgn}(n)} \left(X^2 + |n|^{\frac{1}{2} + \varepsilon} \right) X^\varepsilon.$$

The estimation of the Maass cusp form contribution follows the general structure of the proof of Proposition 5 of [22] and the proof of Theorem 9.1 of [1]. We split the sum \mathcal{K}^m into three ranges

$$\mathcal{K}_1^m = \sum_{1 \leq r_j \ll \frac{a}{x}}, \quad \mathcal{K}_2^m = \sum_{r_j \asymp \frac{a}{x}}, \quad \mathcal{K}_3^m = \sum_{r_j \gg \frac{a}{x}},$$

corresponding to the second, third, and fourth ranges in Theorem 5.1. Recall that $a \asymp \sqrt{|mn|}$. For the first range we use Theorem 5.2 to obtain

$$\begin{aligned} \mathcal{K}_1^m &\ll \sqrt{|mn|} \sum_{r_j \geq 1} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} e^{-r_j/2} \\ &\ll \sqrt{|mn|} \sum_{T=1}^{\infty} e^{-T/2} \sum_{T \leq r_j \leq T+1} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll |dd'|^{\frac{1}{6}} (vw)^\theta |mn|^\varepsilon. \end{aligned}$$

In the second range, again using Theorem 5.2 we have

$$\mathcal{K}_2^m \ll \sqrt{|mn|} \sum_{r_j \asymp \frac{a}{x}} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} r_j^{-1} \ll x \sum_{r_j \asymp \frac{a}{x}} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll |dd'|^{\frac{2}{3}} (vw)^{1+\theta} x^{-1} (|mn|x)^\varepsilon.$$

To estimate \mathcal{K}_3^m we consider the dyadic sums

$$\mathcal{K}^m(A) = \sqrt{|mn|} \sum_{A \leq r_j < 2A} \frac{\overline{\rho_j(m)\rho_j(n)}}{\cosh \pi r_j} \check{\varphi}(r_j)$$

for $A \geq 1$. Applying Cauchy-Schwarz and Theorem 5.3 we obtain the estimate:

$$\sqrt{|mn|} \sum_{r_j \leq A} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll \left(A^2 + |mn|^{\frac{1}{4}} A \right) (|mn|A)^\varepsilon.$$

Together with Theorem 5.2, this implies that

$$\begin{aligned} \sqrt{|mn|} \sum_{A \leq r_j < 2A} \frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} &\ll \min \left(A^2 + |mn|^{\frac{1}{4}} A, |dd'|^{\frac{1}{6}} (vw)^\theta A^2 \right) (|mn|A)^\varepsilon \\ &\ll \left(A^2 + |dd'|^{\frac{5}{24}} (vw)^{\frac{1}{4} + \frac{1}{2}\theta} A^{\frac{3}{2}} \right) (|mn|A)^\varepsilon, \end{aligned}$$

where we have used that $\min(y, z) \leq (yz)^{1/2}$. By Theorem 5.1 it follows that

$$\mathcal{K}^m(A) \ll \min \left(1, \frac{x}{AT} \right) \left(A^{\frac{1}{2}} + |dd'|^{\frac{5}{24}} (vw)^{\frac{1}{4} + \frac{1}{2}\theta} \right) (|mn|A)^\varepsilon,$$

so by summing the dyadic pieces we obtain

$$\mathcal{K}_3^m \ll \left(x^{\frac{1}{2}} T^{-\frac{1}{2}} + |dd'|^{\frac{5}{24}} (vw)^{\frac{1}{4} + \frac{1}{2}\theta} \right) (|mn|x)^\varepsilon.$$

In total,

$$\mathcal{K}^m \ll \left(|dd'|^{\frac{2}{3}} (vw)^{1+\theta} x^{-1} + x^{\frac{1}{2}} T^{-\frac{1}{2}} + |dd'|^{\frac{5}{24}} (vw)^{\frac{1}{4} + \frac{1}{2}\theta} \right) (|mn|x)^\varepsilon. \quad (5.1)$$

