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The subconvexity problem for Artin L–functions

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1. Introduction

Among the arithmetically most interesting L-functions are those of Artin associated to Galois representations. Unfortunately, they are also among

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the least understood analytically. Given a Galois extension of number fields L/K with Galois group G and a complex representation ρ of G the Artin L-function $L(s, \rho)$ of ρ is a product over primes of K of local factors which, at unramified primes $\mathfrak{p}$ of K, have the form

$$\det\left(1 - \rho(\phi_{\mathfrak{p}})N(\mathfrak{p})^{-s}\right)^{-1}$$

where $\phi_{\mathfrak{p}}$ is any Frobenius element over $\mathfrak{p}$. The complete L-function $\Lambda(s, \rho)$, which includes gamma factors from the infinite places, is known to be meromorphic and satisfy the functional equation

$$\Lambda(1 - s, \bar{\rho}) = \epsilon D^{s-\frac{1}{2}}\Lambda(s, \rho)$$

where $\bar{\rho}$ is the contragredient of the representation ρ, D is a positive integer (the conductor) and $|\epsilon| = 1$. The degree of $L(s, \rho)$ is given by $n = \dim(\rho)[K : \mathbb{Q}]$ and, using induced characters, $L(s, \rho)$ may always be realized over $\mathbb{Q}$ by a representation of dimension n. In this case the determinant character $\det(\rho)$ induces, via class field theory, a Dirichlet character χ defined modulo D.

Artin's conjecture that $L(s, \rho)$ is entire unless ρ contains the trivial representation, is known only in special cases. Assuming its truth, the convexity principle of Phragmen-Lindelöf gives the bound

$$L(s, \rho) \ll_{\varepsilon,s} D^{\frac{1}{4}+\varepsilon}$$

for $\mathrm{Re}(s) = \frac{1}{2}$, while the Lindelöf hypothesis, itself a consequence of the GRH for $L(s, \rho)$, would replace the $\frac{1}{4}$ by 0. The subconvexity problem for $L(s, \rho)$ of interest here is to replace $\frac{1}{4}$ by $\frac{1}{4} - \delta$ for some absolute constant $\delta > 0$.

Artin L-functions of degree one over $\mathbb{Q}$ are Dirichlet L -functions modulo D (by the Kronecker-Weber Theorem) and the subconvexity problem was solved by Burgess in 1962. Direct arithmetic applications involve properties of the group $(\mathbb{Z}/D\mathbb{Z})^*$ for large D, a notable example being to estimate the smallest primitive root modulo D, when D is prime. In this paper we shall obtain subconvexity estimates for those Artin L-functions of degree two over $\mathbb{Q}$ which are known to be entire, provided that the conductors of the representation and its determinant coincide. After Artin, Hecke, Langlands and Tunnell, degree two Artin L-functions are known to be entire except possibly for icosahedral ρ. By class field theory, those $L(s, \rho)$ with dihedral ρ over $\mathbb{Q}$ are Hecke L–functions of a quadratic field. In particular, we solve the subconvexity problem for Hecke L-functions associated to class group characters of quadratic fields. In analogy to the degree one case we give applications to the class group, including the distribution of generators of small index cyclic subgroups.

It was conjectured by Langlands that every Artin L-function of degree n comes from an automorphic representation of $GL(n)$. For degree two L-functions this is known except in the icosahedral case and forms the basis for our approach, which is to apply the amplification method and take advantage of the orthogonality of neighboring automorphic forms. Our analysis makes use of the whole spectrum and we simultaneously obtain subconvexity estimates for all associated automorphic L-functions. In particular, the use of neighboring Maass forms appears to be indispensable for obtaining general subconvexity estimates for Hecke L-functions of a quadratic field, at least using presently known techniques. For a survey with references on the general subconvexity problem for L-functions see [IS].

We first succeeded in obtaining a subconvexity bound for Hecke L–functions associated to class group characters of imaginary quadratic fields in [DFI4] but not for every discriminant. For example, we were able to give results when the discriminant has no large prime factors. The method involves the consideration of weighted averages over the characters of the class group. If the class group is small then there are few of these and it should be expected to be difficult to average over them. Actually the class number is never small, but ineffectively so, as shown by Siegel, and the problem remains difficult. In the method of [DFI4] it manifests itself in that we require the existence of many small splitting primes, a problem closely related to the class number problem but even more poorly understood.

The class group L–functions for imaginary quadratic fields correspond to automorphic forms of weight one. In general, if the weight exceeds one, the space of holomorphic cusp forms is isolated from the rest of the spectrum and is sufficiently large that we were able to carry out our arguments within this space. This was accomplished in [DFI3] for forms with trivial nebentypus and in the more difficult case of primitive nebentypus, real or complex, in [DFI7]. The space for weight one is much smaller and it is expected only to be essentially the same size as the class group itself (see the conjecture of Serre [Se], and also [Du]). Fortunately, when considered within the space of all Maass forms, it again becomes part of a much larger family. Furthermore, we may also seamlessly treat the real quadratic Hecke L-functions, which also correspond to the bottom of the continuous spectrum but for weight zero, as well as treating other Artin L-functions. Note that for real quadratic fields the class number is expected to frequently be small so for these an extension of the earlier method of [DFI4] seems hopeless.

Among the new difficulties that arise in treating the problem in the context of automorphic L–functions, two of these appeared to be quite serious and merit specific mention here. The first of these is related to a counting problem for a determinant equation with general integer entries and this in turn requires the estimation of exponential sums of Kloosterman type. These results, which were given in [DFI6], [DFI5], were completely motivated by the current project (although they do have other applications).

In addition to the above-resolved difficulty, we are faced with the appearance of the Eisenstein series (because we are at the bottom of the continuous spectrum). The starting point is a summation formula of Petersson-Kuznetsov type which relates sums of Fourier coefficients of automorphic forms over the spectrum with sums of Kloosterman sums on the other side.

Our problem reduces eventually to deducing from the formula that the contribution from the cuspidal spectrum is small. We are hampered in this by the large contribution now coming from the continuous spectrum. Of course another large contribution must occur on the Kloosterman sum side of the formula. Indeed we were able to identify such a contribution as coming from the determinant equation in the singular case. However, to show that these large contributions match and therefore cancel (up to an admissible error) proved to be a considerable difficulty. This matching is not of a precise combinatorial nature but occurs in Sect. 13 only after both sides are transformed, in Sects. 11 and 12 respectively, by means of harmonic analysis on the relevant character sums.

This paper completes a project which has stretched over several years. During this time we have benefited on many occasions from helpful conversations with and encouragement from P. Michel and P. Sarnak. We are happy to acknowledge them here.

2. Statement of results

Now we proceed to a description of the main theorems. The first of these, from which the rest will follow, may sound a little technical but we wish to describe precisely what we have proved.

Our principal target is the L–function

$$L_j(s) = \sum_1^\infty \lambda_j(n) n^{-s} = \prod_p \left(1 - \lambda_j(p) p^{-s} + \chi(p^2) p^{-2s}\right)^{-1}$$

(see Sect. 8) associated to a Hecke–Maass cusp form $u_j(z)$ (see Sects. 4 and 6).

We begin by defining

$$\sigma_F(n, \chi) = \sum_{n_1 n_2 = n} F(n_1, n_2) \overline{\chi}(n_2) \tag{2.1}$$

where F is a smooth function, see (11.1). We shall refer to F as a bit function and think of σ_F as the multiplicity of terms in the Dirichlet series for $|L_j(s)|^2$. Therefore $|L_j(s)|^2$ is built out of pieces of type

$$\mathcal{N}_j = \sum_n \lambda_j(n) \sigma_F(n, \chi) \tag{2.2}$$

where the pieces which are significant for breaking the convexity bound come from those ranges in which n is near $\sqrt{D}$.

Theorem 2.1. *Let $k \geqslant 0$ and χ an arbitrary primitive character of conductor D with $\chi(-1) = (-1)^k$. Let $\{u_j\}$ be the complete system of Hecke-Maass cusp forms of weight k, level D, and character χ with corresponding Hecke* *eigenvalues* $\lambda_j(n)$ *and Laplace eigenvalue* $\lambda_j = \frac{1}{4} + t_j^2$. *Let* $\ell \geqslant 1$ *be an integer and* $T \geqslant k+1$. *We have*

$$\sum_{|t_j|\leqslant T} \lambda_j(\ell) \mid \mathcal{N}_j \mid^2 \ll \frac{DX_1X_2}{\sqrt{\ell}} P^{10}T^{17}\left\{1 + \ell^4 X^{10-\theta} D^{-5}\right\} X^{\varepsilon} \tag{2.3}$$

for any $\varepsilon > 0$ *and the implied constant depends on k and* ε. *Here* $\theta = \frac{1}{1152}$, P, X_1, X_2 *are defined in (11.1) and* $X = X_1 + X_2 + \sqrt{D}$.

The above should, in view of the previous comment, be thought of as representative of an upper bound for the more natural sum

$$\sum_{|t_j|\leqslant T} \lambda_j(\ell) \mid L_j(s) \mid^4 .$$

In fact a result of this type could have been derived but the complications would be even greater and the result would be of no extra value for our purpose.

The key feature in (2.3) is that the variation in sign of $\lambda_j(\ell)$ produces a cancellation $\ell^{-\frac{1}{2}}$ which is best possible. By means of the best possible bound for the simpler sum $\sum |L_j(s)|^4$ one would obtain the convexity bound and nothing better.

Out of the $\lambda_j(\ell)$ we build our amplifier.

Theorem 2.2. *For any complex numbers* c_ℓ, $1 \leqslant \ell \leqslant L$, *we have*

$$\begin{aligned}\sum_{|t_j|\leqslant T} \Big| \sum_{\ell\leqslant L} c_\ell \lambda_j(\ell) \Big|^2 \mid \mathcal{N}_j \mid^2 \\ \ll DX_1X_2P^{10}T^{17}\left(1 + L^4D^{-5}X^{10-\theta}\right) \|\mathbf{c}\|^2 X^{\varepsilon}\end{aligned} \tag{2.4}$$

where $\|\mathbf{c}\|$ *is the* ℓ_2*–norm of* $\{c_\ell\}$.

As before this may be thought of as representing an upper bound for the corresponding sum with the fourth power of $|L_j(s)|$. In this case however, because of positivity, such a result can be given without much extra work at all. Since we do not apply this result we omit the proof. For $\operatorname{Re} s = \frac{1}{2}$ we have

$$\begin{aligned}\sum_{|t_j|\leqslant T} \Big| \sum_{\ell\leqslant L} c_\ell \lambda_j(\ell) \Big|^2 \mid L_j(s) \mid^4 \\ \ll \|\mathbf{c}\|^2 |s|^{20} T^{37}\left(1 + L^4 D^{-\frac{\theta}{2}}\right) D^{1+\varepsilon} .\end{aligned} \tag{2.5}$$

From an appropriate choice of the amplification coefficients c_ℓ, see Sect. 21, we derive from Theorem 2.2 a bound for the individual sums $\mathcal{N}_j$ with general bit function F.

Theorem 2.3. *For any F satisfying (11.1) we have*

$$\mathcal{N}_j \ll P^5(|t_j|+1)^9(DX_1X_2)^{\frac{1}{2}+\varepsilon}$$
$$\left\{D^{-\frac{1}{2}}(X_1+X_2)^{1-\frac{\theta}{10}} + D^{-5}(X_1+X_2)^{10-\theta} + D^{-\frac{\theta}{20}}\right\}^{\frac{1}{2}}$$

where the implied constant depends on k and ε.

Using the approximation formula (9.7) for $L_j(s)$ we derive our main result.

Theorem 2.4. *Let $k \geqslant 0$ be an integer and $\chi(\bmod D)$ an arbitrary primitive character with $\chi(-1) = (-1)^k$. Let $u_j(z)$ be a Hecke-Maass cusp form of weight k for the group $\Gamma_0(D)$, character χ and with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$. Then the associated L–function satisfies*

$$L_j(s) \ll \left(|t_j| + |s|\right)^{10} D^{\frac{1}{4}-\frac{1}{23041}}$$

for $\mathrm{Re}\, s = \frac{1}{2}$*, where the implied constant depends only on k.*

This result, apart from the smaller saving in the exponent, contains the corresponding bounds from both [DFI4] and [DFI7].

After Artin, Hecke, Langlands and Tunnell, degree two Artin L-functions are known to be automorphic, hence entire, except possibly for icosahedral ρ. Taking $k = 0$ or 1 and $\lambda_j = \frac{1}{4}$ we infer the following.

Theorem 2.5. *Let $L(s, \rho)$ be an Artin L-function of degree two over $\mathbb{Q}$ not of icosahedral type with conductor D such that the determinant character χ is primitive modulo D. Then $L(s, \rho)$ satisfies for* $\mathrm{Re}\, s = \frac{1}{2}$

$$L(s, \rho) \ll |s|^{10} D^{\frac{1}{4}-\frac{1}{23041}} ,$$

where the implied constant is absolute.

It is conjectured that the icosahedral Artin L-functions are automorphic, in which case this estimate holds for them as well. Note that for reducible ρ a better estimate follows since the L-function then factors as a product of two Dirichlet L-functions (Kronecker's decomposition).

Let K be a quadratic field with discriminant d. Let $\mathcal{C}l(K)$ be the (narrow) class group of K and ψ be a character of $\mathcal{C}l(K)$. It is known that the determinant character corresponding to ψ is the quadratic field character χ_d with conductor $|d|$ whose values are given by the Kronecker symbol $\chi_d(a) = (d/a)$ (see [Se], p. 239.). We deduce from Theorem 2.5 the following result.

Theorem 2.6. *Let ψ be a character of $\mathcal{C}l(K)$ and*

$$L_K(s, \psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) N(\mathfrak{a})^{-s}$$

the corresponding L–function. We have for $\operatorname{Re} s = \frac{1}{2}$

$$L_K(s, \psi) \ll |s|^{10} |d|^{\frac{1}{4} - \frac{1}{23041}} ,$$

the implied constant being absolute.

A similar estimate also holds for a general Hecke L-function with a ray class character on K, provided that the associated determinant character is primitive.

We now turn to some applications of Theorem 2.6 to the class group. The first one is best introduced using the language of integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$. It is classical that if Q is primitive with discriminant $d = b^2 - 4ac$ then Q represents a positive integer $\ll |d|^{\frac{1}{2}}$ with an absolute constant. This is in general best possible for both negative and positive d. For negative d one considers the forms $nx^2 + (n + 1)y^2$ while for positive d the Markoff forms [M] only represent positive integers $> \frac{1}{3}|d|^{\frac{1}{2}}$ for an infinite set of $d > 0$. A. Baker and A. Schinzel [BS] treated the problem of improving the estimate when one considers genera of forms instead of classes. They were able to show that every genus of primitive forms represents a positive integer $\ll_\varepsilon |d|^{\frac{3}{8}+\varepsilon}$ for all $\varepsilon > 0$. The implied constant is ineffective since Siegel's theorem is employed. Genera may be analyzed using real characters and the proof of this estimate relies on the Burgess $GL(1)$ subconvexity bounds (the exponent $\frac{3}{8}$ was subsequently reduced to $\frac{1}{4}$ in [H-B]).

This problem can be considered as a special case of a more general question which asks for a non-trivial upper bound for the smallest positive integer represented by every coset of an arbitrary quotient group of the class group, provided that the quotient group is not too large. The quotient of genera is very small, having size which is $\ll |d|^\varepsilon$ for any $\varepsilon > 0$. Theorem 2.7 below provides a non-trivial upper bound provided that the quotient group has size $\ll |d|^{\frac{1}{23041}}$. For this result the full $GL(2)$ theory is needed.

It is more convenient to use the language of ideals and we shall in fact give a slightly more refined result which is interesting also for the coset containing the principal class. Recall that an ideal $\mathfrak{a}$ of K is primitive if it is not divisible by any rational integer > 1. Let G be a quotient group of $\mathcal{C}l(K)$ of size $|G|$. Our first application shows that sufficiently large cosets of the class group will always contain primitive ideals of small norm.

Theorem 2.7. *Every element of a quotient group G of* $\mathcal{C}l(K)$ *contains primitive ideals having norm* > 1 *and* $\ll |G|^2 |d|^{\frac{1}{2} - \frac{1}{11521}}$. *The implied constant is ineffective.*

Turning now to questions about the structure of $\mathcal{C}l(K)$, it follows easily along the same lines as the proof of Theorem 2.7 that $\mathcal{C}l(K)$ is generated by all of the primes of K with norm $\ll |d|^{\frac{1}{2} - \frac{1}{11521}}$ since every nontrivial class character ψ must satisfy $\psi(\mathfrak{a}) \neq 1$ for some $N(\mathfrak{a}) \ll |d|^{\frac{1}{2} - \frac{1}{11521}}$, again with an ineffective constant. A more refined application may be given which is somewhat analogous to bounding the smallest primitive root. It concerns the distribution of generators of a cyclic subgroup of $\mathcal{C}l(K)$ which has small index in $\mathcal{C}l(K)$.

Theorem 2.8. *Every cyclic subgroup of $\mathcal{C}l(K)$ of index k may be generated by an ideal of norm $\ll k^2|d|^{\frac{1}{2}-\frac{1}{11521}}$, the implied constant being ineffective.*

After Gauss, Art. 306 of [Ga], a discriminant d is said to be regular if the principal genus of $\mathcal{C}l(K)$ is cyclic. It follows from Theorem 2.8 that *the principal genus of a regular discriminant d may be generated by an ideal of norm $\ll |d|^{\frac{1}{2}-\frac{1}{11522}}$.*

These results are most interesting if $\mathcal{C}l(K)$ is large. For negative d this is always the case. For positive d this seems not to be the case (though proofs are lacking) but certainly it is true for many d. Some extreme examples are provided by certain quadratic sequences such as the squarefree d of the form $4n^2+1$ in which case this "class number" satisfies $|\mathcal{C}l(K)| \gg d^{\frac{1}{2}-\varepsilon}$. Although the existence of infinitely many regular discriminants is not known, the numerical evidence supports the existence of a large positive proportion of regular discriminants among the negative discriminants [Bue]. In fact, it has been conjectured in [Ge] that the proportion of regular to all negative discriminants is $(\zeta(6)\prod_{n\geqslant 4}\zeta(n))^{-1} \approx .8469$.

3. Structure of the paper

Now we describe the structure of the paper. In Sect. 4 we give a fairly extensive review of the spectral theory of automorphic forms in the generality that is required for this work. This theory in full generality is due to Maass and Selberg and their original papers [Ma1], [Ma2], [S1], [S2] are still valuable references where many proofs can be found. The papers [Ro] are also recommended as are the books [He], [I2], [I3]. In our case we restrict to the Hecke congruence group $\Gamma_0(D)$ and automorphic forms with multiplier given by a primitive Dirichlet character χ mod D. The fact that the conductor of the character coincides with the level of the group makes the theory particularly elegant and yet still allows us to treat the most interesting Artin L-functions, namely those associated to class group characters. For needed background on the theory of Artin L-functions we refer to [Se].

In Sect. 5 we present the summation formula of Petersson-Kuznetsov type which is our starting point. Since there is no easy reference for the formula that is needed (see however the preprint [Pr]), we present complete details.

It is essential for our amplification method that the Fourier coefficients of the automorphic forms be multiplicative. Hence we need to choose the basis of primitive forms (newforms in the terminology of Atkin-Lehner [AL]).

The requisite theory of Hecke operators is presented in Sect. 6. That in turn leads us to a reformulation, in Proposition 6.1, of the summation formula in which the Fourier coefficients are replaced by Hecke eigenvalues.

The Hecke eigenvalues of Maass cusp forms are very poorly understood. On the other hand the corresponding eigenvalues of Eisenstein series can be explicitly computed. In Sect. 7 we provide such computations. The fact that we can do so is essential to the work because we need to match the dominant contribution of these series to the summation formula. By contrast the contribution to this formula from the cusp forms is small, as we shall prove, so luckily we don't require the analogous computations for these. The Fourier coefficients of the Eisenstein series are proportional to the Hecke eigenvalues. This constant of proportionality must also be evaluated; see Proposition 7.1.

We shall also require a result for the analogous proportionality factor between the Fourier coefficients and the Hecke eigenvalues of the cusp forms. This turns out to be given in terms of the symmetric square L–functions. It suffices to have a lower bound (7.16) for the former and as a result we need an upper bound for the latter. The proofs of these are postponed to Sect. 19 just before their application is required.

In Sects. 8 and 9 we introduce various L–functions associated with Maass forms and derive some of their basic properties, in particular their functional equations and the resultant approximation formulae (referred to in the literature as approximate functional equations). These formulae allow us to reduce the L–function to partial sums of length $\ll D^{\frac{1}{2}+\varepsilon}$.

At this point we have a spectral summation formula for each pair of Hecke operators T_m, T_n. However we are not able to separate out the main terms in the formula and see that they match. Indeed although on the spectral side we do know that the cusps at ∞ and 0 are clear suspects, on the Kloosterman sum side we cannot even see a likely candidate. In fact it is only after Fourier analysis with respect to one of the variables, say n, that such a candidate will emerge.

In Sect. 10 we consider an average of our summation formula over the variable n, followed by an application of Poisson-Voronoi summation. We don't have an immediate gain from this operation because the resulting dual variable, say n', is more or less in the same range as n. However the Kloosterman sums $S_\chi(m, n; c)$ collapse to Gauss-Ramanujan sums $S_\chi(m - n', 0; c)$. Now it is possible to point out the main contribution which will come from the singular term $n' = m$. It is worthwhile to emphasize that this collapse of the Kloosterman sum does not occur in general but rather because we have weighted the summation over n by a special arithmetic function (see (10.1)) which is relevant to the main goal. On the spectral side, by the same summation over n we obtain, without any transformation, sums $\mathcal{N}_j$, $\mathcal{N}_\mathfrak{a}(t)$ (see (10.2), (10.3)) which correspond to the cusp forms and Eisenstein series respectively.

In Sect. 11 we evaluate asymptotically $\mathcal{N}_\mathfrak{a}(t)$ for each singular cusp $\mathfrak{a}$. We find that $\mathcal{N}_\mathfrak{a}(t)$ is small unless $\mathfrak{a} \sim \infty$ or 0 and in those two cases we get an equal contribution; see (11.5). In obtaining these results the Burgess subconvexity bound [B2] for Dirichlet L–functions is required. The contribution of the Eisenstein series associated to cusps other than ∞ and 0 is shown to be negligible by means of the Burgess bound whereas for these two cusps the bound is again required, now to separate out a manageable main term which occurs in the form of a definite integral in several variables.

Having completed the computation of $\mathcal{N}_\infty(t)$, $\mathcal{N}_0(t)$ (and hence of the spectral side) we turn in Sect. 12 to the analogous evaluation of the singular determinant contribution, see (12.1). Here the Gauss-Ramanujan sum collapses further to the Euler function $\varphi(c)$. We need to perform the summation over this modulus c in order to make the asymptotic evaluation. To this end we apply the Euler-Maclaurin formula. Once again, as on the other side of the formula, the Burgess subconvexity bound is required since, in separating out the leading term from the singular determinant contribution, short character sums need to be estimated. This leading term, given by (12.9), also occurs as a definite integral in several variables but there is no clear similarity between the two.

In Sect. 13 we finally prove that the main terms from Sects. 11 and 12 coincide, without ever evaluating either one.

In Sect. 14 we input the above results into the summation formula freeing the latter from the main terms. The new formula contains, apart from error terms, cuspidal terms on the one side and non-singular Gauss-Ramanujan sums on the other. At this point it closely resembles in structure the Petersson formula for the subspace of holomorphic cusp forms and as a result our arguments from now on are close to those in [DFI7]. In that paper we assumed $k \geqslant 3$. In order to avoid that assumption here, we integrate our formula over a certain parameter r which yields better test functions and hence improves the rate of convergence in the sum over the modulus c. This integrated version of the summation formula is stated in Proposition 14.2.

The formula in Proposition 14.2 is still valid for each individual $m \geqslant 1$. In Sect. 15 we sum over m in the same way that we summed over n so long ago. The result is stated in Proposition 15.1 but it will take several sections to complete its proof.

The main remaining ingredient in this proof is the determinant problem considered in Sect. 16 which we take largely unchanged from Sect. 8 of [DFI7]. In turn that was derived from the two papers [DFI5], [DFI6]. It is worthwhile to note that the fundamental idea in [DFI5] is an unusual application of the amplification method which is thus occurring not only in our main bound but also three different times in subsidiary roles.

There are some difficulties in verifying the applicability to our problem of the result as stated in Sect. 16. These verifications are carried out in Sect. 17.

In Sect. 18 we combine our estimates, optimize the free parameters and complete the proof of Proposition 15.1.

In Sect. 19 we study some properties of the Rankin–Selberg and symmetric square L–functions. This equips us to give proofs of two results, (7.16) and Proposition 19.6, on the size of the Fourier coefficients of cusp forms, which will be needed to complete the proofs of the main theorems.

In Sect. 20 we prepare for the application of the amplification method. This involves some computations with Hecke eigenvalues which exploit their multiplicativity thus allowing us to separate out the amplification variable. It is in this section that we complete the proof of Theorem 2.1.