We turn to the estimate of the integral

$$\mathcal{K}^e = \int_{-\infty}^{\infty} \left| \frac{d}{d'} \right|^{ir} \frac{L_{d'}(-r)L_d(r)\mathfrak{S}_{d'}(v, 2ir)\mathfrak{S}_d(w, 2ir)}{|\zeta(1+4ir)|^2 \cosh \pi r |\Gamma(\frac{1-k}{2} + ir)\Gamma(\frac{1+k}{2} - ir)|} \check{\varphi}(r) dr,$$

where, for brevity, we have written $L(\frac{1}{2} + 2ir, \chi_d) = L_d(r)$. By symmetry it suffices to estimate the integrals $\mathcal{K}_0^e = \int_0^1$ and $\mathcal{K}_1^e = \int_1^\infty$. Estimating the divisor sums trivially we find that $|\mathfrak{S}_d(w, s)| \leq \sigma_0(w)^2 \ll w^\varepsilon$. For $r \in (0, 1]$ we have

$$|\zeta(1+4ir)|^2 \gg r^{-2} \text{ and } \cosh \pi r |\Gamma(\frac{1-k}{2} + ir)\Gamma(\frac{1+k}{2} - ir)| \gg 1,$$

so by Theorem 5.1 we have the estimate

$$\mathcal{K}_0^e \ll (vw)^\varepsilon \int_0^1 |L_{d'}(r)L_d(r)| dr.$$

Since $\cosh \pi r |\Gamma(\frac{1-k}{2} + ir)\Gamma(\frac{1+k}{2} - ir)| \sim \pi$ and $|\zeta(1+4ir)|^{-1} \ll r^\varepsilon$ for large r we have by Theorem 5.1 that

$$\mathcal{K}_1^e \ll (vw)^\varepsilon \int_1^\infty |L_{d'}(r)L_d(r)| \frac{dr}{r^{3/2-\varepsilon}} + (vw)^\varepsilon \int_{a/(8x)}^{a/x} |L_{d'}(r)L_d(r)| \frac{dr}{r^{1-\varepsilon}}.$$

In the first integral we multiply each Dirichlet L -function by $r^{-3/8}$ and the last factor by $r^{3/4}$. Applying Hölder's inequality in the case $\frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1$ to both integrals, we obtain

$$\begin{aligned} \mathcal{K}_1^e &\ll (vw)^\varepsilon \left(\int_1^\infty |L_{d'}(r)|^6 \frac{dr}{r^{9/4}} \right)^{\frac{1}{6}} \left(\int_1^\infty |L_d(r)|^6 \frac{dr}{r^{9/4}} \right)^{\frac{1}{6}} \left(\int_1^\infty \frac{dr}{r^{9/8-\varepsilon}} \right)^{\frac{2}{3}} \\ &+ (vw)^\varepsilon \sum_{\frac{a}{8x} \leq T < \frac{a}{x}} \left(\int_T^{T+1} |L_{d'}(r)|^6 dr \right)^{\frac{1}{6}} \left(\int_T^{T+1} |L_d(r)|^6 dr \right)^{\frac{1}{6}} \left(\int_T^{T+1} \frac{dr}{r^{3/2-\varepsilon}} \right)^{\frac{2}{3}}. \end{aligned} \quad (5.2)$$

Young [23] proved that

$$\int_T^{T+1} |L_d(r)|^6 dr \ll (|d|(1+T))^{1+\varepsilon},$$

from which it follows that

$$\int_1^\infty |L_d(r)|^6 \frac{dr}{r^{9/4}} \leq \sum_{T=1}^\infty \frac{1}{T^{9/4}} \int_T^{T+1} |L_d(r)|^6 dr \ll |d|^{1+\varepsilon}$$

and

$$\sum_{\frac{a}{8x} \leq T < \frac{a}{x}} \left(\int_T^{T+1} |L_{d'}(r)|^6 dr \right)^{\frac{1}{6}} \left(\int_T^{T+1} |L_d(r)|^6 dr \right)^{\frac{1}{6}} \left(\int_T^{T+1} \frac{dr}{r^{3/2-\varepsilon}} \right)^{\frac{2}{3}} \ll |dd'|^{\frac{2}{3}+\varepsilon} (vw)^{1+\varepsilon} x^{-1+\varepsilon}.$$