In Sect. 21 we perform the amplification itself and derive bounds for the individual $\mathcal{N}_j$, thereby proving Theorems 2.2 and 2.3. Along the way we have a slight problem in the non-holomorphic case due to the lack of the Ramanujan conjectures. As a result we require Proposition 19.6. Now we have non-trivial bounds for the sums $\mathcal{N}_j$ which are the building blocks for our L–functions. We combine these with the partition in Proposition 9.7 to deduce bounds for the L–functions themselves. As a result we complete the proof of Theorem 2.4.

Finally, in Sect. 22, we give the proofs of the applications.

4. Background on Maass forms

In this section we collect those basic facts about Maass forms (real-analytic automorphic forms) which are needed in this paper. Most of these are standard and can be found in the original sources [Ma1], [Ma2], [Ro], [S1], [S2].

The group $SL_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H} = \{z = x + iy;\ x \in \mathbb{R}, y \in \mathbb{R}^+\}$ by the linear-fractional transformations

$$\gamma z = \frac{az+b}{cz+d}\,, \quad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For any $\gamma \in SL_2(\mathbb{R})$ we define $j_\gamma : \mathbb{H} \to \mathbb{C}$ by

$$j_\gamma(z) = \frac{cz+d}{|cz+d|} = e^{i \arg(cz+d)}\,. \tag{4.1}$$

Throughout k is an integer. For any $\gamma \in SL_2(\mathbb{R})$ we introduce the linear operator $R_\gamma^{(k)}$ defined on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$\left(R_\gamma^{(k)} f\right)(z) = j_\gamma(z)^{-k} f(\gamma z)\,. \tag{4.2}$$

These operators satisfy $R_{\gamma_1\gamma_2}^{(k)} = R_{\gamma_1}^{(k)} R_{\gamma_2}^{(k)}$ for all $\gamma_1, \gamma_2 \in SL_2(\mathbb{R})$. A linear operator L is said to be invariant of weight k if L commutes with $R_\gamma^{(k)}$ for every $\gamma \in SL_2(\mathbb{R})$.

Following Maass [Ma2] we consider two first order differential operators

$$K_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = \frac{k}{2} + (z - \bar{z})\frac{\partial}{\partial z} \tag{4.3}$$

$$\Lambda_k = \frac{k}{2} + y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) = \frac{k}{2} + (z - \bar{z})\frac{\partial}{\partial \bar{z}} \tag{4.4}$$

where $\partial/\partial z$, $\partial/\partial \bar{z}$ are the complex partial derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The Maass operators of weight k are not exactly invariant, but they satisfy the following rules

$$K_k R_\gamma^{(k)} = R_\gamma^{(k+2)} K_k \tag{4.5}$$

$$\Lambda_k R_\gamma^{(k)} = R_\gamma^{(k-2)} \Lambda_k\,. \tag{4.6}$$

In other words the operators K_k and Λ_k have effect of changing the weight by 2 up and down respectively.

The Laplace operator of weight k is defined by

$$\begin{aligned}\Delta_k &= y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}\\ &= -(z-\bar{z})^2\frac{\partial^2}{\partial z\partial\bar{z}} - \frac{k}{2}(z-\bar{z})\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right).\end{aligned} \tag{4.7}$$

This can be expressed in terms of Maass operators in two ways;

$$\Delta_k = -K_{k-2}\Lambda_k - \lambda\left(\frac{k}{2}\right), \tag{4.8}$$

$$\Delta_k = -\Lambda_{k+2}K_k - \lambda\left(\frac{-k}{2}\right), \tag{4.9}$$

where

$$\lambda(s) = s(1-s)\,. \tag{4.10}$$

Hence one derives the following commutation rules

$$K_k\Delta_k = \Delta_{k+2}K_k \tag{4.11}$$

$$\Lambda_k\Delta_k = \Delta_{k-2}\Lambda_k\,, \tag{4.12}$$

and that Δ_k commutes with $R_\gamma^{(k)}$ for all $\gamma \in SL_2(\mathbb{R})$; i.e. Δ_k is an invariant operator of weight k.

A smooth function $f : \mathbb{H} \to \mathbb{C}$ is an eigenfunction of Δ_k with eigenvalue $\lambda \in \mathbb{C}$ if

$$(\Delta_k + \lambda)f = 0\,. \tag{4.13}$$

We shall write the eigenvalues in the form (4.10), where $s = \frac{1}{2} + it$ is a complex number, and so is t. Note that $\lambda(s) = \lambda(1-s)$ so the map $s \mapsto \lambda(s)$ covers $\mathbb{C}$ twice except for $\lambda\left(\frac{1}{2}\right) = \frac{1}{4}$.

Since Δ_k is an elliptic operator its eigenfunctions are real-analytic. Suppose f is an eigenfunction of Δ_k with eigenvalue λ. Then

– $K_k f$ is an eigenfunction of Δ_{k+2} with eigenvalue λ,
– $\Lambda_k f$ is an eigenfunction of Δ_{k-2} with eigenvalue λ.

For a smooth function f on $\mathbb{H}$ we have the following facts:

– $K_k f = 0$ if and only if $y^{k/2}\bar{f}(z)$ is holomorphic in z. In this case f is an eigenfunction of Δ_k with eigenvalue $\lambda(-k/2)$,
– $\Lambda_k f = 0$ if and only if $y^{-k/2} f(z)$ is holomorphic in z. In this case f is an eigenfunction of Δ_k with eigenvalue $\lambda(k/2)$.

There is a vast variety of eigenfunctions of Δ_k with eigenvalue λ. Some of them which satisfy special transformation rules can be solved by the separation of variables method. The results, of course, depend on the coordinate system being used. For Fourier analysis of automorphic forms the rectangular coordinates $z = x + iy$ are most suitable. For example, if we seek $f(z)$ which depends only on y we find two linearly independent solutions

$$f^+(z, s) = \frac{1}{2}\left(y^s + y^{1-s}\right) \tag{4.14}$$

$$f^-(z, s) = \frac{1}{2s-1}\left(y^s - y^{1-s}\right) \tag{4.15}$$

(for $s = \frac{1}{2}$ these solutions are $\sqrt{y}$ and $\sqrt{y}\log y$ respectively). If we want $f(z)$ to be periodic in x of period 1 we may set $f(z) = e(\pm x)W(2\pi y)$ and find that $W(y)$ satisfies the ordinary differential equation

$$W''(y) + \left(\lambda y^{-2} \pm k y^{-1} - 1\right) W(y) = 0\,. \tag{4.16}$$

There are two linearly independent solutions, the first of which decays exponentially while the second one grows exponentially as $y \to \infty$. The first solution is given by the Whittaker function $W_{\alpha,\beta}(2y)$ with $\alpha = \pm\frac{k}{2}$ and $\beta = s - \frac{1}{2}$, the corresponding Laplace eigenfunction being

$$f_k^+(z, s) = W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi y)e(x), \tag{4.17}$$

$$f_k^-(z, s) = W_{-\frac{k}{2}, s-\frac{1}{2}}(4\pi y)e(-x)\,. \tag{4.18}$$

For any $\alpha, \beta \in \mathbb{C}$ the Whittaker function $W_{\alpha,\beta}(y)$ satisfies

$$W_{\alpha,\beta}(y) \sim y^\alpha e^{-y/2} \qquad \text{if } y \to \infty\,.$$

We have

$$e^{-y/2} W_{\alpha,\beta}(y) = \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(u-\beta)\Gamma(u+\beta)}{\Gamma\left(u - \alpha + \frac{1}{2}\right)} y^{\frac{1}{2}-u}\,du \tag{4.19}$$

where $\sigma > |\operatorname{Re}\beta|$. Notice that $W_{\alpha,\beta}(y)$ is holomorphic in both of the parameters α, β and is even in β. For $\operatorname{Re}\left(\beta - \alpha + \frac{1}{2}\right) > 0$ we have the integral representation

$$W_{\alpha,\beta}(y) = \frac{y^\alpha e^{-y/2}}{\Gamma\left(\beta - \alpha + \frac{1}{2}\right)} \int_0^\infty e^{-t} t^{\beta-\alpha+\frac{1}{2}} \left(1 + \frac{t}{y}\right)^{\beta+\alpha-\frac{1}{2}} dt\,. \tag{4.20}$$

In particular for $\beta = \alpha - \frac{1}{2}$ we get

$$W_{\alpha,\alpha-\frac{1}{2}}(y) = y^{\alpha} e^{-y/2} \,. \tag{4.21}$$

Hence we obtain by (4.17) and (4.18)

$$f_k^{+}\Big(z, \frac{k}{2}\Big) = y^{\frac{k}{2}} e(z), \tag{4.22}$$

$$f_k^{-}\Big(z, -\frac{k}{2}\Big) = y^{-\frac{k}{2}} e(-\bar{z}), \tag{4.23}$$

which are eigenfunctions of Δ_k for eigenvalues $\lambda\left(\frac{k}{2}\right)$ and $\lambda\left(-\frac{k}{2}\right)$ respectively.

For any $\alpha, \beta \in \mathbb{C}$ the Whittaker functions satisfy the differential recursion formulae (see (9.234) of [GR])

$$\begin{aligned} yW'_{\alpha,\beta}(y) &= \Big(\alpha - \tfrac{1}{2}y\Big) W_{\alpha,\beta}(y) \\ &\quad + \Big(\alpha - \beta - \tfrac{1}{2}\Big)\Big(\alpha + \beta - \tfrac{1}{2}\Big) W_{\alpha-1,\beta}(y) \\ &= -\Big(\alpha - \tfrac{1}{2}y\Big) W_{\alpha,\beta}(y) - W_{\alpha+1,\beta}(y) \,. \end{aligned} \tag{4.24}$$

Hence we deduce that the Maass operators (4.3), (4.4) act on the eigenfunctions (4.17), (4.18) as follows:

$$K_k f_k^{+}(z, s) = -f_{k+2}^{+}(z, s), \tag{4.25}$$

$$K_k f_k^{-}(z, s) = \Big(s + \frac{k}{2}\Big)\Big(1 - s + \frac{k}{2}\Big) f_{k+2}^{-}(z, s), \tag{4.26}$$

and

$$\Lambda_k f_k^{+}(z, s) = -\Big(s - \frac{k}{2}\Big)\Big(1 - s - \frac{k}{2}\Big) f_{k-2}^{+}(z, s), \tag{4.27}$$

$$\Lambda_k f_k^{-}(z, s) = f_{k-2}^{-}(z, s) \,. \tag{4.28}$$

The same equations hold if one replaces z by az for any constant $a > 0$. Also notice that

$$K_k y^s = \Big(\frac{k}{2} + s\Big) y^s, \tag{4.29}$$

$$\Lambda_k y^s = \Big(\frac{k}{2} - s\Big) y^s \,. \tag{4.30}$$

One may write the factors in (4.26) and (4.27) in terms of the corresponding eigenvalues as follows:

$$\Big(s + \frac{k}{2}\Big)\Big(1 - s + \frac{k}{2}\Big) = \lambda(s) - \lambda\Big(-\frac{k}{2}\Big),$$

$$-\Big(s - \frac{k}{2}\Big)\Big(1 - s - \frac{k}{2}\Big) = \lambda\Big(\frac{k}{2}\Big) - \lambda(s) \,.$$

In this survey we consider automorphic forms with respect to the Hecke congruence group $\Gamma = \Gamma_0(D)$ of level $D \geqslant 3$. This group has index

$$\nu(D) = [\Gamma_0(1) : \Gamma_0(D)] = D \prod_{p|D} \left(1 + \frac{1}{p}\right) . \tag{4.31}$$

Let χ (mod D) be a primitive character such that

$$\chi(-1) = (-1)^k . \tag{4.32}$$

This gives rise to a character on Γ by means of

$$\chi(\gamma) = \chi(d) = \overline{\chi}(a) \tag{4.33}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. A function $f : \mathbb{H} \to \mathbb{C}$ which transforms by

$$f(\gamma z) = \chi(\gamma) j_\gamma(z)^k f(z) \tag{4.34}$$

for all $\gamma \in \Gamma$ is called automorphic of weight k and character χ (in other words f is an eigenfunction of $R_\gamma^{(k)}$ with eigenvalue $\chi(\gamma)$). We denote $\mathcal{A}_k(\Gamma, \chi)$ the linear space of such functions.

A smooth function $f \in \mathcal{A}_k(\Gamma, \chi)$ which satisfies (4.13) is called a Maass form. Let $\mathcal{A}_k(\Gamma, \chi; s)$ denote the linear space of Maass forms with eigenvalue $\lambda(s) = s(1 - s)$ which also satisfy the growth condition

$$f(z) \ll y^\sigma + y^{1-\sigma} \tag{4.35}$$

for $z = x + iy \in \mathbb{H}$ with some σ. These forms have the following Fourier expansion

$$\begin{aligned} f(z) = \rho^+ f^+(z, s) + \rho^- f^-(z, s) \\ + \sum_{n=1}^{\infty} \left(\rho(n) f_k^+(nz, s) + \rho(-n) f_k^-(nz, s)\right) \end{aligned} \tag{4.36}$$

where $f^\pm(z, s)$, $f_k^\pm(z, s)$ are basic eigenfunctions of Δ_k given in (4.14), (4.15), (4.17), (4.18) and ρ^+, ρ^-, $\rho(n)$, $\rho(-n)$ are complex numbers, which we shall call the Fourier coefficients of $f(z)$.

By (4.5), (4.6), (4.11), (4.12) it is clear that the Maass operators K_k and Λ_k map $\mathcal{A}_k(\Gamma, \chi; s)$ into $\mathcal{A}_{k+2}(\Gamma, \chi; s)$ and $\mathcal{A}_{k-2}(\Gamma, \chi; s)$, respectively.

Let $\mathcal{L}_k(\Gamma, \chi)$ be the L_2–space of automorphic functions of weight k with respect to the inner product

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z) \bar{g}(z) d\mu z \tag{4.37}$$

where $d\mu z = y^{-2} dx\, dy$ is the hyperbolic measure (the $SL_2(\mathbb{R})$ invariant measure). Let $\mathcal{B}_k(\Gamma, \chi)$ be the linear space of smooth functions $f \in$

$\mathcal{A}_k(\Gamma,\chi)$ such that f and $\Delta_k f$ are both bounded on $\mathbb{H}$. Then $\mathcal{B}_k(\Gamma,\chi)$ is a dense subspace of $\mathcal{L}_k(\Gamma,\chi)$, and we take it as an initial domain for spectral resolution of Δ_k.

Clearly the Maass operators K_k and Λ_k map the space $\mathcal{B}_k(\Gamma,\chi)$ into $\mathcal{L}_{k+2}(\Gamma,\chi)$ and $\mathcal{L}_{k-2}(\Gamma,\chi)$ respectively (see (4.5) and (4.6)). Moreover, by Green's theorem one derives two formulas

$$\begin{aligned} \langle f, -\Delta_k g\rangle &= \langle K_k f, K_k g\rangle + \lambda\Big(-\frac{k}{2}\Big)\langle f, g\rangle \\ &= \langle \Lambda_k f, \Lambda_k g\rangle + \lambda\Big(\frac{k}{2}\Big)\langle f, g\rangle \end{aligned} \tag{4.38}$$

for every $f, g \in \mathcal{B}_k(\Gamma,\chi)$. Either formula shows that Δ_k is symmetric on $\mathcal{B}_k(\Gamma,\chi)$, that is

$$\langle \Delta_k f, g\rangle = \langle f, \Delta_k g\rangle\ . \tag{4.39}$$

Moreover from both formulas one sees that $-\Delta_k$ is bounded from below by $\lambda\big(\frac{|k|}{2}\big)$, that is

$$\langle f, -\Delta_k f\rangle \geqslant \lambda\Big(\frac{|k|}{2}\Big)\langle f, f\rangle \tag{4.40}$$

for every $f \in \mathcal{B}_k(\Gamma,\chi)$. Therefore by a theorem of Friedrichs the operator $-\Delta_k$ admits a self-adjoint extension (which we also denote by $-\Delta_k$), and by a theorem of von Neumann the space $\mathcal{L}_k(\Gamma,\chi)$ has a complete spectral resolution with respect to $-\Delta_k$.

We proceed to a description of the spectral theory in practical terms which is due to Maass and Selberg. It is illuminating to begin with the Eisenstein series. Let $\mathfrak{a}$ be a cusp for $\Gamma = \Gamma_0(D)$ and

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma;\, \gamma\mathfrak{a} = \mathfrak{a}\} \tag{4.41}$$

the stability group. There exists $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$, unique up to translation on the right, such that

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}\ , \qquad \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty \tag{4.42}$$

where $\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & b\\ & 1\end{pmatrix};\, b \in \mathbb{Z}\right\}$ is the stability group for the cusp at ∞. The group $\Gamma_{\mathfrak{a}}$ is generated by two parabolic elements $\pm\gamma_{\mathfrak{a}}$ where

$$\gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\sigma_{\mathfrak{a}}^{-1}.$$

The cusp $\mathfrak{a}$ is said to be singular with respect to χ if

$$\chi(\gamma_{\mathfrak{a}}) = 1\ , \qquad \text{or} \quad \chi(-\gamma_{\mathfrak{a}}) = 1\ . \tag{4.43}$$

Since $\chi \pmod D$ is primitive every singular cusp $\mathfrak{a}$ for $\Gamma = \Gamma_0(D)$ is equivalent to exactly one of type $\frac{1}{v}$ with $vw = D$, $(v, w) = 1$ (the complementary divisor w appears to be the width of the cusp $\frac{1}{v}$). Therefore the number of inequivalent singular cusps equals 2^t where t is the number of distinct prime divisors of D (see Sect. 7 for further details).

For every singular cusp $\mathfrak{a}$ the Eisenstein series $E_{\mathfrak{a}}(z, s)$ is defined by

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} \left(\operatorname{Im} \sigma_{\mathfrak{a}}^{-1} \gamma z\right)^s . \tag{4.44}$$

This definition is independent of the choice of $\mathfrak{a}$ in its equivalence class and also of the choice of $\sigma_{\mathfrak{a}}$. The series (4.44) converges absolutely for $\operatorname{Re} s > 1$ and, as proved by Selberg [S1], the function $E_{\mathfrak{a}}(z, s)$ has analytic continuation to the whole complex s–plane without poles in $\operatorname{Re} s \geqslant \frac{1}{2}$. If s is not a pole then $E_{\mathfrak{a}}(z, s)$ is a Maass form with eigenvalue $\lambda(s)$, but it is not in $\mathcal{L}_k(\Gamma, \chi)$.

Let $\psi(y)$ be a smooth compactly supported function on $\mathbb{R}^+$. Then the so-called incomplete Eisenstein series

$$E_{\mathfrak{a}}(z|\psi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\chi}(\gamma) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} \psi \left(\operatorname{Im} \sigma_{\mathfrak{a}}^{-1} \gamma z\right) \tag{4.45}$$

is in $\mathcal{L}_k(\Gamma, \chi)$, but it fails to be a Maass form ($\psi(y)$ is not an eigenfunction of Δ_k). Nevertheless the incomplete Eisenstein series are spanned by the Eisenstein series in the sense of continuous spectrum, this means

$$E_{\mathfrak{a}}(z|\psi) = \sum_{\mathfrak{b}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle E_{\mathfrak{a}}(*|\psi),\ E_{\mathfrak{b}}\left(z, \tfrac{1}{2} + it\right) \right\rangle E_{\mathfrak{b}}\left(z, \tfrac{1}{2} + it\right) dt$$

where $\mathfrak{b}$ runs over all the singular cusps. Here the integral converges absolutely and uniformly on compacta.

Note that one needs all the Eisenstein series $E_{\mathfrak{b}}\left(z, \frac{1}{2} + it\right)$ to represent one incomplete Eisenstein series $E_{\mathfrak{a}}(z|\psi)$. This representation extends by linearity to the whole space of all incomplete Eisenstein series, which we denote by $\mathcal{E}_k(\Gamma, \chi)$. Clearly Δ_k acts on $\mathcal{E}_k(\Gamma, \chi)$.

Proposition 4.1. *The Laplace operator Δ_k has purely continuous spectrum in $\mathcal{E}_k(\Gamma, \chi)$ which covers $\left[\frac{1}{4}, \infty\right)$ with multiplicity equal to the number of inequivalent singular cusps. The eigenpacket of continuous spectrum consists of the Eisenstein series $E_{\mathfrak{a}}\left(z, \frac{1}{2} + it\right)$ for singular cusps $\mathfrak{a}$ and real t, the spectral measure being $(4\pi)^{-1} dt$. This means that every $f \in \mathcal{E}_k(\Gamma, \chi)$ has the expansion*

$$f(z) = \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle f, E_{\mathfrak{a}}\left(*, \tfrac{1}{2} + it\right) \right\rangle E_{\mathfrak{a}}\left(z, \tfrac{1}{2} + it\right) dt . \tag{4.46}$$

Applying the Maass operators K_k, Λ_k to the Eisenstein series of weight k (see (4.44)) one can verify term-by-term the following relations

$$K_k E_{\mathfrak{a}}^{(k)}(z,s) = \left(\frac{k}{2}+s\right) E_{\mathfrak{a}}^{(k+2)}(z,s) \tag{4.47}$$

$$\Lambda_k E_{\mathfrak{a}}^{(k)}(z,s) = \left(\frac{k}{2}-s\right) E_{\mathfrak{a}}^{(k-2)}(z,s) \tag{4.48}$$

where the superscript displays the weight. Hence $K_k : \mathcal{E}_k(\Gamma,\chi) \to \mathcal{E}_{k+2}(\Gamma,\chi)$ and $\Lambda_k : \mathcal{E}_k(\Gamma,\chi) \to \mathcal{E}_{k-2}(\Gamma,\chi)$ are bijective maps.

The space $\mathcal{E}_k(\Gamma,\chi)$ is a rather small subspace of $\mathcal{L}_k(\Gamma,\chi)$. The orthogonal complement $\mathcal{C}_k(\Gamma,\chi)$ consists of functions f whose zero coefficient in the Fourier expansion at every singular cusp $\mathfrak{a}$ vanishes, i.e. f satisfies

$$\int_0^1 \left(R_{\sigma_{\mathfrak{a}}}^{(k)} f\right)(z)dx = 0\,. \tag{4.49}$$

The Laplace operator Δ_k maps $\mathcal{C}_k(\Gamma,\chi)$ into itself (to be precise Δ_k acts on smooth functions). An eigenfunction of Δ_k which satisfies (4.49) for every singular cusp $\mathfrak{a}$ is called a Maass cusp form. From the Fourier expansion one can see that a cusp form decays exponentially to zero at every cusp (not only those which are singular). As shown by Selberg [S1] Δ_k has purely point (discrete) spectrum on $\mathcal{C}_k(\Gamma,\chi)$, in other words $\mathcal{C}_k(\Gamma,\chi)$ is spanned by Maass cusp forms.

Proposition 4.2. *The spectrum of Δ_k on $\mathcal{C}_k(\Gamma,\chi)$ is discrete and infinite, but of finite multiplicity. Let $\{u_j(z)\}_{j=1}^{\infty}$ be a complete orthonormal system of Maass cusp forms of weight k, the group Γ and character χ. Then every $f \in \mathcal{C}_k(\Gamma,\chi)$ has the spectral expansion*

$$f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle\, u_j(z)\,. \tag{4.50}$$

Combining Proposition 4.1 and Proposition 4.2 one obtains a complete spectral resolution of $\mathcal{L}_k(\Gamma,\chi)$ with respect to Δ_k.

Let $\mathcal{C}_k(\Gamma,\chi;s)$ be the linear space of Maass cusp forms of weight k and eigenvalue $\lambda(s)$. Clearly $K_k : \mathcal{C}_k(\Gamma,\chi;s) \to \mathcal{C}_{k+2}(\Gamma,\chi;s)$ and $\Lambda_k : \mathcal{C}_k(\Gamma,\chi;s) \to \mathcal{C}_{k-2}(\Gamma,\chi;s)$. It follows from (4.38) that for f, g in $\mathcal{C}_k(\Gamma,\chi;s)$

$$\langle K_k f, K_k g \rangle = \left(\lambda(s) - \lambda\left(\frac{-k}{2}\right)\right) \langle f, g \rangle \tag{4.51}$$

$$\langle \Lambda_k f, \Lambda_k g \rangle = \left(\lambda(s) - \lambda\left(\frac{k}{2}\right)\right) \langle f, g \rangle\,. \tag{4.52}$$

Hence $K_k f = 0 \Leftrightarrow \lambda(s) = \lambda\left(-\frac{k}{2}\right) \Leftrightarrow y^{\frac{k}{2}} \bar{f}(z)$ is holomorphic in z. Similarly $\Lambda_k f = 0 \Leftrightarrow \lambda(s) = \lambda\left(\frac{k}{2}\right) \Leftrightarrow y^{-\frac{k}{2}} f(z)$ is holomorphic in z. If $\lambda(s) \neq \lambda\left(-\frac{k}{2}\right)$ then the map

$$\left(\lambda(s) - \lambda\left(-\frac{k}{2}\right)\right)^{-\frac{1}{2}} K_k : \mathcal{C}_k(\Gamma, \chi; s) \to \mathcal{C}_{k+2}(\Gamma, \chi; s)$$

is a bijective isometry. Similarly if $\lambda(s) \neq \lambda\left(\frac{k}{2}\right)$ then the map

$$\left(\lambda(s) - \lambda\left(\frac{k}{2}\right)\right)^{-\frac{1}{2}} \Lambda_k : \mathcal{C}_k(\Gamma, \chi; s) \to \mathcal{C}_{k-2}(\Gamma, \chi; s)$$

is a bijective isometry.