We also have $\mathcal{K}_0^e \ll (vw)^\varepsilon |dd'|^{\frac{1}{6}+\varepsilon}$. These estimates, together with (5.2) show that

$$\mathcal{K}^e \ll |dd'|^{\frac{2}{3}+\varepsilon} (vw)^{1+\varepsilon} x^{-1+\varepsilon}. \quad (5.3)$$

Putting (5.1) and (5.3) together, we find that

$$\sum_{4|c>0} \frac{S_k^+(m, n, c)}{c} \varphi\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \ll \left(|dd'|^{\frac{2}{3}} (vw)^{1+\theta} x^{-1} + x^{\frac{1}{2}} T^{-\frac{1}{2}} + |dd'|^{\frac{5}{24}} (vw)^{\frac{1}{4}+\frac{1}{2}\theta} \right) (|mn|x)^\varepsilon.$$

To unsmooth the sum of Kloosterman sums, we argue as in [22, 1] to obtain

$$\sum_{4|c>0} \frac{S_k^+(m, n, c)}{c} \varphi\left(\frac{4\pi\sqrt{|mn|}}{c}\right) - \sum_{\substack{x \leq c < 2x \\ 4|c}} \frac{S_k^+(m, n, c)}{c} \ll \frac{T \log x}{\sqrt{x}} |mn|^\varepsilon.$$

Choosing $T = x^{\frac{2}{3}}$ we obtain

$$\sum_{x \leq c < 2x} \frac{S_k^+(m, n, c)}{c} \ll \left(x^{\frac{1}{6}} + |dd'|^{\frac{2}{3}}(vw)^{1+\theta}x^{-1} + |dd'|^{\frac{5}{24}}(vw)^{\frac{1}{4}+\frac{1}{2}\theta}\right) (|mn|x)^\varepsilon. \quad (5.4)$$

To prove Theorem 4.1 we sum the initial segment $c \leq |dd'|^\alpha(vw)^\beta$ and apply the Weil bound (4.3), then sum the dyadic pieces for $c \geq |dd'|^\alpha(vw)^\beta$ using (5.4). To balance the resulting terms we take $\alpha = \frac{4}{9}$ and $\beta = \frac{2}{3}(1 + \theta)$, which gives the bound

$$\sum_{c \leq x} \frac{S_k^+(m, n, c)}{c} \ll \left(x^{\frac{1}{6}} + |dd'|^{\frac{2}{9}}(vw)^{\frac{1}{3}(1+\theta)}\right) (|mn|x)^\varepsilon.$$

This completes the proof. \square

APPENDIX A. PROOF OF THEOREM 5.1

In this section we prove the first estimate of Theorem 5.1 and correct some mistakes in the proof given in [1, Section 6] for the other three estimates of that theorem. The mistakes in [1] all stem from a misunderstanding of the choice of branch cut implicit in Balogh's [3] uniform asymptotic expansion for the K -Bessel function $K_{iv}(vz)$ when $z \in (0, 1)$. The paper [13] provides a clear reference for this asymptotic expansion in both regions $z < 1$ and $z > 1$. Here we prove corrected statements of some of the propositions in Section 6 of [1]; the remaining steps of the proof given in that paper follow with only minor changes and are omitted here.