Let $S_k(\Gamma, \chi)$ denote the space of classical cusp forms of weight k with respect to group Γ and character χ, i.e. the space of holomorphic functions $F : \mathbb{H} \to \mathbb{C}$ which satisfy

$$F(\gamma z) = \chi(\gamma)(cz + d)^k F(z) \tag{4.53}$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and which vanish at every cusp. From the above properties of Maass operators one can see that the eigenspace of Δ_k with the lowest eigenvalue $\lambda\left(\frac{|k|}{2}\right)$ is

$$\mathcal{C}_k\left(\Gamma, \chi; -\frac{k}{2}\right) = \left\{ y^{-\frac{k}{2}} \overline{F}(z);\ F \in S_{-k}(\Gamma, \overline{\chi}) \right\} \tag{4.54}$$

if $k \leqslant 0$, and

$$\mathcal{C}_k\left(\Gamma, \chi; \frac{k}{2}\right) = \left\{ y^{\frac{k}{2}} F(z);\ F \in S_k(\Gamma, \chi) \right\} \tag{4.55}$$

if $k \geqslant 0$. Note that for $k = 0$ we have $\mathcal{C}_0(\Gamma, \chi; 0) = 0$, because there is no holomorphic cusp form of weight zero for a non-trivial character χ on Γ. Indeed, if F was such a cusp form then $K_0 F = \Lambda_0 F = 0$, hence $\frac{\partial}{\partial z} F = \frac{\partial}{\partial \bar{z}} F = 0$, $F =$ constant, so $F = 0$.

In a similar fashion one can determine the other eigenspaces of Δ_k in $\mathcal{C}_k(\Gamma, \chi)$ for the eigenvalues $\lambda\left(\frac{m}{2}\right)$ with $m \equiv k \pmod 2$. They all are derived from classical cusp forms of various weights by repeated applications of the Maass operators.

From now on we assume that $k \geqslant 0$.

Proposition 4.3. *Choose a system $\left\{u_j(z)\right\}$ of Maass cusp forms of weight k which is an orthonormal basis of $\mathcal{C}_k(\Gamma, \chi)$, the corresponding eigenvalues being $\lambda(s_j)$. Moreover, choose $\{F_i(z)\}$ an orthonormal basis of $S_{k+2}(\Gamma, \chi)$. Then the collection of functions*

$$\left(\lambda(s_j) - \lambda\left(-\frac{k}{2}\right)\right)^{-\frac{1}{2}} K_k u_j(z) \tag{4.56}$$

together with $y^{\frac{k+2}{2}}F_i(z)$ *yield a system of cusp forms of weight* $k+2$ *which is an orthonormal basis of* $\mathcal{C}_{k+2}(\Gamma,\chi)$.

Remarks. Note that $\lambda(s_j) \geqslant \lambda\big(\frac{k}{2}\big) > \lambda\big(-\frac{k}{2}\big)$ (except for $k=0$, but there is no Maass cusp form of weight $k=0$ with eigenvalue zero), hence the normalization factor of $K_k u_j(z)$ in (4.56) is real. This result shows that the space of Maass cusp forms $\mathcal{C}_k(\Gamma,\chi)$ of weight $k \geqslant 0$ is determined by $\mathcal{C}_0(\Gamma,\chi)$ or $\mathcal{C}_1(\Gamma,\chi)$ according to the parity of k, except for a finite number of additional classical cusp forms which join the space as soon as k goes over their weight.

Corollary 4.4. *Let* $\kappa = 0, 1$ *and* $k \geqslant \kappa$, $k \equiv \kappa \pmod 2$. *Choose a system* $\{u_{j\kappa}(z)\}$ *of Maass cusp forms of weight* κ *which is an orthonormal basis of* $\mathcal{C}_\kappa(\Gamma,\chi)$, *the corresponding eigenvalues being* $\lambda(s_j)$. *Moreover, for any* $0 < m \leqslant k$, $m \equiv \kappa \pmod 2$ *choose* $\{F_{jm}(z)\}$ *an orthonormal basis of* $S_m(\Gamma,\chi)$. *Then, an orthonormal basis of* $\mathcal{C}_k(\Gamma,\chi)$ *is given by the following system of Maass cusp forms of weight k:*

$$u_{jk}(z) = \overrightarrow{\prod_{\substack{\kappa\leqslant\ell<k\\ \ell\equiv\kappa(2)}}}\Big(\lambda(s_j)-\lambda\Big(-\frac{\ell}{2}\Big)\Big)^{-\frac12} K_\ell\left\{u_{j\kappa}(z)\right\} \tag{4.57}$$

$$u_{jmk}(z) = \overrightarrow{\prod_{\substack{m\leqslant\ell<k\\ \ell\equiv\kappa(2)}}}\Big(\lambda\Big(\frac{m}{2}\Big)-\lambda\Big(-\frac{\ell}{2}\Big)\Big)^{-\frac12} K_\ell\left\{y^{\frac{m}{2}}F_{jm}(z)\right\} \tag{4.58}$$

for all $0 < m \leqslant k$ *with* $m \equiv \kappa \pmod 2$. *The arrow indicates that the operators* K_ℓ *are applied in increasing order of* ℓ. *If* $\kappa = k$, *or* $m = k$, *the above products stand for the identity operator, respectively.*

Remarks. All of the points s_j are either on the line $\operatorname{Re} s_j = \frac12$ or, in case $\kappa = 0$, possibly on the segment $0 < s_j < 1$. A conjecture of Selberg [S2] asserts that the latter case does not occur.

The normalization factors in (4.57) and (4.58) are real numbers and can be written in terms of the gamma function. Indeed we have

$$\begin{aligned}\alpha^2(s,k) &= \prod_{\substack{\kappa\leqslant\ell<k\\ \ell\equiv\kappa(2)}}\Big(\lambda(s)-\lambda\Big(-\frac{\ell}{2}\Big)\Big)^{-1}\\ &= \frac{\Gamma\big(s+\frac{\kappa}{2}\big)\Gamma\big(1-s+\frac{\kappa}{2}\big)}{\Gamma\big(s+\frac{k}{2}\big)\Gamma\big(1-s+\frac{k}{2}\big)} \qquad (4.59)\\ &= (-1)^{\frac{k-\kappa}{2}}\frac{\Gamma\big(s-\frac{k}{2}\big)\Gamma\big(s+\frac{\kappa}{2}\big)}{\Gamma\big(s+\frac{k}{2}\big)\Gamma\big(s-\frac{\kappa}{2}\big)},\end{aligned}$$

$$\begin{aligned}\beta^2(m,k) &= \prod_{\substack{m\leqslant \ell<k\\ \ell\equiv\kappa(2)}} \Big(\lambda\Big(\frac{m}{2}\Big)-\lambda\Big(-\frac{\ell}{2}\Big)\Big)^{-1}\\ &= \Gamma(m)/\Gamma\Big(\frac{k+m}{2}\Big)\Gamma\Big(\frac{k-m}{2}+1\Big)\,.\end{aligned}\tag{4.60}$$

We have not specified the sign of $\Big(\lambda(s)-\lambda\big(-\frac{\ell}{2}\big)\Big)^{\frac{1}{2}}$ so the normalization factors $\alpha(s,k)$, $\beta(m,k)$ are determined up to sign, however there is no need to fix these. In the sequel we shall be working with various bases of $\mathcal{C}_k(\Gamma,\chi)$ which are not always of type (4.57), (4.58). Sometimes, however, it will be convenient (for verification of computations) to refer to the basic ancestors coming from $\mathcal{C}_\kappa(\Gamma,\chi)$, $\kappa=0$ or 1, $\kappa\equiv k\ (\mathrm{mod}\,2)$, and $S_m(\Gamma,\chi)$, $0<m\leqslant k$, $m\equiv k\ (\mathrm{mod}\,2)$ via (4.57) and (4.58).

First we consider the eigenspace $\mathcal{C}_k(\Gamma,\chi;s)$ having the eigenvalue $\lambda(s)=s(1-s)$ where $s\neq\frac{\ell}{2}$ for any $\ell\equiv k\ (\mathrm{mod}\,2)$, that is the cusp forms which are not induced from holomorphic forms. Our goal is to construct an involution on $\mathcal{C}_k(\Gamma,\chi;s)$ which acts on Fourier series by interchanging the positive and negative terms (up to a constant factor which corrects a bad normalization of the Whittaker function). We begin by applying Λ_ℓ successively in decreasing order of $\ell\equiv k\ (\mathrm{mod}\,2)$ k–times starting from Λ_k. We obtain the operator

$$P_k=\overleftarrow{\prod_{\substack{-k<\ell\leqslant k\\ \ell\equiv k(2)}}}\Lambda_\ell:\mathcal{C}_k(\Gamma,\chi;s)\to\mathcal{C}_{-k}(\Gamma,\chi;s)\,.\tag{4.61}$$

Note that $\langle\, P_k u,\,P_k u\,\rangle=\gamma(s,k)\,\langle\, u,u\,\rangle$ on $\mathcal{C}_k(\Gamma,\chi;s)$ where

$$\gamma(s,k)=\prod_{\substack{-k<\ell\leqslant k\\ \ell\equiv k(2)}}\Big(\lambda(s)-\lambda\Big(\frac{k}{2}\Big)\Big)=(-1)^k\left(\frac{\Gamma\big(s+\frac{k}{2}\big)}{\Gamma\big(s-\frac{k}{2}\big)}\right)^2\,,$$

see (4.52). Therefore, normalizing by the scalar

$$\delta(s,k)=\Gamma\Big(s-\frac{k}{2}\Big)/\Gamma\Big(s+\frac{k}{2}\Big),\tag{4.62}$$

we get the operator $P_{sk}=\delta(s,k)P_k$ which is a bijective isometry from $\mathcal{C}_k(\Gamma,\chi;s)$ to $\mathcal{C}_{-k}(\Gamma,\chi;s)$ (for $k=0$ this is the identity operator).

Next we introduce the reflection operator X which acts on functions $f:\mathbb{H}\to\mathbb{C}$ by

$$(Xf)(z)=f(-\bar{z})\,.\tag{4.63}$$

Note that $X:\mathcal{C}_{-k}(\Gamma,\chi)\to\mathcal{C}_k(\Gamma,\chi)$ and $X^2=1$. Moreover $\Delta_kX=X\Delta_{-k}$ so $X:\mathcal{C}_{-k}(\Gamma,\chi;s)\to\mathcal{C}_k(\Gamma,\chi;s)$. Also notice that

$$Xf^{\pm}_{-k}(z,s)=f^{\mp}_k(z,s)\,.\tag{4.64}$$

By composing P_k with X we return to the space of forms of weight k. Precisely we obtain the operator

$$Q_k = XP_k : \mathcal{C}_k(\Gamma, \chi; s) \to \mathcal{C}_k(\Gamma, \chi; s) \tag{4.65}$$

such that $Q_{sk} = \delta(s, k)Q_k$ is an isometry. In fact we have (see Hilfssatz 5 of [Ma2]):

Proposition 4.5. *If* $s \neq \frac{\ell}{2}$ *for any* $\ell \equiv k \pmod 2$ *then the operator* $Q_{sk} = \delta(s, k)XP_k$ *is an involution on* $\mathcal{C}_k(\Gamma, \chi; s)$.

Proof. First we show that

$$K_\ell X = -X\Lambda_{-\ell}\ . \tag{4.66}$$

Indeed we have

$$\begin{aligned} XK_kXg(x, y) = XK_kg(-x, y) &= X\left(\frac{k}{2} + iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) g(-x, y) \\ &= \left(\frac{k}{2} - iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) g(x, y) = -\Lambda_{-k}g(x, y) \end{aligned}$$

showing (4.66) because $X^2 = 1$. By (4.66) applied k times we get

$$\begin{aligned} Q_k = X \overleftarrow{\prod_{\substack{-k<\ell\leqslant k \\ \ell\equiv k(2)}}} \Lambda_\ell &= (-1)^k\Bigg(\overleftarrow{\prod_{\substack{-k<\ell\leqslant k \\ \ell\equiv k(2)}}} K_{-\ell}\Bigg)X \\ &= (-1)^k\Bigg(\overrightarrow{\prod_{\substack{-k<\ell\leqslant k \\ \ell\equiv k(2)}}} K_{\ell-2}\Bigg)X\ . \end{aligned}$$

Hence

$$Q_k^2 = (-1)^k\Bigg(\overrightarrow{\prod_{\substack{-k<\ell\leqslant k \\ \ell\equiv k(2)}}} K_{\ell-2}\Bigg)\Bigg(\overleftarrow{\prod_{\substack{-k<\ell\leqslant k \\ \ell\equiv k(2)}}} \Lambda_\ell\Bigg)\ .$$

By (4.8) we get for any $g \in \mathcal{C}_\ell(\Gamma, \chi; s)$

$$(K_{\ell-2}\Lambda_\ell)\, g = \left(-\Delta_\ell - \lambda\left(\frac{\ell}{2}\right)\right)g = \left(\lambda(s) - \lambda\left(\frac{\ell}{2}\right)\right)g\ .$$

Applying this successively for $\ell = 2-k, \ldots, k$ (we may assume that $k \geqslant 1$, because the assertion is trivial for $k = 0$) we derive $Q_k^2 f = \delta(s, k)^{-2} f$ for any $f \in \mathcal{C}_k(\Gamma, \chi; s)$. This completes the proof of Proposition 4.5.

Next we examine the action of Q_{sk} on the Fourier series

$$u(z) = \sum_{n=1}^{\infty} \big(\rho(n) f_k^+(nz, s) + \rho(-n) f_k^-(nz, s)\big) \tag{4.67}$$

where $f_k^+(z, s)$, $f_k^-(z, s)$ are given by (4.17), (4.18). Applying the Maass operator Λ_ℓ successively for $\ell = k, \ldots, 2-k$ we derive by (4.27), (4.28) that

$$P_k u(z) = \sum_{n=1}^{\infty} \big(\delta(s, k)^{-2} \rho(n) f_{-k}^+(nz, s) + \rho(-n) f_{-k}^-(nz, s)\big) .$$

Then we apply the reflection (4.63) and the normalization factor (4.62) getting by (4.64)

$$\begin{aligned} Q_{sk} u(z) = \sum_{n=1}^{\infty} \big(&\delta(s, k)^{-1} \rho(n) f_k^-(nz, s) \\ &+ \delta(s, k) \rho(-n) f_k^+(nz, s)\big) . \end{aligned} \tag{4.68}$$

Since Q_k commutes with Δ_k (to see this use (4.12) and $X\Delta_\ell = \Delta_{-\ell} X$), we may require $u(z)$ to be an eigenfunction of Q_{sk}, that is

$$Q_{sk} u(z) = \varepsilon u(z) \tag{4.69}$$

with $\varepsilon = 1$, or $\varepsilon = -1$. Accordingly we say that $u(z)$ is even or odd. Comparing (4.67) and (4.68) we conclude by (4.69) that

$$\rho(-n) = \varepsilon \frac{\Gamma\big(s + \frac{k}{2}\big)}{\Gamma\big(s - \frac{k}{2}\big)} \rho(n) . \tag{4.70}$$

One can see directly that the Eisenstein series (4.44) is also an eigenfunction of the involution Q_{sk}. Indeed it follows by (4.48) and by $XE_{\mathfrak{a}}^{(-k)}(z, s) = E_{\mathfrak{a}}^{(k)}(z, s)$ that

$$Q_{sk} E_{\mathfrak{a}}(z, s) = E_{\mathfrak{a}}(z, s) . \tag{4.71}$$

In other words $E_{\mathfrak{a}}(z, s)$ is even. The Eisenstein series has Fourier expansion of type

$$\begin{aligned} E_{\mathfrak{a}}(z, s) &= \delta_{\mathfrak{a}} y^s + \varphi_{\mathfrak{a}}(s) y^{1-s} \\ &\quad + \sum_{n \neq 0} \rho_{\mathfrak{a}}(n, t) W_{\frac{kn}{2|n|}, it}(4\pi |n| y) e(nx) \end{aligned} \tag{4.72}$$

(recall that $s = \frac{1}{2} + it$). By (4.71) the coefficients satisfy

$$\rho_{\mathfrak{a}}(-n, t) = \frac{\Gamma\big(s + \frac{k}{2}\big)}{\Gamma\big(s - \frac{k}{2}\big)} \rho_{\mathfrak{a}}(n, t) . \tag{4.73}$$

We assert that if $u_{jk}(z)$ is an eigenfunction of Q_{s_jk} (for $s_j \not\equiv \frac{k}{2} \pmod 1$), then its original ancestor $u_{j\kappa}(z)$ is an eigenfunction of $Q_{s_j\kappa}$ and vice-versa. In fact the eigenvalues do not change, that is the following equations are equivalent

$$Q_{s_j\kappa}u_{j\kappa}(z) = \varepsilon u_{j\kappa}(z)\ ,$$
$$Q_{s_jk}u_{jk}(z) = \varepsilon u_{jk}(z)\ .$$

Proof. Let $k > \kappa$, because the case $k = \kappa$ is obvious. Skipping the subscript j we have the following situation

$$\begin{aligned} Q_{sk}u_k(z) &= \alpha(s,k)\delta(s,k)\,(X\Lambda_{2-k}\dots\Lambda_k)\,(K_{k-2}\dots K_k)\,u_k(z) \\ &= \frac{\delta(s,k)}{\alpha(s,k)}X\Lambda_{2-k}\dots\Lambda_k u_k(z) \end{aligned}$$

by (4.9) and (4.59). Then by (4.66) applied $\frac{k-\kappa}{2}$ times we get

$$\begin{aligned} Q_{sk}u_k(z) &= \frac{\delta(s,k)}{\alpha(s,k)}(-1)^{\frac{k-\kappa}{2}}K_{k-2}\dots K_k Q_k u_k(z) \\ &= \varepsilon(-1)^{\frac{k-\kappa}{2}}\frac{\delta(s,k)}{\delta(s,\kappa)\alpha^2(s,k)}u_k(z) = \varepsilon u_k(z) \end{aligned}$$

by (4.59) and (4.62), which completes the proof.

For the theory of L–functions it is instructive (and helpful for double-checking arguments) to have displayed connections between the coefficients of a cusp form and those of its basic ancestor. Suppose $u_{jk}(z)$ is induced by $u_{j\kappa}(z)$ in $\mathcal{C}_\kappa(\Gamma,\chi;s_j)$ with $\kappa = 0, 1$, $\kappa \equiv k \pmod 2$ and $s_j \not\equiv \frac{k}{2} \pmod 1$ by means of (4.57). Let

$$u_{jk}(z) = \sum_{n=1}^{\infty}\big(\rho_{jk}(n)f_k^+(nz,s_j) + \rho_{jk}(-n)f_k^-(nz,s_j)\big) \tag{4.74}$$

$$u_{j\kappa}(z) = \sum_{n=1}^{\infty}\big(\rho_{j\kappa}(n)f_\kappa^+(nz,s_j) + \rho_{j\kappa}(-n)f_\kappa^-(nz,s_j)\big) \tag{4.75}$$

be the Fourier expansions. By (4.25) and (4.26) applied successively to the terms of (4.75) we find that

$$\begin{aligned} u_{jk}(z) = \sum_{n=1}^{\infty}\Big(&(-1)^{\frac{k-\kappa}{2}}\alpha(s_j;k)\rho_{j\kappa}(n)f_k^+(nz,s_j) \\ &+ \alpha(s_j,k)^{-1}\rho_{j\kappa}(-n)f_k^-(nz,s_j)\Big)\ . \end{aligned} \tag{4.76}$$

Hence the desired connections between the Fourier coefficients are

$$\rho_{jk}(n) = (-1)^{\frac{k-\kappa}{2}} \alpha(s_j; k) \rho_{j\kappa}(n), \tag{4.77}$$

$$\rho_{jk}(-n) = \alpha(s_j; k)^{-1} \rho_{j\kappa}(-n), \tag{4.78}$$

for any $n \geqslant 1$, where $\alpha(s, k)$ is the normalization factor given by (4.59).

Suppose $u(z)$ is in $\mathcal{C}_k(\Gamma, \chi; s)$ with $s = \frac{m}{2}$ for some $m \equiv k \pmod 2$. We can assume without loss of generality that $0 < m \leqslant k$ (change s into $1 - s$ if needed). These forms are obtained from classical forms of weight m by successive application of Maass K–operators and, in view of (4.70), they are characterized as having no negative terms in the Fourier expansion (4.61). Suppose (see (4.58))

$$u(z) = \beta(m, k) \overrightarrow{\prod_{\substack{m \leqslant \ell < k \\ \ell \equiv k(2)}}} K_\ell \left\{ y^{\frac{m}{2}} F(z) \right\} \tag{4.79}$$

with

$$F(z) = \sum_{n=1}^{\infty} a_n n^{\frac{m-1}{2}} e(nz) \in S_m(\Gamma, \chi) \ . \tag{4.80}$$

By (4.25) we get

$$\overrightarrow{\prod_{\substack{m \leqslant \ell < k \\ \ell \equiv k(2)}}} K_\ell \left\{ f_m^+(z, s) \right\} = (-1)^{\frac{k-m}{2}} f_k^+(z, s) \ .$$

In particular for $s = \frac{m}{2}$ we get

$$\overrightarrow{\prod_{\substack{m \leqslant \ell < k \\ \ell \equiv k(2)}}} K_\ell \left\{ y^{\frac{m}{2}} e(z) \right\} = (-1)^{\frac{k-m}{2}} W_{\frac{k}{2}, \frac{m-1}{2}}(4\pi y) e(x) \ . \tag{4.81}$$

This also holds if z is replaced by nz. Introducing (4.80) into (4.79) we obtain by (4.81) the following expansion

$$u(z) = (-1)^{\frac{k-m}{2}} \beta(m, k) \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} W_{\frac{k}{2}, \frac{m-1}{2}}(4\pi n y) e(nx) \ . \tag{4.82}$$

Hence the Fourier coefficients of $u(z)$ are given in terms of the classical series (4.80) by

$$\rho(n) = (-1)^{\frac{k-m}{2}} \beta(m, k) a_n n^{-\frac{1}{2}} \tag{4.83}$$

where $\beta(m, k)$ is the normalization factor given by (4.60).

5. Spectral summation formula

Throughout $\{u_j(z)\}$ is an orthonormal system of Maass cusp forms which is a basis of $\mathcal{C}_k(\Gamma, \chi)$. This includes forms induced from holomorphic cusp forms of weight $0 < m \leqslant k$, $m \equiv k \pmod 2$, however we do not make special notation for them. Each of $u_j(z)$ has expansion in terms of Whittaker functions of type

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) W_{\frac{kn}{2|n|}, it_j}(4\pi|n|y)e(nx) \tag{5.1}$$

where $\lambda_j = s_j(1-s_j) = \frac{1}{4}+t_j^2$ is the Laplace eigenvalue of $u_j(z)$. Recall that at the bottom of discrete spectrum $\lambda_j = \lambda\left(\frac{k}{2}\right) = \frac{k}{2}\left(1-\frac{k}{2}\right)$, $s_j = \frac{k}{2}$, $it_j = \frac{k-1}{2}$, and the $u_j(z)$ corresponds to a holomorphic cusp form $F_j(z) = y^{-\frac{k}{2}}u_j(z)$ from $S_k(\Gamma, \chi)$. By (4.21) the expansion (5.1) becomes

$$F_j(z) = \sum_1^\infty \rho_j(n)(4\pi n)^{\frac{k}{2}} e(nz)\ . \tag{5.2}$$

Moreover each Eisenstein series $E_{\mathfrak{a}}(z, s)$ associated with a singular cusp $\mathfrak{a}$ has a similar expansion

$$\begin{aligned} E_{\mathfrak{a}}(z, s) = \delta_{\mathfrak{a}} y^s + \varphi_{\mathfrak{a}}(s) y^{1-s} \\ + \sum_{n \neq 0} \rho_{\mathfrak{a}}(n, t) W_{\frac{kn}{2|n|}, it}(4\pi|n|y)e(nx) \end{aligned} \tag{5.3}$$

for $s = \frac{1}{2} + it$.

In this section we establish a summation formula for the coefficients $\rho_j(n)$, $\rho_{\mathfrak{a}}(n, t)$ with respect to the spectrum. To this end we introduce two Poincaré series of type

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma) j_\gamma(z)^{-k} P(m\gamma z) \tag{5.4}$$

$$Q_n(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{\chi}(\gamma) j_\gamma(z)^{-k} Q(m\gamma z) \tag{5.5}$$

where $P(z)$, $Q(z)$ are suitable functions on $\mathbb{H}$ such that the series converge (we do not require the absolute convergence). Let $p(y)$, $q(y)$ be smooth functions on $\mathbb{R}^+$ such that $p(y)$, $q(y)$, $yp'(y)$, $yq'(y)$ are all bounded. We take

$$P(z) = 4\pi y p(4\pi y) e(z)$$

$$Q(z) = 4\pi y q(4\pi y) e(z)\ .$$

Let m, n be positive integers. Then $P_m(z)$ and $Q_n(z)$ are in $\mathcal{L}_k(\Gamma, \chi)$. The starting point for our derivation of the formula in question is the inner product

$$\langle\, P_m,\, Q_n \,\rangle \;=\; \int_{\Gamma\backslash\mathbb{H}} P_m(z)\overline{Q}_n(z)d\mu z\ .$$

First we compute $\langle\, P_m,\, Q_n \,\rangle$ by using the spectral expansion

$$P_m(z) = \sum_j \langle\, P_m, u_j \,\rangle\, u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle\, P_m, E_{\mathfrak{a}}(*, s) \,\rangle\, E_{\mathfrak{a}}(z, s)dt$$

where $s = \frac{1}{2} + it$ (add (4.46) to (4.50)). Hence the Parseval formula

$$\langle\, P_m, Q_n \,\rangle \;=\; \sum_j \langle\, P_m, u_j \,\rangle\ \langle\, u_j, Q_n \,\rangle + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle\, P_m, E_{\mathfrak{a}}(*, s) \,\rangle\ \langle\, E_{\mathfrak{a}}(*, s), Q_n \,\rangle\, dt\ .$$

This formula reduces the problem to computing the projections of the Poincaré series on the basic Maass forms. By unfolding the fundamental domain we deduce using the Fourier series (5.1) that

$$\begin{aligned}\langle\, P_m, u_j \,\rangle \;&=\; \int_{\Gamma_\infty\backslash\mathbb{H}} P(mz)\bar{u}_j(z)d\mu z\\ &= 4\pi m\overline{\rho}_j(m) \int_0^{\infty} p(4\pi my)e^{-2\pi my}W_{\frac{k}{2},it_j}(4\pi my)y^{-1}dy\\ &= 4\pi m\overline{\rho}_j(m)h_p(t_j)\end{aligned}$$

where

$$h_p(t) = \int_0^{\infty} p(y)e^{-y/2}W_{\frac{k}{2},it}(y)y^{-1}dy\ . \tag{5.6}$$

The same argument works for the inner product against the Eisenstein series giving

$$\langle\, P_m, E_{\mathfrak{a}}(*, s) \,\rangle \;=\; 4\pi m\overline{\rho}_{\mathfrak{a}}(m, t)h_p(t)\ .$$

Inserting these results into the Parseval formula we conclude that

$$\langle\, P_m, Q_n \,\rangle \;=\; 16\pi^2 mn\Big\{\sum_j h_p(t_j)\bar{h}_q(t_j)\overline{\rho}_j(m)\rho_j(n) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h_p(t)\bar{h}_q(t)\overline{\rho}_{\mathfrak{a}}(m, t)\rho_{\mathfrak{a}}(n, t)dt\Big\}\ . \tag{5.7}$$

Another way of computing the inner product $\langle P_m, Q_n \rangle$ goes by unfolding with respect to Q_n. This leads to

$$\langle P_m, Q_n \rangle = \int_{\Gamma_\infty \backslash \mathbb{H}} P_m(z)\overline{Q}(nz)d\mu z \, . \tag{5.8}$$

The Poincaré series $P_m(z)$ is periodic in x of period one, so it has the Fourier expansion

$$P_m(z) = \sum_{\ell=-\infty}^{\infty} p(m, \ell; y)e(\ell x) \tag{5.9}$$

with the Fourier coefficients $p(m, \ell; y)$ which are functions of y. By (5.8) and (5.9) we arrive at

$$\langle P_m, Q_n \rangle = \int_0^\infty p(m, n; y)\overline{Q}(iny)y^{-2}dy \, . \tag{5.10}$$

Now we refine the Fourier expansion (5.9) using the series (5.4). The identity motion $\gamma = 1$ contributes $P(mz)$. The other cosets are parametrized by pairs of integers $\{c, d\}$ with $c > 0$, $c \equiv 0 \pmod D$ and $(c, d) = 1$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $ad \equiv 1 \pmod c$,

$$\gamma z = \frac{a}{c} - \frac{1}{c(cz + d)}$$

and

$$P(m\gamma z) = e\Big(\frac{am}{c}\Big)P\Big(\frac{-m}{c(cz + d)}\Big) \, .$$

Therefore $P_m(z)$ splits into

$$P_m(z) = P(mz) + \sum_{c\equiv 0(D)} \sum_{(d,c)=1} \overline{\chi}(d)e\Big(\frac{\bar{d}m}{c}\Big)\Big(\frac{|cz + d|}{cz + d}\Big)^k P\Big(\frac{-m}{c(cz + d)}\Big)$$

and here we can further split the inner sum over d prime to c into reduced classes modulo c. For each such class we apply Poisson's formula obtaining

$$P_m(z) = P(mz) + \sum_{c\equiv 0(D)} \sum_{\ell} S_\chi(m, \ell; c) \int_{-\infty}^{\infty} \Big(\frac{|z + u|}{z + u}\Big)^k P\Big(\frac{-mc^{-2}}{z + u}\Big)e(-\ell u)du \, .$$

where $S_\chi(m, \ell; c)$ is the Kloosterman sum

$$S_\chi(m, \ell; c) = \sideset{}{^*}\sum_{d \pmod c} \overline{\chi}(d)e\Big(\frac{\bar{d}m + d\ell}{c}\Big) \, . \tag{5.11}$$

Changing the variable $u = \mathfrak{z} - z$ and transferring the factor $e^{4\pi\ell y}$ we get

$$P_m(z) = P(mz) + \sum_{\ell} \sum_{c\equiv 0(D)} S_\chi(m, \ell; c) P_c(m, \ell; y) e(\ell \bar{z}) \tag{5.12}$$

where

$$P_c(m, \ell; y) = \int_{\mathrm{Im}\,\mathfrak{z}=y} \left(\frac{|\mathfrak{z}|}{\mathfrak{z}}\right)^k P\left(\frac{-m}{\mathfrak{z}c^2}\right) e(-\bar{\mathfrak{z}}\ell) d\mathfrak{z}\ . \tag{5.13}$$

Hence the n^{th} Fourier coefficient of $P_m(z)$ is given by a series of Kloosterman sums

$$p(m, n; y) = \delta_{mn} P(iny) + \sum_{c\equiv 0(D)} S_\chi(m, n; c) P_c(m, n; y) e^{2\pi n y}. \tag{5.14}$$

Inserting (5.14) into (5.10) we derive by a simple change of the variables of integration $\mathfrak{z}$ and y the following expression

$$\begin{aligned} \langle\, P_m, Q_n \,\rangle\ = 4\pi\sqrt{mn} \Big\{ & \delta(m, n) I \\ & + \sum_{c\equiv 0(D)} c^{-1} S_\chi(m, n; c) I\left(\frac{4\pi\sqrt{mn}}{c}\right) \Big\} \end{aligned} \tag{5.15}$$

where

$$I = \int_0^\infty e^{-y} p(y) \bar{q}(y) dy \tag{5.16}$$

and $I(x)$ is given by the double integral

$$x \int_{\mathrm{Im}\,\mathfrak{z}=1} \left(\frac{|\mathfrak{z}|}{\mathfrak{z}}\right)^k \int_0^\infty p\left(\frac{x}{y|\mathfrak{z}|}\right) \bar{q}\left(\frac{xy}{|\mathfrak{z}|}\right) e\left(\frac{-x\bar{\mathfrak{z}}}{4\pi|\mathfrak{z}|}\left(y + \frac{1}{y}\right)\right) \frac{dy\, d\mathfrak{z}}{y|\mathfrak{z}|^2}.$$

Next we transform the horizontal line $\mathrm{Im}\,\mathfrak{z} = 1$ into the semi-circle $|\zeta| = 1$, $\mathrm{Re}\,\zeta > 0$ by the change of variables $\mathfrak{z} = 2i(\zeta^2 + 1)^{-1}$. We have $\mathfrak{z} = i(\zeta\,\mathrm{Re}\,\zeta)^{-1}$, $|\mathfrak{z}| = (\mathrm{Re}\,\zeta)^{-1} = -i\mathfrak{z}\zeta$ and $|\mathfrak{z}|^{-2} d\mathfrak{z} = -i\zeta^{-1} d\zeta$. Hence

$$\begin{aligned} I(x) = -x \int_{-i}^{i} (-i\zeta)^{k-1} \int_0^\infty & e^{-\frac{1}{2}\zeta x(y+y^{-1})} \\ & p\left(xy^{-1}\,\mathrm{Re}\,\zeta\right) \bar{q}\left(xy\,\mathrm{Re}\,\zeta\right) y^{-1} dy\, d\zeta \end{aligned} \tag{5.17}$$

where the integration over ζ runs counter-clockwise along right half of the unit circle.

Finally combining (5.8) and (5.15) we conclude the following

Proposition 5.1. *Let $p(y)$, $q(y)$ be smooth functions on $\mathbb{R}^+$ such that $p(y)$, $q(y)$ and $yp'(y)$, $yq'(y)$ are bounded. For any positive integers m, n we have*

$$4\pi^2\sqrt{mn}\Big(\sum_j h(t_j)\overline{\rho}_j(m)\rho_j(n) + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\overline{\rho}_{\mathfrak{a}}(m,t)\rho_{\mathfrak{a}}(n,t)dt\Big)$$

$$= \delta(m,n)I + \sum_{c\equiv 0(D)} c^{-1}S_\chi(m,n;c)I\Big(\frac{4\pi\sqrt{mn}}{c}\Big) \quad (5.18)$$

where

$$h(t) = \frac{1}{\pi}h_p(t)\bar{h}_q(t)\ ,$$

I is given by (5.16), $I(x)$ is given by (5.17) and $S_\chi(m,n;c)$ denotes the Kloosterman sum.

The same formula (5.18) remains true if both m, n are negative integers provided that one changes k into $-k$ in (5.6) and (5.17). The arguments of the proof are very much the same so we do not repeat them here. When m, n have different sign then again a formula similar to (5.18) is true, but with a somewhat different function $I(x)$, however we are able to avoid its use in our applications.

In fact we shall only require Proposition 5.1 for the special test functions

$$p(y) = q(y) = y^{ir}$$

where r is a real parameter. This choice gives rise to the standard Poincaré series

$$P_m(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(d)\Big(\frac{cz+d}{|cz+d|}\Big)^{-k}\Big(\frac{4\pi m y}{|cz+d|^2}\Big)^{1+ir} e(m\gamma z)\ .$$

In this case (5.6) gives by (7.621.11) of [GR]

$$h_p(t) = \Gamma\Big(\frac{1}{2}+i(r-t)\Big)\Gamma\Big(\frac{1}{2}+i(r+t)\Big)\Gamma\Big(1-\frac{k}{2}-ir\Big)^{-1}.$$

Hence by the functional equation for the gamma function

$$h(t) = \pi\Big|\Gamma\Big(1-\frac{k}{2}-ir\Big)\Big|^{-2}(\operatorname{ch}\pi(r-t)\operatorname{ch}\pi(r+t))^{-1}\ . \quad (5.19)$$

Next the integral (5.17) simplifies to

$$I(x) = -x\int_{-i}^{i}(-i\zeta)^{k-1}\int_0^\infty e^{-\frac{1}{2}\zeta x(y+y^{-1})}y^{-1-2ir}dy\,d\zeta\ , \quad (5.20)$$

and the inner integral is $2K_{2ir}(\zeta x)$ where $K_\nu(y)$ is the K–Bessel function. Therefore

$$I(x) = I(x,r) = -2x \int_{-i}^{i} (-i\zeta)^{k-1} K_{2ir}(\zeta x) d\zeta \ . \tag{5.21}$$

In Lemma 17.2 we shall give another integral representation for $I(x,r)$ in terms of the J–Bessel function.

Finally (5.16) gives $I = 1$. Thus, for our special test functions, Proposition 5.1 becomes

Proposition 5.2. *For any positive integers m, n and any real number r we have*

$$\sum_j \frac{\overline{\rho}_j(m)\rho_j(n)}{\operatorname{ch}\pi(r-t_j)\operatorname{ch}\pi(r+t_j)} + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\overline{\rho}_{\mathfrak{a}}(m,t)\rho_{\mathfrak{a}}(n,t)dt}{\operatorname{ch}\pi(r-t)\operatorname{ch}\pi(r+t)}$$

$$= \frac{\left|\Gamma\left(1-\frac{k}{2}-ir\right)\right|^2}{4\pi^3\sqrt{mn}} \left(\delta(m,n) + \sum_{c\equiv 0(D)} c^{-1} S_\chi(m,n;c) I\left(\frac{4\pi\sqrt{mn}}{c}\right)\right)$$

where $I(x)$ is given by (5.21).

Having the real parameter r at our disposal we could create a large class of test functions $h(t)$ in (5.18) (similar to that in the Selberg trace formula) by integrating over r against a properly chosen distribution $q(r)$. This integration will be needed to improve our estimates for the resulting function $I(x)$ (for small x) which is required in order to accelerate the convergence of the sums of Kloosterman sums on the right side of (5.18). Nevertheless, because a substantial part of our arguments can be carried out more clearly for individual r we have chosen to postpone the integration in question to the last possible moment (see Sect. 14).

6. Hecke operators

For any $n \geqslant 1$ the Hecke operator T_n is defined on functions $f : \mathbb{H} \to \mathbb{C}$ which are periodic of period one by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \pmod d} f\left(\frac{az+b}{d}\right) . \tag{6.1}$$

These operators are multiplicative, precisely they satisfy

$$T_m T_n = \sum_{d|(m,n)} \chi(d) T_{mnd^{-2}} \ . \tag{6.2}$$

Hence the Hecke operators commute with each other.

Note that due to our normalization T_n does not depend on k, but we are mostly interested on its action on automorphic functions of weight k. Clearly $T_n : \mathcal{A}_k(\Gamma, \chi) \to \mathcal{A}_k(\Gamma, \chi)$. Since T_n commutes with Δ_k it also acts on eigenspaces; $T_n : \mathcal{A}_k(\Gamma, \chi; s) \to \mathcal{A}_k(\Gamma, \chi; s)$. For $(n, D) = 1$ the operator T_n is normal, precisely we have

$$\langle T_n f, g \rangle = \langle f, \overline{\chi}(n) T_n g \rangle \tag{6.3}$$

for every f, g in $\mathcal{L}_k(\Gamma, \chi)$. Therefore we can choose an orthonormal basis $\{u_j(z)\}$ of $\mathcal{C}_k(\Gamma, \chi)$ which consists of common eigenfunctions of all T_n with $(n, D) = 1$. We call such functions the Hecke-Maass cusp forms.

Because $\chi \pmod D$ is primitive the Hecke operators enjoy the so called multiplicity-one property. This means that any two functions in $\mathcal{L}_k(\Gamma, \chi)$ which are eigenfunctions of all T_n for $(n, D) = 1$ with the same eigenvalues are equal up to a constant factor. Consequently if $T : \mathcal{L}_k(\Gamma, \chi) \to \mathcal{L}_k(\Gamma, \chi)$ is a linear operator over $\mathbb{C}$ which commutes with all T_n for $(n, D) = 1$, then every common eigenfunction of all T_n for $(n, D) = 1$ is also an eigenfunction of T. This property is the key fact in the Atkin-Lehner theory of newforms, see [AL], [L], [I3]. We would like to emphasize that the multiplicity-one property fails for automorphic functions which are not square-integrable!

By the multiplicity-one property it follows that the Hecke-Maass cusp forms $u_j(z)$ are eigenfunctions of all T_n, not just those with $(n, D) = 1$, never mind that they are not all normal operators. Therefore we have for all $n \geqslant 1$

$$T_n u_j = \lambda_j(n) u_j \,. \tag{6.4}$$

By (6.2) these eigenvalues satisfy the multiplicativity formula

$$\lambda_j(m)\lambda_j(n) = \sum_{d|(m,n)} \chi(d)\lambda_j(mnd^{-2}) \tag{6.5}$$

for all $m, n \geqslant 1$. If $(n, D) = 1$ then (6.3) implies that

$$\lambda_j(n) = \chi(n)\overline{\lambda}_j(n) \,. \tag{6.6}$$

For every $p \mid D$ we have $\lambda_j(p^\ell) = \lambda_j(p)^\ell$ with $|\lambda_j(p)| = 1$.

Define the operator W_k acting on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(W_k f)(z) = \left(\frac{|z|}{z}\right)^k f\left(\frac{-1}{zD}\right) . \tag{6.7}$$

Note that $W_k : \mathcal{A}_k(\Gamma, \chi) \to \mathcal{A}_k(\Gamma, \overline{\chi})$. We have

$$y^{-\frac{k}{2}} W_k\Big(y^{\frac{k}{2}} F(z)\Big) = \big(z\sqrt{D}\big)^{-k} F\big(-1/zD\big) \tag{6.8}$$

which gives the action of W_k in the more familiar format of holomorphic forms.

In order to return to the original character we compose W_k with the operator $\overline{X}$ defined by

$$(\overline{X} f)(z) = \bar{f}(-\bar{z}) \tag{6.9}$$

(this is the complex conjugation of the operator X). Put $\overline{W}_k = \overline{X} W_k$ so

$$\left(\overline{W}_k f\right)(z) = \left(\frac{|z|}{-z}\right)^k \bar{f}(1/\bar{z}D) \, . \tag{6.10}$$

Obviously $\overline{W}_k$ is an involution, i.e. $\overline{W}_k^2 = 1$. But note that $\overline{W}_k$ is not linear over $\mathbb{C}$, precisely we have

$$\overline{W}_k(\lambda f) = \bar{\lambda}\, \overline{W}_k f \qquad \text{for } \lambda \in \mathbb{C} \, . \tag{6.11}$$

One can show that $\overline{W}_k : \mathcal{A}_k(\Gamma, \chi) \to \mathcal{A}_k(\Gamma, \chi)$ (which is not obvious, see p. 112 of [I3]) and that $\overline{W}_k$ is an isometry on $\mathcal{L}_k(\Gamma, \chi)$. Since W_k and $\overline{X}$ commute with Δ_k, so does $\overline{W}_k$. Therefore $\overline{W}_k : \mathcal{C}_k(\Gamma, \chi; s) \to \mathcal{C}_k(\Gamma, \chi; s)$.

The operator $\overline{W}_k$ does not precisely commute with the Hecke operators, but it satisfies

$$T_n \, \overline{W}_k = \chi(n) \overline{W}_k \, T_n \qquad \text{if } (n, D) = 1 \, . \tag{6.12}$$

The three properties (6.6), (6.11), (6.12) suffice to imply by the multiplicity-one property that the Hecke-Maass cusp forms $u_j(z)$ are also eigenfunctions of the involution $\overline{W}_k$. We have

$$\overline{W}_k \, u_j = \eta_j \, u_j \tag{6.13}$$

for a complex number η_j having $|\eta_j| = 1$ (the eigenvalue η_j does not need to be real even if the character χ is real!).

Applying the Hecke operator (6.1) to the Fourier expansion (5.1) one can see quickly that the Fourier coefficients $\rho_j(n)$ are proportional to the Hecke eigenvalues $\lambda_j(|n|)$, specifically

$$\rho_j(n) = \rho_j(1) \lambda_j(n) n^{-\frac{1}{2}} \tag{6.14}$$

$$\rho_j(-n) = \rho_j(-1) \lambda_j(n) n^{-\frac{1}{2}} \tag{6.15}$$

for any $n \geqslant 1$, the factor $n^{-\frac{1}{2}}$ appearing due to our earlier normalization.

The above theory of Hecke operators acting on Maass cusp forms does not exactly apply to the Eisenstein series. Later, in Sect. 7, we shall show that the Eisenstein series are eigenfunctions of Hecke operators T_n with $(n, D) = 1$, specifically

$$T_n \, E_{\mathfrak{a}}(z, s) = \lambda_{\mathfrak{a}}(n, t) E_{\mathfrak{a}}(z, s) \tag{6.16}$$

with

$$\lambda_{\mathfrak{a}}(n,t)=\sum_{ad=n}\chi_v(a)\chi_w(d)\Big(\frac{a}{d}\Big)^{it} \tag{6.17}$$

for $s=\frac{1}{2}+it$. Here $\mathfrak{a}$ is the cusp equivalent to $1/v$ with $vw=D$, $(v,w)=1$ and $\chi=\chi_v\,\chi_w$ is the decomposition of the primitive character χ (mod D) into characters of moduli v, w. In spite of the property (6.16) the multiplicity-one principle does not apply because $E_{\mathfrak{a}}(z,s)$ is not square-integrable. Therefore we cannot argue that $E_{\mathfrak{a}}(z,s)$ is an eigenfunction of the involution $\overline{W}_k$. In fact we shall show in Sect. 7 the following relation

$$\overline{W}_k\,E_{\mathfrak{a}}(z,s)=E_{\bar{\mathfrak{a}}}(z,s) \tag{6.18}$$

where $\bar{\mathfrak{a}}$ stands for the cusp "dual" to $\mathfrak{a}$ in the sense that if $\mathfrak{a}\sim 1/v$ then $\bar{\mathfrak{a}}\sim 1/w$ with $vw=D$, $(v,w)=1$. This reveals that the square-integrability plays a significant role in the arithmetic of Maass forms.

In Sect. 7, rather than using Hecke operators in the fragile space of the continuous spectrum, we shall perform direct and explicit computations on the Eisenstein series. We shall establish the following formulas for the Fourier coefficients:

$$\rho_{\mathfrak{a}}(n,t)=\rho_{\mathfrak{a}}(1,t)\lambda_{\mathfrak{a}}(n,t)n^{-\frac{1}{2}}, \tag{6.19}$$

$$\rho_{\mathfrak{a}}(-n,t)=\rho_{\mathfrak{a}}(-1,t)\lambda_{\mathfrak{a}}(n,t)n^{-\frac{1}{2}}, \tag{6.20}$$

where $\lambda_{\mathfrak{a}}(n,t)$ is given by (6.17) for all $n\geqslant 1$.

By means of (6.14) and (6.19) we can rephrase our spectral summation formula (5.18) in terms of Hecke eigenvalues.

Proposition 6.1. *For any positive integers m,n we have*

$$\begin{aligned}\sum_j h(t_j)\nu_j\bar{\lambda}_j(m)\lambda_j(n)&\\ +\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}h(t)\nu_{\mathfrak{a}}(t)\bar{\lambda}_{\mathfrak{a}}(m,t)\lambda_{\mathfrak{a}}(n,t)dt&\\ =\delta(m,n)I+\sum_{c\equiv 0(D)}c^{-1}S_\chi(m,n;c)I\Big(\frac{4\pi}{c}\sqrt{mn}\Big)&\end{aligned} \tag{6.21}$$

where we put

$$\nu_j=\mid 2\pi\rho_j(1)\mid^2 \tag{6.22}$$

$$\nu_{\mathfrak{a}}(t)=\mid 2\pi\rho_{\mathfrak{a}}(1,t)\mid^2, \tag{6.23}$$

while the other notation and conditions are as in Proposition 5.1.

Finally notice that the Hecke operators T_n commute with the Maass operators Λ_ℓ, K_ℓ and with the reflection operator X. Hence all T_n commute with the involution Q_{sk} on $\mathcal{C}_k(\Gamma, \chi; s)$ for any $s \not\equiv \frac{k}{2} \pmod 1$. Therefore the Hecke-Maass cusp forms $u_j(z)$ of weight $k \geqslant 0$ and eigenvalue $\lambda(s_j)$ with $s_j \not\equiv \frac{k}{2} \pmod 1$ (i.e. those whose ancestors are not holomorphic forms) are automatically eigenfunctions of Q_{sk} (by the multiplicity-one property), say

$$Q_{s_j k} u_j(z) = \varepsilon_j\, u_j(z) \tag{6.24}$$

with $\varepsilon_j = \pm 1$. Consequently for $s_j \not\equiv \frac{k}{2} \pmod 1$ we conclude the orthogonal decomposition

$$\mathcal{C}_k(\Gamma, \chi; s_j) = \mathcal{C}_k^+(\Gamma, \chi; s_j) \oplus \mathcal{C}_k^-(\Gamma, \chi; s_j) \tag{6.25}$$

where $\mathcal{C}_k^+$, $\mathcal{C}_k^-$ denote the subspaces of even and odd forms respectively.

7. Explicit computations on Eisenstein series

In this section we prove (6.16), (6.18) and we compute the coefficients $\rho_{\mathfrak{a}}(n, t)$ in the Fourier expansion (4.72) for the Eisenstein series $E_{\mathfrak{a}}(z, s)$. The results are quite explicit and will be needed in Sect. 13 to match and cancel a large contribution coming from the other side of the formula.