Balogh [3] (see also [13, Section 4]) gives a uniform asymptotic expansion for $K_{iv}(vz)$ in terms of the Airy function Ai and its derivative Ai' . For $z \in (0, 1)$ define

$$w(z) = \text{arccosh}\left(\frac{1}{z}\right) - \sqrt{1 - z^2} \quad \text{and} \quad \zeta = \left[\frac{3}{2}w(z)\right]^{\frac{2}{3}} > 0.$$

Taking $m = 1$ in equation (2) of [3] (see also [13, (12)]) we have

$$\begin{aligned} e^{\frac{\pi v}{2}} K_{iv}(vz) &= \frac{\pi\sqrt{2}}{v^{\frac{1}{3}}} \left(\frac{\zeta}{1 - z^2}\right)^{\frac{1}{4}} \left\{ \text{Ai}\left(-v^{\frac{2}{3}}\zeta\right) \left[1 + \frac{A(z)}{v^2}\right] \right. \\ &\quad \left. + \left(\frac{2}{3}\right)^{\frac{1}{3}} \text{Ai}'\left(-v^{\frac{2}{3}}\zeta\right) \frac{B(z)}{v(vw(z))^{\frac{1}{3}}}\right\} + O\left(\frac{v^{-\frac{7}{2}}}{(1 - z^2)^{\frac{1}{4}}}\right), \end{aligned} \quad (\text{A.1})$$

uniformly for $v \in [1, \infty)$, where

$$\begin{aligned} A(z) &= \frac{455}{10368 w(z)^2} - \frac{7(3z^2 + 2)}{1728(1 - z^2)^{\frac{3}{2}} w(z)} - \frac{81z^4 + 300z^2 + 4}{1152(1 - z^2)^3}, \\ B(z) &= \frac{3z^2 + 2}{24(1 - z^2)^{\frac{3}{2}}} - \frac{5}{72w(z)}. \end{aligned} \quad (\text{A.2})$$

Both $A(z)$ and $B(z)$ are $O(1)$ for $z \in (0, 1)$.

Proposition A.1. *Suppose that $z \in (0, 3/4]$ and that $v \geq 1$. Then*

$$e^{\frac{\pi v}{2}} K_{iv}(vz) = \frac{\sqrt{2\pi}}{v^{\frac{1}{2}}(1-z^2)^{\frac{1}{4}}} \left[\cos\left(vw(z) - \frac{\pi}{4}\right) + \sin\left(vw(z) - \frac{\pi}{4}\right) \frac{3z^2 + 2}{24v(1-z^2)^{\frac{3}{2}}} \right] + O(v^{-\frac{5}{2}}).$$

Proof. Let $c(v, z) = \cos(vw(z) - \pi/4)$ and $s(v, z) = \sin(vw(z) - \pi/4)$. By the asymptotic expansions [7, (9.7.9) and §9.7(iii)] we have

$$\text{Ai}(-v^{\frac{2}{3}}\zeta) = \pi^{-\frac{1}{2}} \left(\frac{3}{2}vw(z)\right)^{-\frac{1}{6}} \left\{ c(v, z) + \frac{5s(v, z)}{72vw(z)} + O(v^{-2}) \right\}, \quad (\text{A.3})$$

$$\left(\frac{2}{3}\right)^{\frac{1}{3}} \frac{\text{Ai}'(-v^{\frac{2}{3}}\zeta)}{(vw(z))^{\frac{1}{3}}} = \pi^{-\frac{1}{2}} \left(\frac{3}{2}vw(z)\right)^{-\frac{1}{6}} \{s(v, z) + O(v^{-1})\}. \quad (\text{A.4})$$

Thus by (A.1) we have

$$e^{\frac{\pi v}{2}} K_{iv}(vz) = \frac{\sqrt{2\pi}}{v^{\frac{1}{2}}(1-z^2)^{\frac{1}{4}}} \left\{ c(v, z) + \frac{s(v, z)}{v} \left(\frac{5}{72w(z)} + B(z) \right) \right\} + O(v^{-\frac{5}{2}}).$$