Recall that $\mathfrak{a} \sim \frac{1}{v}$ with $vw = D$, $(v, w) = 1$ so w is the "width" of $\mathfrak{a}$. We begin by writing in more explicit form the series (4.44). The scaling matrix can be chosen as

$$\sigma_{\mathfrak{a}} = \begin{pmatrix} \sqrt{w} & 0 \\ v\sqrt{w} & 1/\sqrt{w} \end{pmatrix}. \tag{7.1}$$

Hence

$$\begin{aligned} \sigma_{\mathfrak{a}}^{-1}\Gamma = \Bigg\{ \tau = \begin{pmatrix} a/\sqrt{w} & b/\sqrt{w} \\ c\sqrt{w} & d\sqrt{w} \end{pmatrix}; \\ \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad c + av \equiv 0 (vw) \Bigg\} \end{aligned} \tag{7.2}$$

and for $\tau \in \sigma_{\mathfrak{a}}^{-1}\Gamma$ we have

$$\sigma_{\mathfrak{a}}\tau = \begin{pmatrix} a & b \\ c + av & d + bv \end{pmatrix}. \tag{7.3}$$

Hence, factoring the character $\chi = \chi_v\, \chi_w$, we find that $\overline{\chi}(\sigma_{\mathfrak{a}}\tau) = \chi(a) = \chi_v(a)\chi_w(a) = \overline{\chi}_v(d)\chi_w(-c/v)$. The coset $\Gamma_\infty \setminus \sigma_{\mathfrak{a}}^{-1}\Gamma$ is parametrized by pairs of integers $(c, d) = 1$, $c \equiv 0 \pmod v$, $(c/v, w) = 1$, because a runs modulo cw and is determined by the congruences $a \equiv \bar{d} \pmod c$, $a + c/v \equiv$

0 (mod w) (note that $(c,w)=1$ since $(v,w)=1$ and $(c/v,w)=1$). Hence we deduce that

$$E_{\mathfrak{a}}(z,s)=\frac{1}{2}\Big(\frac{y}{w}\Big)^{s}\sum\sum_{(c,d)=1}\overline{\chi}_v(d)\chi_w(-c)J_s(c,d;vz) \tag{7.4}$$

where

$$J_s(c,d;z)=\Big(\frac{cz+d}{|cz+d|}\Big)^{-k}|cz+d|^{-2s}\,. \tag{7.5}$$

From the explicit formula (7.4) it is easy to see that

$$\overline{X}\,E_{\mathfrak{a}}^{\chi}(z,s)=\chi_w(-1)E_{\mathfrak{a}}^{\overline{\chi}}(z,s) \tag{7.6}$$

where $\overline{X}$ is the reflection operation (6.9) and the superscript in the Eisenstein series indicates (temporarily) the nebentypus character. We also see from (7.4) that

$$E_{\tilde{\mathfrak{a}}}^{\overline{\chi}}\Big(\frac{-1}{Dz},s\Big)=\chi_w(-1)\Big(\frac{z}{|z|}\Big)^{k}E_{\mathfrak{a}}^{\chi}(z,s)\,.$$

In other words

$$W_kE_{\mathfrak{a}}^{\chi}(z,s)=\chi_w(-1)E_{\tilde{\mathfrak{a}}}^{\overline{\chi}}(z,s)\,. \tag{7.7}$$

Applying (7.7) followed by (7.6) we arrive at (6.18).

Next we proceed to the derivation of (6.16). For convenience we remove the condition $(c,d)=1$ in (7.4) by means of Möbius inversion, getting

$$E_{\mathfrak{a}}(z,s)=\frac{(y/w)^s}{2L(2s,\overline{\chi}_v\,\chi_w)}\sum\sum_{(c,d)\neq(0,0)}\overline{\chi}_v(d)\chi_w(-c)J_s(c,d;vz)\,. \tag{7.8}$$

Applying the Hecke operator T_n (to avoid confusion of notation we rename a, d, b in (6.1) as α, δ, β respectively) we obtain

$$\begin{aligned}T_n\,E_{\mathfrak{a}}(z,s)=\frac{n^{s-\frac{1}{2}}(y/w)^s}{2L(2s,\overline{\chi}_v\,\chi_w)}\sum_{\alpha\delta=n}\chi(\alpha)\sum_{\beta\pmod{\delta}}&\\ \sum\sum_{(c,d)\neq(0,0)}\overline{\chi}_v(d)\chi_w(-c)J_s(c,\ell;\alpha vz)&\end{aligned} \tag{7.9}$$

where $\ell=\beta cv+\delta d$. Note that the condition $(c,d)\neq(0,0)$ is equivalent to $(c,\ell)\neq(0,0)$.

Now suppose $(n,v)=1$. Thus $(\delta,v)=1$ and $\overline{\chi}_v(d)=\overline{\chi}_v(\ell)\chi_v(\delta)$. Put $(\delta,c)=\eta$, $\delta=\eta\delta_1$, $c=\eta c_1$ (so that $(\delta_1,c_1)=1$) and $\ell=\eta\ell_1$. We have $\beta c_1 v + \delta_1 d = \ell_1$ so $\beta c_1 v \equiv \ell_1 \pmod{\delta_1}$ and the number of $\beta \pmod{\delta}$ satisfying this congruence is η. Therefore we have

$$T_n\, E_{\mathfrak{a}}(z,s) = \frac{n^{s-\frac{1}{2}}(y/w)^s}{2L(2s,\overline{\chi}_v\chi_w)} \sum_{\alpha\eta\delta_1=n} \chi(\alpha)\eta^{1-2s}$$

$$\sum\sum_{\substack{(c_1,\ell_1)\neq(0,0)\\ (c_1,\delta_1)=1}} \overline{\chi}_v(\ell_1)\chi_v(\delta_1)\chi_w(-\eta c_1) J_s(c_1,\ell_1;\alpha v z)\ .$$

We relax the condition $(c_1,\delta_1)=1$ by Möbius inversion getting

$$T_n\, E_{\mathfrak{a}}(z,s) = \frac{n^{s-\frac{1}{2}}(y/w)^s}{2L(2s,\overline{\chi}_v\chi_w)} \sum_{\alpha\eta\delta_2\delta_3=n} \chi(\alpha)\eta^{1-2s}\mu(\delta_2)\chi_v(\delta_2\delta_3)\chi_w(\eta\delta_2)$$

$$\sum\sum_{(c_2,\ell_1)\neq(0,0)} \overline{\chi}_v(\ell_1)\chi_w(-c_2) J_s(c_2,\ell_1;\gamma v z)$$

where $\gamma=\alpha\delta_2$. Here $\chi(\alpha)\chi_v(\delta_2)\chi_w(\delta_2)=\chi(\gamma)$ and the sum $\sum\mu(\delta_2)$ over $\alpha\delta_2=\gamma$ vanishes unless $\gamma=1$ in which case $\alpha=\delta_2=1$, giving

$$T_n\, E_{\mathfrak{a}}(z,s) = \frac{n^{s-\frac{1}{2}}(y/w)^s}{2L(2s,\overline{\chi}_v\chi_w)} \Bigg(\sum_{\eta\delta_3=n} \eta^{1-2s}\chi_v(\delta_3)\chi_w(\eta)\Bigg)$$

$$\sum\sum_{(c_2,\ell_1)\neq(0,0)} \overline{\chi}_v(\ell_1)\chi_w(-c_2) J_s(c_2,\ell_1;vz)\ ,$$

and by (7.8) this is just (6.16) with the eigenvalue given by (6.17).

Recall that the above computations are only valid if $(n,v)=1$. The fact that $E_{\mathfrak{a}}(z,s)$ is an eigenfunction for all the other T_n with eigenvalues $\lambda_{\mathfrak{a}}(n,t)$ given by (6.17) as well will be seen as a by–product of our computation of the Fourier coefficients $\rho_{\mathfrak{a}}(n,t)$ of $E_{\mathfrak{a}}(z,s)$ which we give next.

We begin from (7.8). Let $Z(c)$ denote the contribution for given c. The case of $c=0$ is special. In this case $w=1$ so $v=D$ which means $\mathfrak{a}\sim\infty$ and we find that

$$Z(0) = \delta_{\mathfrak{a}\infty} y^s\ . \tag{7.10}$$

For $c\neq 0$ we have $Z(c)=Z(-c)$ so it suffices to consider $c\geqslant 1$. In this case d runs over all integers so we can apply Poisson's formula

$$\sum_d \overline{\chi}_v(d)G(d) = v^{-1}\tau_{\overline{\chi}_v}\sum_h \chi_v(h)\hat{G}\Big(\frac{h}{v}\Big) \tag{7.11}$$

where $\hat{G}$ is the Fourier transform of G

$$\hat{G}(y) = \int_{-\infty}^{\infty} G(x)e(-xy)dx\ .$$

This gives

$$Z(c) = \frac{\chi_w(-c)(y/w)^s}{2vL(2s, \overline{\chi}_v\chi_w)}\tau_{\overline{\chi}_v}$$
$$\sum_h \chi_v(h)\int_{-\infty}^{\infty} J_s(c, \xi; vz)e\Big(\frac{-\xi h}{v}\Big)d\xi\ .$$

Here the integral is equal to $(cv)^{1-2s}e(hcx)I_s(hc, y)$, where

$$I_s(n, y) = \int_{-\infty}^{\infty}\Big(\frac{u+iy}{|u+iy|}\Big)^{-k}|u+iy|^{-2s}e(-nu)du$$

by (7.5) and by a change of variable $\xi = cvu$. For $n = 0$ we have

$$I_s(0, y) = 2\pi i^{-k}\Gamma(2s-1)\Gamma\Big(s-\frac{k}{2}\Big)^{-1}\Gamma\Big(s+\frac{k}{2}\Big)^{-1}(2y)^{1-2s}$$

by (8.381.1) of [GR], and for $n \neq 0$ we have

$$I_s(n, y) = i^{-k}\Gamma\Big(s+\frac{kn}{2|n|}\Big)^{-1}W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y)\Big(\frac{\pi|n|}{y}\Big)^s|n|^{-1}$$

where $W_{\alpha,\beta}(y)$ is the Whittaker function by (3.384.9) of [GR].

Note that for $n = hc = 0$ with $c \geqslant 1$ we have $h = 0$ and $\chi_v(0) = 0$ unless $v = 1$, i.e. $\mathfrak{a} \sim 0$. From the above computations we obtain the Fourier expansion (4.72) with

$$\varphi_{\mathfrak{a}}(s) = \delta_{\mathfrak{a}0}\frac{4\pi i^k}{(4D)^s}\ \frac{\Gamma(2s-1)}{\Gamma\big(s-\frac{k}{2}\big)\Gamma\big(s+\frac{k}{2}\big)}\ \frac{L(2s-1, \chi)}{L(2s, \chi)} \tag{7.12}$$

and for $n \neq 0$

$$\rho_{\mathfrak{a}}(n, t) = \frac{i^k}{\sqrt{|n|}}\Big(\frac{\pi}{vD}\Big)^s\chi_v\Big(\frac{n}{|n|}\Big)\overline{\tau}_{\chi_v}\frac{\lambda_{\mathfrak{a}}(n, t)}{\Gamma\big(s+\frac{kn}{2|n|}\big)L(2s, \overline{\chi}_v\chi_w)} \tag{7.13}$$

where $\lambda_{\mathfrak{a}}(n, t)$ is given by (6.17) for any $n \neq 0$ (recall that $s = \frac{1}{2} + it$).

Using (7.13) we obtain an explicit formula for $\nu_{\mathfrak{a}}(t)$ defined in (6.23).

Proposition 7.1. *Let* $\mathfrak{a} \sim \frac{1}{v}$ *with* $vw = D$, $(v, w) = 1$ *and let* $\chi = \chi_v\chi_w$ *be the factorization of* $\chi \pmod D$ *into primitive characters* $\chi_v \pmod v$, $\chi_w \pmod w$. *Then for real t we have*

$$\nu_{\mathfrak{a}}(t) = \frac{4\pi^3}{D}\ \Big|\ \Gamma\Big(\frac{k+1}{2}+it\Big)L\big(1+2it, \overline{\chi}_v\chi_w\big)\Big|^{-2}\ . \tag{7.14}$$

Remarks. The above formula yields a rather precise lower bound

$$\nu_{\mathfrak{a}}(t) D(|t|+1)^k e^{-\pi|t|} \gg (\log(D+|t|))^{-2} . \tag{7.15}$$

In Sect. 19 we shall establish a result analogous to Proposition 7.1 for the cuspidal case and derive an estimate for ν_j, defined in (6.22), specifically

$$\nu_j D(|t_j|+1)^k e^{-\pi|t_j|} \gg (D+|t_j|)^{-\varepsilon}, \tag{7.16}$$

for any $\varepsilon > 0$, the implied constant depending only on ε and k.

Upper bounds for $\nu_{\mathfrak{a}}(t)$ and ν_j are also available but these are not needed for our applications.

8. L-functions

In this section we introduce the L–functions in which we are interested as well as some others which we shall require along the way. Let $\{u_j(z)\}$ be an orthonormal basis of the space $\mathcal{C}_k(\Gamma,\chi)$ which are Hecke-Maass cusp forms, and let $\left\{E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right)\right\}$ be the eigenpacket of the space $\mathcal{E}_k(\Gamma,\chi)$ which are Eisenstein series on the critical line.

To every $u_j(z)$ we associate the L–function

$$L_j(s) = \sum_1^\infty \lambda_j(n) n^{-s} \tag{8.1}$$

where $\lambda_j(n)$ are the corresponding eigenvalues of Hecke operators, and analogously, to every Eisenstein series $E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right)$ we associate the L–function

$$L_{\mathfrak{a}}(s,t) = \sum_1^\infty \lambda_{\mathfrak{a}}(n,t) n^{-s} \tag{8.2}$$

where $\lambda_{\mathfrak{a}}(n,t)$ are given by (6.17). Note that t does not here denote the imaginary part of s, but rather the spectral parameter.

By the multiplicativity of Hecke eigenvalues (see (6.5)) we obtain the Euler products

$$L_j(s) = \prod_p \left(1 - \lambda_j(p) p^{-s} + \chi(p^2) p^{-2s}\right)^{-1}, \tag{8.3}$$

$$L_{\mathfrak{a}}(s,t) = \prod_p \left(1 - \lambda_{\mathfrak{a}}(p,t) p^{-s} + \chi(p^2) p^{-2s}\right)^{-1}, \tag{8.4}$$

for $\operatorname{Re} s > 1$. We require analytic continuation and functional equations for each of these.

The case of Eisenstein series is slightly unusual, because $E_{\mathfrak{a}}(z,s)$ is not exactly an eigenfunction of the involution $\overline{W}_k$. However using the explicit formula for $\lambda_{\mathfrak{a}}(n, t)$ we can play directly. First we see from (6.17) that $L_{\mathfrak{a}}(s, t)$ factors into Dirichlet L–functions

$$L_{\mathfrak{a}}(s, t) = L(s - it, \chi_v)L(s + it, \chi_w) \tag{8.5}$$

and inherits the analytic continuation and the functional equation therefrom. Here recall that the singular cusp $\mathfrak{a}$ is equivalent to $1/v$ with $vw = D$, $(v, w) = 1$ and $\chi = \chi_v \chi_w$ is the factorization of the primitive character $\chi \pmod D$ into primitive characters $\chi_v \pmod v$, $\chi_w \pmod w$. Note that $L_{\mathfrak{a}}(s, t) = L_{\bar{\mathfrak{a}}}(s, -t)$, where $\bar{\mathfrak{a}}$ is the dual cusp. For $\mathfrak{a} \sim \infty$ or $\mathfrak{a} \sim 0$ we have

$$L_\infty(s, t) = L(s - it, \chi)\zeta(s + it) \tag{8.6}$$
$$L_0(s, t) = L(s + it, \chi)\zeta(s - it) \tag{8.7}$$

so these L–functions have a simple pole at $s = 1 \mp it$ which has residue $L(1 \mp 2it, \chi)$ respectively. If $\mathfrak{a}$ is not equivalent to ∞ or 0, then $L_{\mathfrak{a}}(s, t)$ is entire. For any $\mathfrak{a}$ and real t we put

$$\Lambda_{\mathfrak{a}}(s, t) = \Big(\frac{\sqrt{D}}{\pi}\Big)^s \Gamma\Big(\frac{s - it}{2} + \frac{1 - \chi_v(-1)}{4}\Big) \Gamma\Big(\frac{s + it}{2} + \frac{1 - \chi_w(-1)}{4}\Big) L_{\mathfrak{a}}(s, t) \tag{8.8}$$

and we find the following functional equation

$$\Lambda_{\mathfrak{a}}(s, t) = \omega \overline{\Lambda}_{\mathfrak{a}}(1 - \bar{s}, -t) \tag{8.9}$$

with the root number ω given by Gauss sums

$$\omega = i^{\frac{1}{2}(\chi_v(-1)+\chi_w(-1)-2)} \Big(\frac{v}{w}\Big)^{it} \tau_{\chi_v} \tau_{\chi_w} D^{-\frac{1}{2}} . \tag{8.10}$$

Next we examine a Hecke-Maass cusp form $u_j(z)$ in $\mathcal{C}_k\left(\Gamma, \chi; \frac{m}{2}\right)$ with $0 < m \leqslant k$, $m \equiv k \pmod 2$, i.e. having the Laplace eigenvalue $\lambda\left(\frac{m}{2}\right) = \frac{m}{2}\left(1 - \frac{m}{2}\right)$. Such a form is induced by a holomorphic cusp form $F(z)$ in $S_m(\Gamma, \chi)$. The Fourier coefficients $\rho_j(n)$ of $u_j(z)$ are connected to the coefficients a_n of $F(z)$ by the formula (4.83). Here $u_j(z)$ and $y^{\frac{m}{2}} F(z)$ are both assumed to have L_2–norm one so the factor $(-1)^{\frac{k-m}{2}} \beta(m, k)$ in (4.83) shows a discrepancy in normalization of the Whittaker function and the exponential function. This formula shows that $\rho_j(n) = \rho_j(1)\lambda_j(n)$ and $a_n = a_1\lambda(n)$ with Hecke eigenvalues $\lambda_j(n) = \lambda(n)$ for all $n \geqslant 1$. Therefore the L–function of $u_j(z)$ agrees with that of $F(z)$. In this case Hecke showed that

$$\Lambda_j(s) = \Big(\frac{\sqrt{D}}{2\pi}\Big)^s \Gamma\Big(s + \frac{m-1}{2}\Big) L_j(s) \tag{8.11}$$

is entire and satisfies the functional equation

$$\Lambda_j(s) = \omega_j \overline{\Lambda}_j(1-\bar{s}) \tag{8.12}$$

with the root number ω_j given by the corresponding eigenvalue of the involution $i^m W_m$. This can be expressed in terms of the Hecke eigenvalue $\lambda_j(D)$ and the Gauss sum τ_χ, precisely

$$\omega_j = i^m \bar{\lambda}_j(D) \bar{\tau}_\chi D^{-\frac{1}{2}} \ . \tag{8.13}$$

However, all we need to know about ω_j in this paper is that

$$|\omega_j| = 1 \ . \tag{8.14}$$

Now we proceed to the case of a Hecke-Maass cusp form $u_j(z)$ in $\mathcal{C}_k(\Gamma, \chi; s_j)$ with $s_j \not\equiv \frac{k}{2} \pmod 1$. In this case the form is induced by a non-holomorphic cusp form of weight $\kappa = 0, 1$, $\kappa \equiv k \pmod 2$ with the same Laplace eigenvalue $\lambda(s_j) = s_j(1-s_j)$ so we have one of the following two possibilities:

$$\mathrm{Re}\, s_j = \tfrac{1}{2} \ , \qquad s_j \neq \tfrac{1}{2} \tag{8.15}$$

$$0 < s_j < 1 \ , \qquad \text{if } k \equiv 0 \pmod 2 \ . \tag{8.16}$$

Actually the Selberg conjecture predicts that (8.16) never happens, but so far it resists proof.

The associated L–function (8.1) can be examined by arguments similar to these applied by Hecke to the classical (holomorphic) cusp forms. Unfortunately we could not find in the published literature satisfactory results so we are going to sketch proofs of what is needed in this paper. Our first objective is to establish a functional equation with appropriate gamma factors. Put

$$\Lambda_j(s) = \Big(\frac{\sqrt{D}}{\pi}\Big)^s \Gamma\Big(\frac{s+it_j}{2} + \frac{1-\varepsilon_j}{4}\Big) \Gamma\Big(\frac{s-it_j}{2} + \frac{1-\varepsilon_j(-1)^k}{4}\Big) L_j(s) \tag{8.17}$$

where $\varepsilon_j = \pm 1$ is the eigenvalue of the involution $Q_{s_j k}$ (see (6.24)).

Proposition 8.1. *The function $\Lambda_j(s)$ is entire and it satisfies*

$$\Lambda_j(s) = \omega_j \overline{\Lambda}_j(1-\bar{s}) \tag{8.18}$$

with the root number ω_j which depends on the corresponding eigenvalue η_j of the involution $\overline{W}_k$ (see (6.13)) as follows

$$\omega_j = i^{k+2\left[\frac{k}{2} - \frac{1-\varepsilon_j}{4}\right]} \bar{\eta}_j \, \bar{\rho}_j(1)/\rho_j(1) \ . \tag{8.19}$$

Remark. The appearance of the first Fourier coefficient $\rho_j(1)$ in the root number ω_j is superficial, the reason being that $\overline{W}_k$ is not linear over $\mathbb{C}$ and the form $u_j(z)$ is not perfectly normalized. Multiplying by a unitary factor we could assume that $\rho_j(1)$ is real so it would not have appeared in ω_j. Observe that the functional equation (8.18) does not, in the case of holomorphic forms, immediately resemble (8.11). To bring it to the same shape one requires applications of the recurrence and duplication formulae for the gamma function and this introduces a polynomial factor which can then be cancelled out since it enjoys the same symmetry.

We begin by writing the equation (6.13) at $z = iy/\sqrt{D}$ which becomes

$$u_j(iy/\sqrt{D}) = i^k \bar{\eta}_j\, \bar{u}_j(i/y\sqrt{D})\ . \tag{8.20}$$

Let $\Psi_j(s)$ denote the Mellin transform of $y^{-\frac{1}{2}} u_j(iy/\sqrt{D})$,

$$\Psi_j(s) = \int_0^\infty u_j(iy/\sqrt{D}) y^{s-\frac{3}{2}} dy\ . \tag{8.21}$$

By (8.20) we arrange this integral as follows

$$\Psi_j(s) = \int_1^\infty \left\{ u_j\Big(\frac{iy}{\sqrt{D}}\Big) y^{s-\frac{1}{2}} + i^k \bar{\eta}_j\, \bar{u}_j\Big(\frac{iy}{\sqrt{D}}\Big) y^{\frac{1}{2}-s} \right\} \frac{dy}{y}\ . \tag{8.22}$$

Hence $\Psi_j(s)$ is entire and it satisfies the functional equation

$$\Psi_j(s) = i^k \bar{\eta}_j\, \overline{\Psi}_j(1-\bar{s})\ . \tag{8.23}$$

Next we compute $\Psi_j(s)$ by integrating the Fourier expansion

$$\begin{aligned} u_j\Big(\frac{iy}{\sqrt{D}}\Big) &= \sum_{n\neq 0} \rho_j(n) W_{\frac{kn}{2|n|}, it_j}\Big(\frac{4\pi|n|y}{\sqrt{D}}\Big) \\ &= \sum_1^\infty \frac{\lambda_j(n)}{\sqrt{n}} \left\{ \rho_j(1) W_{\frac{k}{2}, it_j} + \rho_j(-1) W_{-\frac{k}{2}, it_j} \right\} \Big(\frac{4\pi n y}{\sqrt{D}}\Big)\ , \end{aligned}$$

see (6.14) and (6.15). Note that

$$\rho_j(-1) = \varepsilon_j \rho_j(1) \frac{\Gamma\left(s_j + \frac{k}{2}\right)}{\Gamma\left(s_j - \frac{k}{2}\right)}$$

by (4.70). Hence we derive by a change of variable

$$\Psi_j(s) = 4D^{-\frac{1}{4}} \Big(\frac{\sqrt{D}}{\pi}\Big)^s \Phi_k^\varepsilon(s, it_j) \rho_j(1) L_j(s) \tag{8.24}$$

where we put

$$\Phi_k^\varepsilon(s,\beta) = \frac{\sqrt{\pi}}{4}\int_0^\infty \Big\{W_{\frac{k}{2},\beta}(4y) + \varepsilon\frac{\Gamma\left(\beta+\frac{1+k}{2}\right)}{\Gamma\left(\beta+\frac{1-k}{2}\right)}W_{-\frac{k}{2},\beta}(4y)\Big\}y^{s-\frac{3}{2}}dy\,. \tag{8.25}$$

We shall prove by induction on k the following result.

Lemma 8.2. *For $\varepsilon = \pm 1$ and any complex numbers s, β with $\operatorname{Re} s > |\operatorname{Re}\beta|$ we have*

$$\Phi_k^\varepsilon(s,\beta) = p_k^\varepsilon(s,\beta)\Gamma\Big(\frac{s+\beta}{2}+\frac{1-\varepsilon}{4}\Big)\Gamma\Big(\frac{s-\beta}{2}+\frac{1-\varepsilon(-1)^k}{4}\Big) \tag{8.26}$$

where the $p_k^\varepsilon(s,\beta)$ are the polynomials in s recursively defined below.

Proof. First we establish a recursion formula for

$$V_{k,\beta}^\varepsilon(y) = W_{\frac{k}{2},\beta}(4y) + \varepsilon\frac{\Gamma\left(\beta+\frac{1+k}{2}\right)}{\Gamma\left(\beta+\frac{1-k}{2}\right)}W_{-\frac{k}{2},\beta}(4y)\,. \tag{8.27}$$

Combining (9.234.1) and (9.234.2) of [GR] we derive the following two recursion formulae for the Whittaker function:

$$\begin{aligned}\frac{2\beta}{\sqrt{y}}W_{\alpha,\beta}(y) &= W_{\alpha+\frac{1}{2},\beta-\frac{1}{2}}(y) - W_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(y)\\ &= \left(\beta-\alpha+\tfrac{1}{2}\right)W_{\alpha-\frac{1}{2},\beta+\frac{1}{2}}(y) + \left(\beta+\alpha-\tfrac{1}{2}\right)W_{\alpha-\frac{1}{2},\beta-\frac{1}{2}}(y)\,.\end{aligned}$$

Hence we get

$$\begin{aligned}\frac{2\beta}{\sqrt{y}}V_{\kappa,\beta}^\varepsilon\Big(\frac{y}{4}\Big) &= \Big(\beta+\frac{1-k}{2}\Big)W_{\frac{k-1}{2},\beta+\frac{1}{2}}(y) + \Big(\beta+\frac{k-1}{2}\Big)W_{\frac{k-1}{2},\beta-\frac{1}{2}}(y)\\ &\quad + \varepsilon\frac{\Gamma\left(\beta+\frac{1+k}{2}\right)}{\Gamma\left(\beta+\frac{1-k}{2}\right)}\Big(W_{\frac{1-k}{2},\beta-\frac{1}{2}}(y) - W_{\frac{1-k}{2},\beta+\frac{1}{2}}(y)\Big)\,.\end{aligned}$$

Using $s\Gamma(s) = \Gamma(s+1)$ we rearrange the above terms into the recursion formula

$$\begin{aligned}\frac{\beta}{\sqrt{y}}V_{k,\beta}^\varepsilon(y) &= \Big(\beta+\frac{k-1}{2}\Big)V_{k-1,\beta-\frac{1}{2}}^\varepsilon(y)\\ &\quad + \Big(\beta+\frac{1-k}{2}\Big)V_{k-1,\beta+\frac{1}{2}}^{-\varepsilon}(y)\,.\end{aligned} \tag{8.28}$$

Integrating this we obtain by (8.25) the following recursion formula

$$\begin{aligned}\Phi_k^\varepsilon(s,\beta) &= \Big(1+\frac{k-1}{2\beta}\Big)\Phi_{k-1,\beta-\frac{1}{2}}^\varepsilon\Big(s+\frac{1}{2}\Big)\\ &\quad + \Big(1-\frac{k-1}{2\beta}\Big)\Phi_{k-1,\beta+\frac{1}{2}}^{-\varepsilon}\Big(s+\frac{1}{2}\Big)\end{aligned} \tag{8.29}$$

for any $k \geqslant 1$ and $\beta \neq 0$.

Now we are ready to prove (8.26) by induction on k. For $k = 0$ we have

$$V_{0,\beta}^{\varepsilon}(y) = (1+\varepsilon)W_{0,\beta}(4y) = (1+\varepsilon)2\left(\frac{y}{\pi}\right)^{\frac{1}{2}} K_{\beta}(2y)\,.$$

Hence

$$\Phi_0^{\varepsilon}(s,\beta) = \frac{1+\varepsilon}{2}\Gamma\left(\frac{s+\beta}{2}\right)\Gamma\left(\frac{s-\beta}{2}\right) \tag{8.30}$$

by (6.581.16) of [GR]. This proves (8.26) with

$$p_0^{\varepsilon}(s,\beta) = \frac{1+\varepsilon}{2}\,. \tag{8.31}$$

Suppose $k \geqslant 1$. Introducing (8.26) for $\Phi_{k-1}^{\varepsilon}(s,\beta)$ into the right side of (8.29) we obtain

$$\begin{aligned}\Phi_k^{\varepsilon}(s,\beta) = \left(1+\frac{k-1}{2\beta}\right)&p_{k-1}^{\varepsilon}\left(s+\frac{1}{2},\beta-\frac{1}{2}\right)\\ &\Gamma\left(\frac{s+\beta}{2}+\frac{1-\varepsilon}{4}\right)\Gamma\left(\frac{s-\beta+1}{2}+\frac{1+\varepsilon(-1)^k}{4}\right)\\ +\left(1-\frac{k-1}{2\beta}\right)&p_{k-1}^{-\varepsilon}\left(s+\frac{1}{2},\beta+\frac{1}{2}\right)\\ &\Gamma\left(\frac{s+\beta+1}{2}+\frac{1+\varepsilon}{4}\right)\Gamma\left(\frac{s-\beta}{2}+\frac{1-\varepsilon(-1)^k}{4}\right)\,.\end{aligned}$$

Here we write

$$\Gamma\left(\frac{s-\beta+1}{2}+\frac{1+\varepsilon(-1)^k}{4}\right) = \Gamma\left(\frac{s-\beta}{2}+\frac{1-\varepsilon(-1)^k}{4}\right)\begin{cases}1\\ \frac{s-\beta}{2}\end{cases}$$

according as $\varepsilon = \mp(-1)^k$ and

$$\Gamma\left(\frac{s+\beta+1}{2}+\frac{1+\varepsilon}{4}\right) = \Gamma\left(\frac{s+\beta}{2}+\frac{1-\varepsilon}{4}\right)\begin{cases}1\\ \frac{s+\beta}{2}\end{cases}$$

according as $\varepsilon = \mp 1$. Hence we obtain the assertion (8.26) with

$$\begin{aligned}p_k^{\varepsilon}(s,\beta) = \left(1+\frac{k-1}{2\beta}\right)p_{k-1}^{\varepsilon}\left(s+\tfrac{1}{2},\beta-\tfrac{1}{2}\right)&\begin{cases}1 & \text{if } \varepsilon = -(-1)^k\\ \frac{s-\beta}{2} & \text{if } \varepsilon = (-1)^k\end{cases}\\ +\left(1-\frac{k-1}{2\beta}\right)p_{k-1}^{-\varepsilon}\left(s+\tfrac{1}{2},\beta+\tfrac{1}{2}\right)&\begin{cases}1 & \text{if } \varepsilon = -1\\ \frac{s+\beta}{2} & \text{if } \varepsilon = 1.\end{cases}\end{aligned}$$

This recursion formula shows that $p_k^{\varepsilon}(s,\beta)$ is a polynomial in s.

From this formula it is easy to find by induction on $k \geqslant 1$ the degree and the leading coefficient of the polynomials, specifically

$$p_k^1(s,\beta) = 2^{[\frac{k-1}{2}]} s^{[\frac{k}{2}]} + \text{lower degree terms} \tag{8.32}$$
$$p_k^{-1}(s,\beta) = 2^{[\frac{k}{2}]} s^{[\frac{k-1}{2}]} + \text{lower degree terms.} \tag{8.33}$$

Therefore $p_k^\varepsilon(s,\beta)$ are not identically zero, except for $p_0^{-1}(s,\beta) \equiv 0$.

The first four polynomials are

$$\begin{aligned}
&p_1^1(s,\beta) = 1 && p_1^{-1}(s,\beta) = 1\\
&p_2^1(s,\beta) = s - \tfrac{1}{2} && p_2^{-1}(s,\beta) = 2\\
&p_3^1(s,\beta) = 2s + \beta - 1 && p_3^{-1}(s,\beta) = 2s - \beta - 1\\
&p_4^1(s,\beta) = 2s^2 - 2s - \beta^2 + \tfrac{3}{4} && p_4^{-1}(s,\beta) = 4s - 2
\end{aligned}$$

while the next, which we don't actually require, are

$$p_5^{\pm 1}(s,\beta) = 4s^2 - 2(2 \mp \beta)s - \beta^2 \mp \beta + 2.$$

The polynomials $p_k^\varepsilon(s,\beta)$ have numerous symmetries. It is easy to see that

$$\Phi_k^\varepsilon(s,\beta) = \Phi_k^{\varepsilon(-1)^k}(s,-\beta)\ . \tag{8.34}$$

This follows from $W_{\alpha,\beta}(y) = W_{\alpha,-\beta}(y)$ and

$$\frac{\Gamma\big(\beta + \frac{1+k}{2}\big)}{\Gamma\big(\beta + \frac{1-k}{2}\big)} = (-1)^k \frac{\Gamma\big(-\beta + \frac{1+k}{2}\big)}{\Gamma\big(-\beta + \frac{1-k}{2}\big)}$$

by the functional equation for the gamma function. Then (8.34) implies

$$p_k^\varepsilon(s,\beta) = p_k^{\varepsilon(-1)^k}(s,-\beta)\ . \tag{8.35}$$

Less obvious is the functional equation

$$p_k^\varepsilon(s,\beta) = \nu p_k^\varepsilon(1-s,-\beta) \tag{8.36}$$

with $\nu = \pm 1$. This can be verified for $1 \leqslant k \leqslant 4$ from the following expressions

$$\begin{aligned}
&p_1^1\big(s + \tfrac{1}{2},\beta\big) = 1 && p_1^{-1}\big(s + \tfrac{1}{2},\beta\big) = 1\\
&p_2^1\big(s + \tfrac{1}{2},\beta\big) = s && p_2^{-1}\big(s + \tfrac{1}{2},\beta\big) = 2\\
&p_3^1\big(s + \tfrac{1}{2},\beta\big) = 2s + \beta && p_3^{-1}\big(s + \tfrac{1}{2},\beta\big) = 2s - \beta\\
&p_4^1\big(s + \tfrac{1}{2},\beta\big) = 2s^2 - \beta^2 + 1 && p_4^{-1}\big(s + \tfrac{1}{2},\beta\big) = 4s.
\end{aligned}$$

We do not need to compute any more of these polynomials. To establish the functional equation (8.36) for all $k \geqslant 0$ and to complete the proof of Proposition 8.1 we argue as follows. First from (8.23) we get (8.18) if $1 \leqslant k \leqslant 4$, because the polynomials $p_k^\varepsilon(s, it_j)$ and $p_k^\varepsilon(1-s, -it_j)$ cancel out by virtue of (8.36). Note that $\beta = it_j$ has the property $-\beta = \bar{\beta}$ unless β is real; that can happen only if k is even in which case we also use (8.35). For larger k we have already obtained the functional equation (8.18) but only with ω_j replaced by a rational function of s. On the other hand we know that $L_j(s)$ agrees with the L–function of a cusp form of weight $\kappa = 0, 1$ which induces $u_j(z)$, so (8.18) holds for $L_j(s)$ with the same gamma factors (because $k \equiv \kappa \pmod 2$ and the parity of the form, i.e. the eigenvalue $\varepsilon_j = \pm 1$ does not change). Comparing both functional equations we deduce that the rational function in question must be constant. This means that the functional equation (8.36) holds for some constant ν. Then by (8.32) and (8.33) we find that this constant is given by

$$\nu = (-1)^{\deg p_k^\varepsilon} = (-1)^{\left[\frac{k}{2} - \frac{1-\varepsilon}{4}\right]} .$$

Finally, having the functional equation for the polynomial, we can apply (8.26), (8.24) and (8.23) to obtain the functional equation (8.18) for the L–function with the root number given by

$$\omega_j = i^k \nu \bar{\eta}_j \, \bar{\rho}_j(1)/\rho_j(1)$$

as claimed.

We still have to justify (8.18) for $u_j(z)$ in $\mathcal{C}_0^-(\Gamma, \chi; s_j)$ because $p_0^{-1}(s, \beta_j) \equiv 0$. This case can be reduced to one already established by mapping $u_j(z)$ to its successor.

We remark that although the polynomials p_k seem quite basic we were unable to find any reference to them in the literature.

9. Partitioning the L–function

In this section we break up our L–function $L_j(s)$ into partial sums. Quite generally L–functions can be approximated by partial sums of length about the square root of the conductor and use of the functional equation is the standard way to achieve this. In our case we begin with (8.18). However, if u_j comes from a holomorphic form we prefer to use (8.11) with $m = k$ for technical reasons (to avoid poles of the gamma factors).

We choose a function $G(u)$ holomorphic in $|\operatorname{Re} u| < 2$ such that

$$\begin{aligned} G(u) &= G(-u) \, , \\ G(0) &= 1 \, , \\ G(u) &\ll 1 \, . \end{aligned} \tag{9.1}$$

Consider the integral

$$I_j(s) = \frac{1}{2\pi i} \int_{(1)} \Lambda_j(s+u) G(u) u^{-1} du \, .$$

Moving the integration to the line $\mathrm{Re}\, u = -1$ and applying (8.18) we get

$$\Lambda_j(s) = I_j(s) + \omega_j \overline{I}_j(1-\bar{s}) \; . \tag{9.2}$$

On the other hand, introducing the Dirichlet series (8.1) and integrating termwise we obtain

$$I_j(s) = \sum_1^\infty \lambda_j(n) \frac{1}{2\pi i} \int_{(1)} \Big(\frac{\sqrt{D}}{\pi n}\Big)^{s+u} \gamma_j(s+u) G(u) u^{-1} du$$

where

$$\gamma_j(s) = \Gamma\Big(\frac{s+it_j}{2} + \frac{1-\varepsilon_j}{4}\Big)\Gamma\Big(\frac{s-it_j}{2} + \frac{1-\varepsilon_j(-1)^k}{4}\Big) \tag{9.3}$$

or $\gamma_j(s) = 2^{-s}\Gamma\left(s + \frac{k-1}{2}\right)$ in case $u_j(z)y^{-\frac{k}{2}}$ is holomorphic. Note that in both cases $\gamma_j(s)$ is holomorphic for $\mathrm{Re}\, s \geqslant \frac{1}{2}$ by (8.15), (8.16). Inserting this into (9.2) and dividing by $\left(\frac{\sqrt{D}}{\pi}\right)^s \gamma_j(s)$ we arrive at the following exact approximate functional equation.

Lemma 9.1. *For s with* $\mathrm{Re}\, s = \frac{1}{2}$ *we have*

$$\begin{aligned} L_j(s) = &\sum_1^\infty \lambda_j(n) n^{-s} V_s\Big(\frac{\pi n}{\sqrt{D}}\Big) \\ &+ \omega_j(s) \sum_1^\infty \bar{\lambda}_j(n) n^{s-1} V_{1-s}\Big(\frac{\pi n}{\sqrt{D}}\Big) \end{aligned} \tag{9.4}$$

where

$$\omega_j(s) = \omega_j \Big(\frac{\sqrt{D}}{\pi}\Big)^{1-2s} \frac{\gamma_j(1-s)}{\gamma_j(s)}$$

and $V_s(y)$ is given by the Mellin integral

$$V_s(y) = \frac{1}{2\pi i} \int_{(1)} \frac{\gamma_j(s+u)}{\gamma_j(s)} \, \frac{G(u)}{u} y^{-u} du \; . \tag{9.5}$$

Remark. For $\mathrm{Re}\, s = \frac{1}{2}$ we have $|\omega_j(s)| = 1$.

In our applications of Lemma 9.1 we shall need to control the size of $V_s(y)$ and its derivatives. The following result suffices for this.

Lemma 9.2. *Let $A \geqslant 3$ be an integer. There exists $G(u)$ satisfying (9.1) such that for any integer $a \geqslant 0$ we have*

$$V_s^{(a)}(y) \ll \Big(\frac{|s| + |t_j|}{y}\Big)^a \Big(1 + \frac{y}{|s| + |t_j|}\Big)^{-A} , \tag{9.6}$$

the implied constant depending on a, A and k.

Proof. We choose

$$G(u) = \left(\cos \frac{\pi u}{A}\right)^{-A}.$$

Move the integration to $\operatorname{Re} u = B$ where $B = -\frac{1}{4}$ or $B = A$, and differentiate a times getting

$$V_s^{(a)}(y) \ll \int_{(B)} \Big| \frac{\gamma_j(s+u)}{\gamma_j(s)} G(u) u^{a-1} y^{-B} du \Big| + \begin{cases} 1 & \text{if } B < 0, \\ 0 & \text{if } B > 0. \end{cases}$$

Consider the case where $\gamma_j(s)$ is given by (9.3), so

$$\frac{\gamma_j(s+u)}{\gamma_j(s)} = \frac{\Gamma\left(s' + \frac{u}{2}\right)}{\Gamma(s')} \frac{\Gamma\left(s'' + \frac{u}{2}\right)}{\Gamma(s'')}$$

where

$$s' = \frac{s}{2} + \frac{it_j}{2} + \frac{1-\varepsilon_j}{4},$$

$$s'' = \frac{s}{2} - \frac{it_j}{2} + \frac{1-\varepsilon_j(-1)^k}{4}.$$

Note that $\sigma' = \operatorname{Re} s' > \frac{1}{4}$ and $\sigma'' = \operatorname{Re} s'' > \frac{1}{4}$ by (8.15), (8.16). By Stirling's formula

$$\begin{aligned} \frac{\Gamma\left(s' + \frac{u}{2}\right)}{\Gamma(s')} &\asymp \frac{\left|s' + \frac{u}{2}\right|^{\sigma' - \frac{1}{2} + \frac{B}{2}}}{|s'|^{\sigma' - \frac{1}{2}}} \exp\left(\frac{\pi}{2}\left(|s'| - \left|s' + \frac{u}{2}\right|\right)\right) \\ &\ll_{a,A} |u|^{-\frac{a}{2}} |s'|^{\frac{a+B}{2}} \exp\left(\frac{\pi}{4}|u|\right) \end{aligned}$$

and a similar bound holds with s'' in place of s'. Hence we deduce the estimate

$$V_s^{(a)}(y) \ll y^{-B}\left(|s| + |t_j|\right)^{a+B} \int_{(B)} |G(u)| \exp\left(\frac{\pi}{2}|u|\right) \Big| \frac{du}{u} \Big| + \begin{cases} 1 \text{ if } B < 0 \\ 0 \text{ if } B > 0 \end{cases}$$

and this last integral is bounded. The lemma follows on choosing $B = A$ if $y > |s| + |t_j|$ and $B = -\frac{1}{4}$ otherwise. In the case of holomorphic forms $it_j = \frac{k-1}{2}$ and the result is given in Lemma 3.2 of [DFI7].

Applying a smooth partition of unity we derive by (9.4) and (9.5) that

$$L_j(s) \ll \sum_n \frac{G_j(N)}{\sqrt{N}} \left(1 + \frac{N}{(|s| + |t_j|)\sqrt{D}}\right)^{-A} \tag{9.7}$$

where $G_j(N)$ are sums of type

$$G_j(N) = \sum_n \lambda_j(n) g(n)$$

with $g(x)$ a smooth function supported on $[N, 2N]$ for $N = 2^{\nu/2}$, $\nu \geqslant -1$, such that

$$g^{(a)}(x) \ll \left(\frac{|s| + |t_j|}{N}\right)^a$$

for every $a \geqslant 0$, the implied constant depending on a, A and k.

By Cauchy's inequality applied to (9.7) we have

$$L_j(s)^2 \ll \sum_N \frac{|G_j(N)|^2}{N} \left(1 + \frac{N}{(|s| + |t_j|)\sqrt{D}}\right)^{-2A} \log 3N \ .$$

It remains to estimate $|G_j(N)|^2$. We have

$$|G_j(N)|^2 = \sum_{n_1} \sum_{n_2} \lambda_j(n_1) \bar{\lambda}_j(n_2) g(n_1) \bar{g}(n_2)$$

and from (6.6) and (6.5) we get

$$|G_j(N)|^2 = \sum_{d | D^\infty} \bar{\lambda}_j(d) \sum_{(\delta, D)=1} \sum_{n_1} \sum_{n_2} \lambda_j(n_1 n_2) g(\delta n_1) \bar{g}(\delta d n_2) \overline{\chi}(n_2) \ .$$

Grouping the terms in accordance with the product $n_1 n_2 = n$ we write the above expression as

$$|G_j(N)|^2 = \sum_{d | D^\infty} \bar{\lambda}_j(d) \sum_{(\delta, D)=1} \sum_n \lambda_j(n) \sigma_F(n, \chi) \tag{9.8}$$

where $\sigma_F(n, \chi)$ is defined in (10.1) for general $F(x_1, x_2)$ and here we have specifically

$$F(x_1, x_2) = g(\delta x_1) \bar{g}(\delta d x_2) \ . \tag{9.9}$$

Thus $F(x_1, x_2)$ is supported in the dyadic box $[X_1, 2X_1] \times [X_2, 2X_2]$ with

$$X_1 = N/\delta \ , \qquad X_2 = N/\delta d \ . \tag{9.10}$$

Moreover the partial derivatives of F satisfy (11.1) with

$$P = |s| + |t_j| \ . \tag{9.11}$$

10. Averaging the spectral sum

In this chapter we are going to develop a formula for the average over n of the spectral sum in Proposition 6.1 weighted by a special function of arithmetic nature designed for the problem of bounding the fourth power of our L–function.

Specifically we consider

$$\sigma_F(n,\chi) = \sum_{n_1 n_2 = n} F(n_1, n_2)\overline{\chi}(n_2) \tag{10.1}$$

where $F(x_1, x_2)$ is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and is of Schwartz type. A product $L_j(s)L_j(s,\overline{\chi})$ may be built out of bits of type $\sigma_F(n,\chi)$ twisted against the coefficients of L_j. The product $L_j(s)L_j(s,\overline{\chi})$ is just $|L_j(s)|^2$ apart from local factors at the ramified primes.

To each Hecke-Maass cusp form $u_j(z)$ we attach the corresponding sum

$$\mathcal{N}_j = \sum_n \lambda_j(n)\sigma_F(n,\chi) \tag{10.2}$$

and, to each Eisenstein series $E_{\mathfrak{a}}(z, \frac{1}{2} + it)$, the corresponding sum

$$\mathcal{N}_{\mathfrak{a}}(t) = \sum_n \lambda_{\mathfrak{a}}(n,t)\sigma_F(n,\chi). \tag{10.3}$$

Needless to say these also depend on the character χ and the bit function F but we do not display this since χ will be fixed throughout and most of our arguments will deal with individual F.

Proposition 10.1. *For $m \geqslant 1$ we have*

$$\begin{aligned} &\sum_j h(t_j)\nu_j\bar{\lambda}_j(m)\mathcal{N}_j + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\nu_{\mathfrak{a}}(t)\bar{\lambda}_{\mathfrak{a}}(m,t)\mathcal{N}_{\mathfrak{a}}(t)dt \\ &= \sigma_F(m,\chi) + \chi(-1) \sum_{c \equiv 0 \pmod D} \sum_n \sigma_G(n,\chi)S(m-n,0;c) \end{aligned} \tag{10.4}$$

where $S(h,0;c)$ is the Ramanujan sum and G is the Fourier integral

$$\begin{aligned} G(y_1,y_2) = \iint F(cx_1,cx_2)I\big(4\pi\sqrt{mx_1x_2}\big) \\ e\big(-x_1y_2 - x_2y_1\big)dx_1dx_2\ . \end{aligned} \tag{10.5}$$

Here $I(x)$ is the integral transform of $h(t)$ given in (5.21).

Proof. This follows immediately from Proposition 6.1 by opening the Kloosterman sums and executing the summation over n (but not m) by means of the Poisson-Voronoi formula, Proposition 6.1 of [DFI7].

Note that the effect of this averaging is to replace the Kloosterman sums by the simpler Ramanujan sums. We may recall that the spectral sum formula Proposition 5.1 is essentially a formula for the inner product $\langle P_n, P_m \rangle$ of two Poincaré series. The left side comes from the spectral expansion and the right side comes from the unfolding of P_n over the cosets of the group followed by an application of Poisson summation over the stability group. At that point one could have performed a summation over n first, giving an alternate route to Proposition 10.1. In this way we would have avoided the appearance of any Kloosterman sums whatsoever.

11. Contribution of the continuous spectrum

Although our main objective is to estimate the sums $\mathcal{N}_j$ for individual Hecke-Maass cusp forms u_j we are able to evaluate asymptotically the corresponding sums $\mathcal{N}_{\mathfrak{a}}(t)$ for the Eisenstein series $E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right)$ due to the explicit formula (6.17) for $\lambda_{\mathfrak{a}}(n,t)$. We shall show that sums $\mathcal{N}_{\mathfrak{a}}(t)$ are small except for the two cusps $\mathfrak{a} \sim 0$ and $\mathfrak{a} \sim \infty$ and for each of these we get a main term. This is not surprising because in these cases $\lambda_{\mathfrak{a}}(n,t)$ is very much like $\sigma_F(n,\chi)$ so there is no orthogonality.

We assume that the bit function $F(x_1, x_2)$ is smooth, supported on the box $[X_1, 2X_1] \times [X_2, 2X_2]$ with $X_1, X_2 \geqslant \frac{1}{2}$ and satisfying

$$\frac{\partial^{(\alpha_1,\alpha_2)}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} F(x_1, x_2) \ll P^{\alpha_1+\alpha_2} X_1^{-\alpha_1} X_2^{-\alpha_2} \tag{11.1}$$

for some $P \geqslant 1$, and all $\alpha_1, \alpha_2 \geqslant 0$, the implied constant depending only on α_1, α_2.

This is a natural assumption for a function which does not oscillate too rapidly where the extra factor P provides needed flexibility, for example it will allow us to achieve polynomial growth in our L–function bound in both the s and spectral aspects.