Then (A.2) shows that

$$\frac{5}{72w(z)} + B(z) = \frac{3z^2 + 2}{24(1-z^2)^{\frac{3}{2}}},$$

as desired. \square

We require some notation for the next proposition. Let $J_\nu(x)$ and $Y_\nu(x)$ denote the J and Y -Bessel functions, and define

$$M_\nu(x) = \sqrt{J_\nu^2(x) + Y_\nu^2(x)}.$$

Proposition A.2. *Suppose that $c > 0$. Suppose that $v \geq 1$ and that $\frac{3}{16} \leq z \leq 1 - cv^{-\frac{2}{3}}$. Then*

$$e^{\frac{\pi v}{2}} K_{iv}(vz) = \pi \frac{w(z)^{\frac{1}{2}}}{(1-z^2)^{\frac{1}{4}}} M_{\frac{1}{3}}^2(vw(z)) \sin\left(\theta_{\frac{1}{3}}(vw(z))\right) + O_c(v^{-4/3}),$$

where $\theta_{\frac{1}{3}}(x)$ is a real-valued continuous function satisfying

$$\theta'_{\frac{1}{3}}(x) = \frac{2}{\pi x M_{\frac{1}{3}}^2(x)}.$$

Proof. By [7, (9.8.1) and (9.8.9)] we have

$$\text{Ai}(-v^{\frac{2}{3}}\zeta) = \frac{1}{\sqrt{3}} v^{\frac{1}{3}} \zeta^{\frac{1}{2}} M_{\frac{1}{3}}^2(vw(z)) \sin\left[\theta\left(-\left(\frac{3}{2}vw(z)\right)^{\frac{2}{3}}\right)\right],$$

where $\theta(x)$ is given in [7, (9.8.11)]. Letting $\theta_{1/3}(x) = \theta(-(\frac{3}{2}x)^{2/3})$ we find from [7, (9.8.14)] that

$$\theta'_{\frac{1}{3}}(x) = -\left(\frac{3}{2}x\right)^{-\frac{1}{3}} \theta'\left(-\left(\frac{3}{2}vw(z)\right)^{\frac{2}{3}}\right) = \frac{2}{\pi M_{\frac{1}{3}}^2(x)}.$$

For $z \leq 1 - cv^{-2/3}$ we have $vw(z) \gg_c 1$. Thus it follows from (A.3) and (A.4) that after expanding (A.1), the terms containing $A(z)$ and $B(z)$ are both $\ll v^{-4/3}$. The proposition follows. \square

It remains to prove the first estimate of Theorem 5.1.

Proposition A.3. *Let φ be as in Theorem 5.1. If $r \leq 1$ then*

$$\check{\varphi}(r) \ll r^{-\frac{3}{2}}.$$

Proof. We begin by splitting the integral as

$$\check{\varphi}(r) = \check{\varphi}_1(r) + \check{\varphi}_2(r),$$

where

$$\check{\varphi}_j(r) = \cosh \pi r \int_{I_j} K_{2ir}(u) \varphi(u) \frac{du}{u},$$

with $I_1 = [\frac{a}{2(x+T)}, \frac{a}{x-T}] \cap [0, r]$ and $I_2 = [\frac{a}{2(x+T)}, \frac{a}{x-T}] \cap (r, \infty)$. The second piece $\check{\varphi}_2(r)$ can be estimated exactly as in the proof of Proposition 6.5 of [1] since $u \geq r$. For the first piece we use [7, (10.27.4) and (10.25.2)] to get

$$|K_{2ir}(u)| \leq \frac{\pi \cosh(2\pi r)^{1/2}}{\sinh(2\pi r)} I_0(u).$$

From this and the assumption that $r \leq 1$ we obtain

$$\check{\varphi}_1(r) \ll \frac{1}{r} \int_{\frac{a}{2(x+T)}}^{\frac{a}{x-T}} \frac{du}{u} \ll \frac{1}{r},$$

where in the last inequality we used that $T \leq x/3$. This completes the proof. \square

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