Opening $\lambda_{\mathfrak{a}}(n,t)$ and $\sigma_F(n,\chi)$ in (10.3) we get

$$\mathcal{N}_{\mathfrak{a}}(t) = \sum_{n_1 n_2 = ad} F(n_1, n_2) \overline{\chi}(n_2) \left(\frac{a}{d}\right)^{it} \chi_v(a) \chi_w(d) \,. \tag{11.2}$$

To evaluate this we consider the generating Dirichlet series

$$L_{\mathfrak{a}}(s_1, s_2) = \sum\sum_{n_1 n_2 = ad} n_1^{-s_1} n_2^{-s_2}\, \overline{\chi}(n_2) \left(\frac{a}{d}\right)^{it} \chi_v(a) \chi_w(d) \,.$$

All solutions to $n_1 n_2 = ad$ are given by $n_1 = \alpha\delta$, $n_2 = \beta\gamma$, $a = \alpha\gamma$, $d = \beta\delta$ where $\alpha, \beta, \gamma, \delta$ run over positive integers with $(\gamma, \delta) = 1$. Hence

$$L_{\mathfrak{a}}(s_1, s_2) = \sum_{\alpha} \sum_{(\beta,w)=1} \sum\sum_{(\gamma,\delta v)=1} (\alpha\delta)^{-s_1} (\beta\gamma)^{-s_2} \left(\frac{\alpha\gamma}{\beta\delta}\right)^{it} \chi_v(\alpha) \overline{\chi}_v(\beta) \overline{\chi}_w(\gamma) \chi_w(\delta) \,.$$

We relax the condition $(\gamma, \delta) = 1$ by Möbius inversion getting

$$L_{\mathfrak{a}}(s_1, s_2) = \Big(\sum_{\alpha} \alpha^{-s_1+it} \chi_v(\alpha)\Big)\Big(\sum_{(\beta,w)=1} \beta^{-s_2-it}\, \overline{\chi}_v(\beta)\Big)$$
$$\Big(\sum_{(\gamma,v)=1} \gamma^{-s_2+it}\, \overline{\chi}_w(\gamma)\Big)\Big(\sum_{\delta} \delta^{-s_1-it} \chi_w(\delta)\Big)\Big(\sum_{(\rho,D)=1} \mu(\rho)\rho^{-s_1-s_2}\Big) .$$

To simplify, for any Dirichlet series $D(s) = \sum a_n n^{-s}$ we introduce the notation $D_q(s)$ for the same series restricted to the terms with $(n, q) = 1$. Therefore if $D(s)$ has an Euler product then $D_q(s)$ has the same Euler product with the local factors at the prime divisors of q omitted.

Now the above formula becomes

$$L_{\mathfrak{a}}(s_1, s_2) = \zeta_D(s_1 + s_2)^{-1} L(s_1 - it, \chi_v) L(s_1 + it, \chi_w)$$
$$L_D(s_2 + it, \overline{\chi}_v) L_D(s_2 - it, \overline{\chi}_w) .$$

By contour integration we have

$$\mathcal{N}_{\mathfrak{a}}(t) = -\frac{1}{4\pi^2} \int_{(2)} \int_{(2)} \hat{F}(s_1, s_2) L_{\mathfrak{a}}(s_1, s_2) ds_1 ds_2$$

where $\hat{F}(s_1, s_2)$ is the Mellin transform of $F(x_1, x_2)$

$$\hat{F}(s_1, s_2) = \iint F(x_1, x_2) x_1^{s_1-1} x_2^{s_2-1} dx_1 dx_2 .$$

By (11.1) and integration by parts we deduce that

$$\hat{F}(s_1, s_2) \ll X_1^{\sigma_1} X_2^{\sigma_2} \Big(\frac{P^2}{|s_1 s_2|}\Big)^A \tag{11.3}$$

where $\sigma_1 = \operatorname{Re} s_1$, $\sigma_2 = \operatorname{Re} s_2$, $\frac{1}{2} \leqslant \sigma_1, \sigma_2 \leqslant 2$, and any $A > 0$, the implied constant depending on A. Moving the integration to $\operatorname{Re} s_1 = \frac{1}{2}$, $\operatorname{Re} s_2 = \frac{1}{2}$ we meet simple poles at $s_1 = 1 + it$ and $s_2 = 1 - it$ in case $v = 1$ (that is $\mathfrak{a} \sim 0$) and at $s_1 = 1 - it$ and $s_2 = 1 + it$ in case $w = 1$ (that is $\mathfrak{a} \sim \infty$). To describe this we use the notation $\delta_v = 1$ if $\nu = 1$ and $= 0$ otherwise, and similarly for δ_w. Note that

$$\zeta_D(2) \prod_{p|D} \Big(1 - \frac{1}{p}\Big)^{-1} = \zeta(2) \frac{\nu(D)}{D}$$

where $\nu(D)$ is given by (4.1). We get

$$\begin{aligned}\mathcal{N}_{\mathfrak{a}}(t) = {}& \frac{\delta_v D}{\zeta(2)\nu(D)} \hat{F}(1+it, 1-it)\big|L(1+2it,\chi)\big|^2 \\ & + \frac{\delta_w D}{\zeta(2)\nu(D)} \hat{F}(1-it, 1+it)\big|L(1-2it,\chi)\big|^2 \\ & + \delta_v I(s_2, -t, \overline{\chi}) L(1+2it,\chi) + \delta_w I(s_2, t, \overline{\chi}) L(1-2it,\chi) \\ & + \delta_v I(s_1, t, \chi) L(1-2it,\overline{\chi}) + \delta_w I(s_1, -t, \chi) L(1+2it,\overline{\chi}) \\ & - \frac{1}{4\pi^2} \int_{(\frac{1}{2})} \int_{(\frac{1}{2})} \hat{F}(s_1, s_2) L_{\mathfrak{a}}(s_1, s_2)\, ds_1 ds_2\end{aligned}$$

where

$$I(s,t,\chi) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \hat{F}(s, 1-it) \frac{\zeta_D(s-it)}{\zeta_D(s+1-it)} L(s+it,\chi)\, ds \,.$$

Next we estimate all but the first two terms above using the classical estimates

$$\begin{aligned} &L(s,\chi) \ll |s| D^{\frac{3}{16}+\varepsilon}, && \zeta(s) \ll |s|, && \text{for } \operatorname{Re} s = \tfrac{1}{2}\,, \\ &L(s,\chi) \ll \log(|s|D), && \zeta(s)^{-1} \ll \log|s|, && \text{for } \operatorname{Re} s = 1\,. \end{aligned}$$

When taken together with (11.3) for $A = 2$, these bounds yield

$$\begin{aligned}\mathcal{N}_{\mathfrak{a}}(t) = {}& \frac{\delta_v D}{\zeta(2)\nu(D)} \hat{F}(1+it, 1-it)\big|L(1+2it,\chi)\big|^2 \\ & + \frac{\delta_w D}{\zeta(2)\nu(D)} \hat{F}(1-it, 1+it)\big|L(1-2it,\chi)\big|^2 \\ & + O\Big(P^4(|t|+1)^{4+\varepsilon}(X_1+X_2+\sqrt{D})^{\frac{1}{2}}(X_1X_2)^{\frac{1}{2}} D^{\frac{3}{16}+\varepsilon}\Big)\,.\end{aligned} \tag{11.4}$$

Now we are ready to estimate the contribution of the continuous spectrum to the left hand side of the spectral sum (10.4). This is

$$L_E(m) = \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t)\nu_{\mathfrak{a}}(t)\bar{\lambda}_{\mathfrak{a}}(m,t)\mathcal{N}_{\mathfrak{a}}(t)\,dt\,.$$

We insert the formula (11.4) for $\mathcal{N}_{\mathfrak{a}}(t)$ and write

$$L_E(m) = L_E^{(1)}(m) + L_E^{(2)}(m)$$

where $L_E^{(1)}(m)$ gives the contribution from the two main terms in (11.4) and $L_E^{(2)}(m)$ gives the rest.

Note that only the cusps $\mathfrak{a} \sim \infty$ and $\mathfrak{a} \sim 0$ contribute to $L_E^{(1)}(m)$ and in view of (7.13) these contributions are equal, as they are for each pair of complementary cusps. Therefore

$$L_E^{(1)}(m) = \frac{12}{\nu(D)} \int_{-\infty}^{\infty} \bar{\lambda}(m,t) h(t) \frac{\hat{F}(1+it, 1-it)}{\left|\Gamma\left(\frac{k+1}{2}+it\right)\right|^2} dt \ , \tag{11.5}$$

where

$$\lambda(m,t) = \lambda_\infty(m,t) = \sum_{b_1 b_2 = m} \chi(b_1) \left(\frac{b_1}{b_2}\right)^{it} . \tag{11.6}$$

For the error term we obtain

$$\begin{aligned} L_E^{(2)}(m) \ll (|r|+1)^3 P^4 (X_1 + X_2 + \sqrt{D})^{\frac{1}{2}} \\ (X_1 X_2)^{\frac{1}{2}} D^{-\frac{13}{16}} (D+|r|)^{\varepsilon} \end{aligned} \tag{11.7}$$

because by (5.19) and Stirling's formula,

$$\int_{-\infty}^{\infty} |h(t)|(|t|+1)^{4+\varepsilon} \left|\Gamma\left(\frac{k+1}{2}+it\right)\right|^{-2} dt \ll (|r|+1)^{3+\varepsilon} \ .$$

Here the factor $(D+|r|)^{\varepsilon}$ incorporates the bound,

$$L(1+2it,\chi) \gg D^{-\varepsilon} \big(\log(|t|+2)\big)^{-1}$$

which is required due to (7.13). Therefore, if χ is real our estimate is ineffective and this carries over to our final bounds. For the purpose of our final bounds for L–functions we could rearrange our intermediate arguments so that (by positivity) only upper bounds, not lower bounds, are required leading to effective results.

Turning to the main term $L_E^{(1)}(m)$ it will be convenient to write (11.5) in somewhat more explicit form before we leave this section. Specifically we insert (11.6) and

$$\check{F}(t) = \hat{F}(1+it, 1-it) = \iint F(x_1, x_2) \left(\frac{x_1}{x_2}\right)^{it} dx_1 dx_2$$

giving

$$L_E^{(1)}(m) = \frac{12}{\nu(D)} \sum_{b_1 b_2 = m} \overline{\chi}(b_2) \iint F(x_1, x_2) \mathcal{L}\left(\sqrt{\frac{b_1 x_2}{b_2 x_1}}\right) dx_1 dx_2 \tag{11.8}$$

where we have denoted

$$\mathcal{L}(y) = \int_{-\infty}^{\infty} h(t) y^{2it} \left|\Gamma\left(\frac{k+1}{2}+it\right)\right|^{-2} dt \ . \tag{11.9}$$

12. Contribution of the singular determinants

We single out from the right hand side of (10.4) the contribution from the terms $n = m$. These 'singular' terms give rise to a large contribution which will asymptotically cancel the large contribution $\mathcal{L}_E(m)$ on the left side coming from the continuous spectrum.

This singular contribution is just

$$R_s(m) = \chi(-1) \sum_{c \equiv 0 (\mathrm{mod}\, D)} \sigma_G(m, \chi)\varphi(c) \tag{12.1}$$

where $G(y_1, y_2)$ is given by (10.5). Recall that G depends on c although this is not displayed by the notation.

Opening $\sigma_G(m, \chi)$ (see (10.1)) and interchanging the order of summation we obtain

$$R_s(m) = \iint I\big(4\pi\sqrt{mx_1x_2}\big) \sum_{b_1b_2=m} \overline{\chi}(b_2)E_k(b_1x_2 + b_2x_1) \sum_{c\equiv 0(D)} \varphi(c)F(cx_1, cx_2)dx_1dx_2$$

where

$$E_k(x) = \begin{cases} 2\cos 2\pi x & \text{if } k \text{ even,} \\ 2i \sin 2\pi x & \text{if } k \text{ odd.} \end{cases} \tag{12.2}$$

Originally b_1, b_2 ran over all integers so, after we have grouped the pairs $\pm(b_1, b_2)$, $E_k(x)$ compensates for the sum now running over b_1, b_2 positive.

For the sum over c we apply the Euler-Maclaurin formula (see Lemma 7.1 of [DFI7]) getting

$$\sum_{c\equiv 0(D)} \varphi(c)F(cx_1, cx_2) = \frac{1}{\zeta(2)\nu(D)} \int tF(tx_1, tx_2)dt + \int \xi_D(t)\frac{\partial}{\partial t}\big(tF(tx_1, tx_2)\big)dt\,. \tag{12.3}$$

Recall that $\nu(D)$ is given by (4.1) and, as proved in [DFI7]

$$\xi_D(t) \ll \log\Big(1 + \frac{t}{D}\Big) \ll \frac{t}{D}\,.$$

In accordance with (12.3) we split

$$R_s(m) = R_s^{(1)}(m) + R_s^{(2)}(m)$$

where

$$R_s^{(1)}(m) = \frac{1}{\zeta(2)\nu(D)} \sum_{b_1 b_2 = m} \overline{\chi}(b_2) \iint \Big(\int_0^\infty tF(tx_1, tx_2) dt \Big) \\ I\big(4\pi\sqrt{mx_1x_2}\big) E_k\big(b_1x_2 + b_2x_1\big) dx_1 dx_2 \tag{12.4}$$

and

$$R_s^{(2)}(m) = \sum_{b_1 b_2 = m} \overline{\chi}(b_2) \iint \Big(\int_0^\infty \xi_D(t) \frac{\partial}{\partial t}\big(tF(tx_1, tx_2)\big) dt \Big) \\ I\big(4\pi\sqrt{mx_1x_2}\big) E_k\big(b_1x_2 + b_2x_1\big) dx_1 dx_2 \ . \tag{12.5}$$

First we estimate $R_s^{(2)}(m)$. As in [DFI7, §7] we make the change of variable x_2 into x/x_1 and we use the bound

$$\int \frac{\partial}{\partial t}\big(tF(tx_1, tx/x_1\big) E_k\big(b_1x_2 + b_2x_1\big) dx_1 \ll P^2\big(1 + b_1b_2x\big)^{-\frac{1}{4}} \ .$$

Note that by the support of F, t is in the range $T(x) \leqslant t \leqslant 2T(x)$ where $T(x) = \sqrt{X_1X_2/x}$. Hence

$$R_s^{(2)}(m) \ll \tau(m) P^2 \int_0^\infty \big|I(4\pi\sqrt{mx})\big| (1 + mx)^{-\frac{1}{4}} \int_{T(x)}^{2T(x)} \log\Big(1 + \frac{t}{D}\Big) dt\, dx \ . \tag{12.6}$$

We shall show that

$$I(x) \ll \min\Big\{ x\big(1 + |\log x|\big),\ x^{-\frac{1}{2}}(|r| + 1)^{\frac{1}{2}} \Big\} \ , \tag{12.7}$$

the implied constant depending only on k.

First recall the integral representation (5.21). For the K–Bessel function therein we use the formula

$$K_{2ir}(z) = \int_0^\infty e^{-z\,\mathrm{ch}\,x} \cos(2rx) dx$$

which yields the bound

$$\Big| K_{2ir}(z) \Big| \leqslant K_0(\mathrm{Re}\, z) \ll 1 + |\log \mathrm{Re}\, z|$$

if $\mathrm{Re}\, z > 0$. Hence we obtain the first part of (12.7),

$$I(x) \ll x \int_{-i}^{i} \big(1 + |\log \mathrm{Re}\, \zeta x|\big) |d\zeta| \ll x(1 + |\log x|) \ .$$

For the proof of the second part of (12.7) we appeal instead to the integral representation (5.20). We start from the formula

$$\int_{-i}^{i} e^{-\zeta z} d\zeta = 2i \frac{\sin z}{z} .$$

Differentiating this $k-1$ times in z we get

$$\begin{aligned}\int_{-i}^{i} (-i\zeta)^{k-1} e^{-\zeta z} d\zeta &= 2i^k \Big(\frac{\sin z}{z}\Big)^{(k-1)} \\ &= \sum_{\substack{0\leqslant\alpha<k\\ \alpha \text{ even}}} A(\alpha) z^{\alpha-k} \sin z + \sum_{\substack{0\leqslant\alpha<k\\ \alpha \text{ odd}}} A(\alpha) z^{\alpha-k} \cos z .\end{aligned}$$

Hence (5.20) expands into

$$\begin{aligned}I(x) &= -x \sum_{0\leqslant\alpha<k} A(\alpha)\Big(\frac{x}{2}\Big)^{\alpha-k} \\ &\qquad \int_0^\infty \Big(y+\frac{1}{y}\Big)^{\alpha-k} y^{-1-2ir} \left\{\begin{matrix}\sin\Big(\frac{x}{2}\big(y+\frac{1}{y}\big)\Big)\\ \cos\Big(\frac{x}{2}\big(y+\frac{1}{y}\big)\Big)\end{matrix}\right\} dy \\ &= -\sum_{0\leqslant\alpha<k} A(\alpha) x^{1+\alpha-k} \int_0^\infty (\operatorname{ch} t)^{\alpha-k} \left\{\begin{matrix}\cos(x \operatorname{ch} t)\\ \sin(x \operatorname{ch} t)\end{matrix}\right\} \cos(2rt) dt .\end{aligned}$$

We estimate each integral separately as follows:

$$\int_0^\infty = \int_0^\varepsilon + \int_\varepsilon^\infty \ll \varepsilon + \frac{|r|+1}{\varepsilon x} = 2\Big(\frac{|r|+1}{x}\Big)^{\frac{1}{2}}$$

on choosing ε optimally. Here the first bound is trivial and the second bound comes out by partial integration. Hence we derive the second part of (12.7).

Using (12.7) we conclude from (12.6) that the remainder term (12.5) satisfies

$$R_s^{(2)}(m) \ll P^2 \tau(m) \Big(\frac{|r|+1}{m} X_1 X_2\Big)^{\frac{1}{2}} (\log 5m X_1 X_2)^2 . \tag{12.8}$$

Next we elaborate the main term $R_s^{(1)}(m)$. In [DFI7] this main term vanished due to an orthogonality of $E_k(bx)$ to the Bessel function $J_{k-1}(ax)$. The corresponding special function $I(ax)$ here is no longer orthogonal to $E_k(bx)$ and the integral in (12.4) gives rise to a main term which (as we shall see after considerable effort) matches the corresponding main term $L_E^{(1)}(m)$ coming from the continuous spectrum (which did not exist in [DFI7]) and cancels out in the spectral sum formula in the averaged form (10.4).

Precisely, the main term is after a change of variables in (12.4),

$$R_s^{(1)}(m) = \frac{12}{\nu(D)} \sum_{b_1 b_2 = m} \overline{\chi}(b_2) \iint F(x_1, x_2) \mathcal{R}\Big(\sqrt{\frac{b_1 x_2}{b_2 x_1}}\Big) dx_1 dx_2 \qquad (12.9)$$

where we have denoted

$$\mathcal{R}(y) = \frac{1}{2\pi^2} \int_0^\infty I(2\pi t) E_k\Big(\frac{t}{2}\Big(y + \frac{1}{y}\Big)\Big) \frac{dt}{t}\,. \qquad (12.10)$$

13. Matching the integrals

Our goal in this chapter is to show that

$$L_E^{(1)}(m) = R_s^{(1)}(m) \qquad (13.1)$$

where these are the main terms in the contribution from the continuous spectrum (given in (11.8)) and that from the singular determinants (given in (12.9)). We shall actually be able to show somewhat more, that the inner integrals agree, after which (13.1) follows trivially. In other words, for any $y > 0$,

$$\mathcal{L}(y) = \mathcal{R}(y)\,. \qquad (13.2)$$

We begin by making some computations on $\mathcal{R}(y)$. From (5.21) and (12.10) we have

$$\mathcal{R}(y) = -\frac{2}{\pi} \int_{-i}^{i} (-i\zeta)^{k-1} \int_0^\infty K_{2ir}(2\pi\zeta t) E_k\Big(\frac{t}{2}\Big(y + \frac{1}{y}\Big)\Big) dt\, d\zeta\,.$$

After insertion of (12.2) this becomes

$$-\frac{2}{\pi^2}\Big(\frac{y+\frac{1}{y}}{2i}\Big)^{k-1} \int_{-2i/\left(y+\frac{1}{y}\right)}^{2i/\left(y+\frac{1}{y}\right)} \zeta^{k-1} \int_0^\infty K_{2ir}(\zeta t) \begin{Bmatrix} \cos t \\ i \sin t \end{Bmatrix} dt d\zeta$$

where, by $\begin{Bmatrix} A \\ B \end{Bmatrix}$, we mean A in case k even and B in case k odd. Next, by formulae (6.671.6) and (6.671.5) of [GR] we have

$$\int_0^\infty K_{2ir}(\zeta t) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} dt = \frac{\pi}{4\sqrt{1+\zeta^2}} \begin{Bmatrix} 1/\cos \pi i r \\ 1/\sin \pi i r \end{Bmatrix}$$
$$\Big[\Big(\frac{1}{\zeta} + \sqrt{1 + \frac{1}{\zeta^2}}\Big)^{2ir} \pm \Big(\frac{1}{\zeta} + \sqrt{1 + \frac{1}{\zeta^2}}\Big)^{-2ir}\Big].$$

Inserting these and changing ζ to $1/\xi$ we get

$$\mathcal{R}(y) = -\frac{1}{2\pi}\Big(\frac{y+\frac{1}{y}}{2i}\Big)^{k-1} \begin{Bmatrix} 1/\operatorname{ch}\pi r \\ 1/\operatorname{sh}\pi r \end{Bmatrix} \int_{-\frac{i}{2}\left(y+\frac{1}{y}\right)}^{\frac{i}{2}\left(y+\frac{1}{y}\right)} \left(\xi^2+1\right)^{-\frac{1}{2}} \xi^{-k}\Big[\Big(\xi+\sqrt{1+\xi^2}\Big)^{2ir} \pm \Big(\xi+\sqrt{1+\xi^2}\Big)^{-2ir}\Big]d\xi. \tag{13.3}$$

Our next step requires $k \geqslant 1$. The argument for $k=0$ will be given later.

We replace the integral over a semi-circle by integrals over two segments of the imaginary axis using Cauchy's theorem. (For the convergence of the relevant integral we require $k>0$.) This gives

$$\mathcal{R}(y) = \frac{1}{2\pi}\Big(\frac{y+\frac{1}{y}}{2i}\Big)^{k-1} \begin{Bmatrix} 1/\operatorname{ch}\pi r \\ 1/\operatorname{sh}\pi r \end{Bmatrix} \Big[\int_{\frac{i}{2}\left(y+\frac{1}{y}\right)}^{i\infty} + \int_{-i\infty}^{-\frac{i}{2}\left(y+\frac{1}{y}\right)}\Big].$$

Change ξ to $i\operatorname{ch} x$ in the first integral and to $-i\operatorname{ch} x$ in the second integral getting

$$\begin{aligned}\mathcal{R}(y) = \frac{1}{2\pi i}\Big(\frac{y+\frac{1}{y}}{2}\Big)^{k-1} &\begin{Bmatrix} 1/\operatorname{ch}\pi r \\ 1/\operatorname{sh}\pi r \end{Bmatrix} \\ &\Big[(-1)^{k-1}\int_{\log y}^{\infty} \left(e^{-\pi r+2irx} \pm e^{\pi r-2irx}\right)\frac{dx}{(\operatorname{ch} x)^k} \\ &\quad + \int_{\log y}^{\infty} \left(e^{\pi r+2irx} \pm e^{-\pi r-2irx}\right)\frac{dx}{(\operatorname{ch} x)^k}\Big].\end{aligned}$$

On combining the exponential terms this simplifies to

$$\mathcal{R}(y) = \frac{2}{\pi}\Big(\frac{y+\frac{1}{y}}{2}\Big)^{k-1} \begin{Bmatrix} \operatorname{th}\pi r \\ \operatorname{cth}\pi r \end{Bmatrix} \int_{\log y}^{\infty} \frac{\sin 2rx}{(\operatorname{ch} x)^k}\,dx\,. \tag{13.4}$$

For $k=0$ we compute $\mathcal{R}(y)$ explicitly starting from (13.3). We have

$$\begin{aligned}\mathcal{R}(y) &= \frac{\left(y+\frac{1}{y}\right)^{-1}}{\pi i\operatorname{ch}\pi r}\int_{-\frac{i}{2}\left(y+\frac{1}{y}\right)}^{\frac{i}{2}\left(y+\frac{1}{y}\right)}\Big[\Big(\xi+\sqrt{1+\xi^2}\Big)^{2ir} \\ &\qquad\qquad + \Big(\xi+\sqrt{1+\xi^2}\Big)^{-2ir}\Big]\frac{d\xi}{\sqrt{1+\xi^2}} \\ &= -\frac{\left(y+\frac{1}{y}\right)^{-1}}{2\pi r\operatorname{ch}\pi r}\int_{-\frac{i}{2}\left(y+\frac{1}{y}\right)}^{\frac{i}{2}\left(y+\frac{1}{y}\right)}\Big[\Big(\Big(\xi+\sqrt{1+\xi^2}\Big)^{2ir}\Big)' \\ &\qquad\qquad - \Big(\Big(\xi+\sqrt{1+\xi^2}\Big)^{-2ir}\Big)'\Big]d\xi\,.\end{aligned}$$

Assume without loss of generality that $y \geqslant 1$ (otherwise change y into $1/y$). Then for $\xi = \frac{i}{2}\left(y + \frac{1}{y}\right)$ we have $\sqrt{1+\xi^2} = \frac{i}{2}\left(y - \frac{1}{y}\right)$ so

$$\left(\xi + \sqrt{1+\xi^2}\right)^{2ir} = y^{2ir} e^{-\pi r} .$$

Similarly for $\xi = -\frac{i}{2}\left(y + \frac{1}{y}\right)$ we get

$$\left(\xi + \sqrt{1+\xi^2}\right)^{2ir} = y^{2ir} e^{\pi r} .$$

Hence the above integral is equal to

$$y^{2ir} e^{-\pi r} + y^{-2ir} e^{\pi r} - y^{2ir} e^{\pi r} - y^{-2ir} e^{-\pi r} = -2\left(y^{2ir} + y^{-2ir}\right) \operatorname{sh} \pi r .$$

Therefore, for $k = 0$ we have

$$\mathcal{R}(y) = \frac{\operatorname{th} \pi r}{\pi r} \, \frac{y^{2ir} + y^{-2ir}}{y + y^{-1}} . \tag{13.5}$$

Now we compute $\mathcal{L}(y)$. Inserting (5.19) into (11.9) we get

$$\mathcal{L}(y) = \pi \left| \Gamma\left(1 - \frac{k}{2} - ir\right) \right|^{-2} \int_{-\infty}^{\infty} \frac{y^{2it}\, dt}{\left|\Gamma\left(\frac{k+1}{2} + it\right)\right|^2 \operatorname{ch} \pi(r-t) \operatorname{ch} \pi(r+t)} . \tag{13.6}$$

Let $k \geqslant 1$; the case $k = 0$ will be treated later. Since, by the recurrence formula and functional equation for the gamma function,

$$\left|\Gamma\left(\frac{k+1}{2} + it\right)\right|^2 = \pi i^{-k} \begin{Bmatrix} 1/\operatorname{ch} \pi t \\ 1/\operatorname{sh} \pi t \end{Bmatrix} \prod_{1 \leqslant \nu \leqslant k} \left(\frac{k+1}{2} - \nu + it\right)$$

we have

$$\mathcal{L}(y) = i^k \left| \Gamma\left(1 - \frac{k}{2} - ir\right) \right|^{-2} \int_{-\infty}^{\infty} \frac{\begin{Bmatrix} \operatorname{ch} \pi t \\ \operatorname{sh} \pi t \end{Bmatrix} y^{2it}\, dt}{\prod_{1 \leqslant \nu \leqslant k} \left(\frac{k+1}{2} - \nu + it\right) \operatorname{ch} \pi(r-t) \operatorname{ch} \pi(r+t)} . \tag{13.7}$$

Hence

$$\left(y^{k-1}\mathcal{L}(y)\right)' = 2i^k\left|\Gamma\left(1-\frac{k}{2}-ir\right)\right|^{-2}$$
$$\int_{-\infty}^{\infty}\frac{\begin{Bmatrix}\operatorname{ch}\pi t\\ \operatorname{sh}\pi t\end{Bmatrix} y^{2it+k-2}dt}{\prod\limits_{1<\nu\leqslant k}\left(\frac{k+1}{2}-\nu+it\right)\operatorname{ch}\pi(r-t)\operatorname{ch}\pi(r+t)}\ . \tag{13.8}$$

Now move the integration to the horizontal line $t-i$. We meet poles at $t=r-\frac{i}{2}$ and $t=-r-\frac{i}{2}$ and find that

$$\left(y^{k-1}\mathcal{L}(y)\right)' = V(y) - 2i^k\left|\Gamma\left(1-\frac{k}{2}-ir\right)\right|^{-2}$$
$$\int_{-\infty}^{\infty}\frac{\begin{Bmatrix}\operatorname{ch}\pi t\\ \operatorname{sh}\pi t\end{Bmatrix} y^{2it+k}dt}{\prod\limits_{1<\nu\leqslant k}\left(\frac{k+3}{2}-\nu+it\right)\operatorname{ch}\pi(r-t)\operatorname{ch}\pi(r+t)}$$

where $V(y)$ denotes the contribution from these two poles.

From this we see that $\mathcal{L}(y)$ satisfies the first order differential equation

$$\left(y^{k-1}\mathcal{L}(y)\right)' = -y^{2k}\left(y^{1-k}\mathcal{L}(y)\right)' + V(y)\ .$$

This equation can be written as

$$\left(\left(y+\frac{1}{y}\right)^{1-k}\mathcal{L}(y)\right)' = y^{-k}\left(y+\frac{1}{y}\right)^{-k}V(y)\ . \tag{13.9}$$

Integrating both sides we get

$$\mathcal{L}(y) = -\left(y+\frac{1}{y}\right)^{k-1}\int_y^{\infty}\left(\eta+\frac{1}{\eta}\right)^{-k}\eta^{-k}V(\eta)d\eta + \text{const.} \tag{13.10}$$

The convergence of this integral will be seen once we have computed $V(y)$ which we now proceed to do. The residue of the integrand in (13.8) at $t=r-\frac{i}{2}$ is

$$\operatorname*{res}_{r-\frac{i}{2}} = \frac{-i\begin{Bmatrix}\operatorname{sh}\pi r\\ \operatorname{ch}\pi r\end{Bmatrix} y^{2ir+k-1}}{\Pi_k(r)(-\pi i)(-i\operatorname{sh}2\pi r)}$$

and at $t=-r-\frac{i}{2}$ is

$$\operatorname*{res}_{-r-\frac{i}{2}} = \frac{-i\begin{Bmatrix}-\operatorname{sh}\pi r\\ \operatorname{ch}\pi r\end{Bmatrix} y^{-2ir+k-1}}{\Pi_k(-r)(i\operatorname{sh}2\pi r)(-\pi i)}$$

where

$$\Pi_k(r) = \prod_{1<\nu\leqslant k}\left(\frac{k}{2}+1-\nu+ir\right).$$

Note that $\Pi_k(r) = (-1)^{k-1}\Pi_k(-r)$. Therefore we have

$$\begin{aligned}V(y) &= (-2\pi i)2i^k\left|\Gamma\left(1-\frac{k}{2}-ir\right)\right|^{-2}\left(\operatorname*{res}_{r-\frac{i}{2}} + \operatorname*{res}_{-r-\frac{i}{2}}\right)\\ &= \frac{2i^k}{\Pi_k(r)}\left|\Gamma\left(1-\frac{k}{2}-ir\right)\right|^{-2}\left\{\begin{matrix}1/\operatorname{ch}\pi r\\ 1/\operatorname{sh}\pi r\end{matrix}\right\}y^{k-1}\left(y^{2ir}-y^{-2ir}\right).\end{aligned}$$

Using the recurrence formula and the functional equation for the gamma function one can check that

$$\left|\Gamma\left(1-\frac{k}{2}-ir\right)\right|^2\Pi_k(r) = \pi i^{k-1}\left\{\begin{matrix}1/\operatorname{sh}\pi r\\ 1/\operatorname{ch}\pi r\end{matrix}\right\}.$$

This gives

$$V(y) = \frac{2i}{\pi}\left\{\begin{matrix}\operatorname{th}\pi r\\ \operatorname{cth}\pi r\end{matrix}\right\}y^{k-1}\left(y^{2ir}-y^{-2ir}\right). \tag{13.11}$$

Hence the integral in (13.10) converges for $k > 0$. Moreover the non–constant part of the right side is clearly $O(y^{-1})$. The left side $\mathcal{L}(y)$ is also $O(y^{-1})$ as follows from (13.7) by moving the horizontal line of integration above $\operatorname{Im} t = \frac{1}{2}$. Therefore by letting y tend to infinity we see that (13.10) holds with the constant zero. Introducing (13.11) into (13.10) and changing the variable of integration η into e^x we get

$$\mathcal{L}(y) = \frac{2}{\pi}\left(\frac{y+y^{-1}}{2}\right)^{k-1}\left\{\begin{matrix}\operatorname{th}\pi r\\ \operatorname{cth}\pi r\end{matrix}\right\}\int_{\log y}^{\infty}\frac{\sin 2rx}{(\operatorname{ch}x)^k}\,dx. \tag{13.12}$$

This agrees with the outcome (13.4) of our calculation of $\mathcal{R}(y)$.

For $k = 0$ we compute $\mathcal{L}(y)$ directly from (13.6). We have

$$\begin{aligned}\mathcal{L}(y) &= \left|\Gamma(1-ir)\right|^{-2}\int_{-\infty}^{\infty}\frac{y^{2it}\operatorname{ch}\pi t\,dt}{\operatorname{ch}\pi(r-t)\operatorname{ch}\pi(r+t)}\\ &= \frac{2\operatorname{sh}\pi r}{\pi r}\int_0^{\infty}\frac{\cos(2t\log y)\operatorname{ch}\pi t}{\operatorname{ch}\pi(r-t)\operatorname{ch}\pi(r+t)}\,dt\end{aligned}$$

and by (3.984.4) of [GR] we get

$$\mathcal{L}(y) = \frac{\operatorname{th}\pi r}{\pi r}\,\frac{y^{2ir}+y^{-2ir}}{y+y^{-1}}. \tag{13.13}$$

Again this agrees with the corresponding result (13.5) given for $\mathcal{R}(y)$.

14. Spectral sum formula without Eisenstein series

Having matched the integrals from Sects. 11 and 12 we find that the formula (10.4) reduces to the following.

Proposition 14.1. *For $m \geqslant 1$ we have*

$$\begin{aligned}\sum_j h(t_j)\nu_j\bar{\lambda}_j(m)\mathcal{N}_j &= \sigma_F(m,\chi)\\ &+ \chi(-1)\sum_{c\equiv 0(\operatorname{mod} D)}\sum_{n\neq m}\sigma_G(n,\chi)S(m-n,0;c) \qquad (14.1)\\ &+ R_s^{(2)}(m) - L_E^{(2)}(m)\end{aligned}$$

where the remainders are bounded in (12.8) and (11.7).

One can see that both the series on the left and the sum of Ramanujan sums on the right converge absolutely (and quickly).

We should emphasize that the test function $h(t)$ defined in (5.19) depends on the real parameter r and consequently so does the function which contains $I(x)$ defined by (5.21). The estimates for $R_s^{(2)}(m)$, $L_E^{(2)}(m)$, in (12.8) and (11.7) are uniform in r, in fact only depending on the weight k (and on ε in (11.7)). These two estimates will be sufficient for our applications. On the other hand when it comes to the bounds we intend to develop for the sums of Ramanujan sums on the right side of (14.1), these would not be sufficient. We are able to resolve this problem by exploiting the parameter r. Specifically we integrate (14.1) over r against a suitable test function $q(r)$ which improves the situation.

Fix a large real number A and choose

$$q(r) = \frac{r\operatorname{sh} 2\pi r}{(r^2+A^2)^8}\left(\operatorname{ch}\frac{\pi r}{2A}\right)^{-4A}. \qquad (14.2)$$

Of course other choices are possible. Note that $q(r)$ is positive for r real, but we can also consider it as a function of a complex variable. As such it is even and holomorphic in the horizontal strip $|\operatorname{Im} r| < A$. Moreover, on the line $\operatorname{Im} r = \pm\frac{A}{2}$ it satisfies the bound

$$|q(r)| \leqslant 2^{4A}|r|^{-15}. \qquad (14.3)$$

Indeed, for u real, $u \geqslant 0$, we have

$$\begin{aligned}\left|\operatorname{sh} 2\pi\left(u+\frac{iA}{2}\right)\right| &\leqslant \operatorname{ch} 2\pi u,\\ \left|\operatorname{ch}\frac{\pi}{2A}\left(u+\frac{iA}{2}\right)\right| &\geqslant \frac{1}{2}\exp\left(\frac{\pi u}{2A}\right),\\ \left|\left(u+\frac{iA}{2}\right)^2+A^2\right| &\geqslant u^2+\frac{3}{4}A^2 \geqslant \left|u+\frac{iA}{2}\right|^2.\end{aligned}$$

Combining these we obtain (14.3).

Multiplying (14.1) by $q(r)$ and integrating from $-\infty$ to ∞ we obtain a sum formula with the new test function

$$\mathfrak{H}(t) = \int_{-\infty}^{\infty} q(r) h(t, r) dr \tag{14.4}$$

where (see (5.19))

$$h(t, r) = \pi \left| \Gamma\left(1 - \frac{k}{2} - ir\right) \right|^{-2} (\operatorname{ch} \pi(r - t) \operatorname{ch} \pi(r + t))^{-1} .$$

On the right side the function G is replaced by $\mathfrak{G}$, still given by the integral (10.5), but now with $I(x) = I(x, r)$ replaced by (see (5.21))

$$\begin{aligned} \mathcal{I}(x) &= \int_{-\infty}^{\infty} q(r) I(x, r) dr \\ &= -2x \int_{-i}^{i} (-i\zeta)^{k-1} \left(\int_{-\infty}^{\infty} q(r) K_{2ir}(\zeta x) dr \right) d\zeta . \end{aligned} \tag{14.5}$$

The integration over r of the error terms leaves the bounds unchanged (put $r = 0$ in (12.8) and (11.7)).

Summing up we obtain the following variant of (14.1).

Proposition 14.2. *For $m \geqslant 1$ we have*

$$\begin{aligned} \sum_j \mathfrak{H}(t_j) \nu_j \bar{\lambda}_j(m) \mathcal{N}_j &= c_A \sigma_F(m, \chi) \\ &+ \chi(-1) \sum_{c \equiv 0 (\operatorname{mod} D)} \sum_{n \neq m} \sigma_{\mathfrak{G}}(n, \chi) S(m - n, 0; c) \\ &+ \mathcal{R}_s^{(2)}(m) - \mathcal{L}_E^{(2)}(m) \end{aligned} \tag{14.6}$$

where the new error terms $\mathcal{R}_s^{(2)}(m)$, $\mathcal{L}_E^{(2)}(m)$ are obtained from the old ones by integration over r. Therefore they satisfy (12.8) and (11.7) with $r = 1$. The constant c_A is just the integral of $q(r)$.

We conclude this section by giving a lower bound for $\mathfrak{H}(t)$ which we shall later require. We are interested only in t real or purely imaginary, and for these, $h(t, r)$ is real and positive. Hence

$$\mathfrak{H}(t) \geqslant \int_{|t|+1}^{|t|+2} q(r) h(t, r) dr .$$

In this range $q(r) \asymp r^{-15}$ and by Stirling's formula

$$h(t, r) \asymp r^{k-1} e^{-\pi r} .$$

Therefore, for t real or purely imaginary

$$\mathfrak{H}(t) \gg \left(|t| + 1\right)^{k-16} e^{-\pi |t|} . \tag{14.7}$$

15. A double average of the spectral sum

In the previous sections we obtained formulae for a number of spectral sums valid for each individual $m \geqslant 1$. In the case of the last formula, without Eisenstein series, we are going to perform an additional averaging over the integer m, weighted by an arithmetic function of a particular type.

Let, for a given integer $\ell \geqslant 1$,

$$\alpha(m) = \sum_{\substack{m_1 m_2 = m \\ m_2 \equiv 0 (\mathrm{mod}\, \ell)}} H(m_1, m_2)\delta(m_2) \tag{15.1}$$

where $|\delta(m_2)| \leqslant 1$ and where $H(y_1, y_2)$ is supported on $[Y_1, 2Y_1] \times [Y_2, 2Y_2]$ with $Y_1, Y_2 \geqslant \frac{1}{2}$, is smooth and satisfies

$$\frac{\partial^{\nu_1+\nu_2} H(y_1, y_2)}{\partial y_1^{\nu_1} \partial y_2^{\nu_2}} \ll P^{\nu_1+\nu_2} Y_1^{-\nu_1} Y_2^{-\nu_2} \tag{15.2}$$

for each $\nu_1, \nu_2 \geqslant 0$, the implied constant depending on ν_1, ν_2.

We think of Y_2 as being slightly larger than Y_1 so that $\alpha(m)$ appears as a Dirichlet convolution in which one of the variables, the slightly shorter one, is smooth. It is the fact that non-smooth one is the larger which makes the problem deeper but also gives room for the amplifier and hence makes the application possible.

For each cusp form f_j from our Hecke basis we define

$$\mathcal{M}_j = \sum_m \lambda_j(m)\alpha(m) \; . \tag{15.3}$$

We do not require the corresponding sum $\mathcal{M}_{\mathfrak{a}}(t)$ for the continuous spectrum since the latter has been eliminated from the formula in the previous chapter.

Proposition 15.1. *Let $\mathcal{N}_j$ and $\mathcal{M}_j$ be given by (10.2) and (15.3) with bit functions $F(x_1, x_2)$ satisfying (11.1) and $H(y_1, y_2)$ satisfying (15.2). Let $\mathfrak{H}(t)$ be the test function given by (14.4). We have*

$$\sum_j \mathfrak{H}(t_j)\nu_j \overline{\mathcal{M}}_j \mathcal{N}_j \ll \left\{ D^{\frac{3}{8}} + (Y_1 + Y_2)^{1-\frac{1}{24\cdot 48}} \right\} (Y_1 + Y_2)$$
$$\left(\frac{X_1}{X_2} + \frac{X_2}{X_1}\right)^{\frac{3}{2}} \left(\frac{Y_1}{Y_2} + \frac{Y_2}{Y_1}\right)^{\frac{1}{2}} \left(X_1 X_2 Y_1 Y_2\right)^{\frac{5}{2}} D^{-5} P^{10} \left(D X_1 X_2 Y_1 Y_2\right)^{\varepsilon}$$
$$+ P^4 \ell^{-1} \left(X_1 X_2 Y_1 Y_2\right)^{\frac{1}{2}}$$
$$\left\{ 1 + \left(X_1 + X_2 + \sqrt{D}\right)^{\frac{1}{2}} \left(Y_1 Y_2\right)^{\frac{1}{2}} D^{-\frac{13}{16}} \right\} \left(\ell D X_1 X_2 Y_1 Y_2\right)^{\varepsilon} .$$

The proof of Proposition 15.1 will occupy several sections. From this point on some of our arguments will be rather similar to those in [DFI7] from Sect. 5 onward. The point is that our formula (14.6) without Eisenstein series is similar to the Petersson formula for holomorphic cusp forms and the test functions in both cases enjoy similar analytic properties. Actually we are already one step ahead having eliminated the contribution to the right side coming from the singular determinants.

By Proposition 14.2 we obtain

$$\sum_j \mathfrak{H}(t_j)\nu_j \overline{\mathcal{M}}_j \mathcal{N}_j = c_A \sum_m \bar{\alpha}_m \sigma_F(m,\chi)$$
$$+ \chi(-1) \sum_{c\equiv 0(D)} \sum_{h\neq 0} S(h,0;c)V(h) + \sum_m \bar{\alpha}(m)\Big(\mathcal{R}_s^{(2)}(m) - \mathcal{L}_E^{(2)}(m)\Big)$$

where

$$V(h) = \sum_{m-n=h} \bar{\alpha}(m)\sigma_{\mathfrak{G}}(n,\chi)\ . \tag{15.4}$$

Note that $V(h)$ also depends on c by way of $\mathfrak{G}$ (see (10.5) and (14.5)). A more general sum of this type will be evaluated in the next section.

Here the first sum on the right side (the diagonal terms) is estimated trivially by

$$\sum_{m\equiv 0(\ell)} \alpha(m)\sigma_F(m,\chi)$$
$$\ll \tau^2(\ell)\ell^{-1}\min\{X_1X_2, Y_1Y_2\}(\log 9X_1X_2Y_1Y_2)^3\ .$$

The remainder terms may also be estimated at once, as follows.

$$\sum_{m\equiv 0(\ell)} \alpha(m)\Big(\mathcal{R}_s^{(2)}(m) - \mathcal{L}_E^{(2)}(m)\Big) \ll P^2\ell^{-1+\varepsilon}\big(X_1X_2Y_1Y_2\big)^{\frac12+\varepsilon}$$
$$+ P^4\ell^{-1}\big(X_1+X_2+\sqrt{D}\big)^{\frac12}\big(X_1X_2\big)^{\frac12}Y_1Y_2D^{-\frac{13}{16}}\big(\ell D Y_1Y_2\big)^{\varepsilon}$$

by (12.8) and (11.7) applied to $\mathcal{R}$ and $\mathcal{L}$ (rather than R and L).

Introducing these estimates for the diagonal terms and the remainder terms we obtain

$$\sum_j \mathfrak{H}(t_j)\nu_j \overline{\mathcal{M}}_j \mathcal{N}_j = \chi(-1) \sum_{c\equiv 0(D)} \sum_{h\neq 0} S(h,0;c)V(h)$$
$$+ O\Big(P^4\ell^{-1}\Big\{1 + \big(X_1+X_2+\sqrt{D}\big)^{\frac12}\big(Y_1Y_2\big)^{\frac12}D^{-\frac{13}{16}}\Big\} \tag{15.5}$$
$$\big(X_1X_2Y_1Y_2\big)^{\frac12}\big(\ell D X_1X_2Y_1Y_2\big)^{\varepsilon}\Big)\ .$$

16. Representations by the determinant

In this section we consider the determinant equation

$$a_1b_2 - a_2b_1 = h \tag{16.1}$$

where $h \neq 0$ is a fixed integer and a_1, a_2, b_1, b_2 are integer variables. We are interested in counting the solutions of these with rather general weights. Let F be a smooth function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ such that, for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \leqslant 8$

$$\begin{aligned} a_1^{\alpha_1} a_2^{\alpha_2} b_1^{\beta_1} b_2^{\beta_2} \frac{\partial^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}}{\partial a_1^{\alpha_1} \partial a_2^{\alpha_2} \partial b_1^{\beta_1} \partial b_2^{\beta_2}} F(a_1, a_2; b_1, b_2) \ll \qquad\qquad \\ Z^{\alpha_1+\alpha_2+\beta_1+\beta_2} \Big(1 + \frac{|a_1|}{A}\Big)^{-4} \Big(1 + \frac{|a_2|}{A}\Big)^{-4} \Big(1 + \frac{b_1}{B}\Big)^{-4} \Big(1 + \frac{b_2}{B}\Big)^{-4} \end{aligned} \tag{16.2}$$

where $A, B, Z \geqslant 1$. This indicates that the support of F is ‘essentially’ in the box

$$[-A, A] \times [-A, A] \times [0, B] \times [0, B]\,.$$

In practice Z will be small which indicates that F is not highly oscillatory. Concerning the variables b_1, b_2 we wish to have completely general weights so we let $\gamma_{b_1}, \delta_{b_2}$ be any complex numbers, for $b_1, b_2 > 0$. Our goal is to evaluate the sum

$$V(h) = \sum\sum\sum\sum_{a_1b_2 - a_2b_1 = h} \gamma_{b_1} \delta_{b_2} F(a_1, a_2; b_1, b_2)\,. \tag{16.3}$$

The expected main term is

$$W(h) = \sum\sum_{(b_1,b_2)|h} \gamma_{b_1} \delta_{b_2} \frac{(b_1, b_2)}{b_1 b_2} I(b_1, b_2) \tag{16.4}$$

where

$$I(b_1, b_2) = \int F\Big(\frac{x}{b_2}, \frac{x-h}{b_1}; b_1, b_2\Big) dx\,. \tag{16.5}$$

Our main result in this section is

Theorem 16.1. *Let $|\gamma_{b_1}| \leqslant 1$ and $|\delta_{b_2}| \leqslant 1$. For any $h \neq 0$ we have*

$$\begin{aligned} V(h) &= W(h) \\ &+ O\left(\tau(h)\Big(1 + \frac{|h|}{AB}\Big)^{-2} \Big(Z^8 A^{-1} B^{\frac{47}{48}}\Big)^{\frac{1}{24}} (AB)^{1+\varepsilon}\right) \end{aligned} \tag{16.6}$$

with any $\varepsilon > 0$, the implied constant depending on ε.

The main ingredient in the proof and one of the main ingredients in this paper is the following corollary of Theorem 2 of [DFI5].

Proposition 16.2. *Let α_m for $M < m \leqslant 2M$, and β_n for $N < n \leqslant 2N$ be arbitrary complex numbers and $h \neq 0$ an integer. Then, for any $\varepsilon > 0$, we have*

$$\sum\sum_{(m,n)=1} \alpha_m \beta_n e\Big(h\frac{\overline{m}}{n} + \frac{X}{mn}\Big)$$
$$\ll \|\alpha\|\,\|\beta\|\Big(1 + \frac{|X|}{MN}\Big)\big(|h| + MN\big)^{\frac{3}{8}}\big(M + N\big)^{\frac{11}{48}+\varepsilon}$$

where $e(t) = e^{2\pi i t}$, $\overline{m}m \equiv 1 \pmod{n}$, and the implied constant depends only on ε.

Proof. Theorem 2 [DFI5] is precisely the case $X = 0$. To derive the general case we separate the variables m, n in $e(X/mn)$ by Fourier inversion, interchange the order of integration and estimate using the theorem together with trivial bounds.

Proof of Theorem 13.1. By applying a smooth partition of unity on $\mathbb{R}^4$ we may assume that F is supported in one of the following sets:

$$\begin{aligned}
\mathcal{B}_1 &= [-1, 1] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\\
\mathcal{B}_2 &= \mathbb{R} \times [-1, 1] \times \mathbb{R}^+ \times \mathbb{R}^+,\\
\mathcal{B}^\sigma &= \sigma_1[X_1, 4X_1] \times \sigma_2[X_2, 4X_2] \times [Y_1, 4Y_1] \times [Y_2, 4Y_2],
\end{aligned}$$

where $\sigma = (\sigma_1, \sigma_2) = (\pm, \pm)$ and X_1, X_2, Y_1, Y_2 take values $2^n \geqslant \frac{1}{2}$.

If F is supported in $\mathcal{B}_1$ then the left side of (16.6) is

$$\begin{aligned}
V(h) &= \sum_{-a_2 b_1 = h} \gamma_{b_1} \delta_{b_2} F(0, a_2; b_1, b_2)\\
&\ll B \sum_{ab=h} \Big(1 + \frac{|a|}{A}\Big)^{-4}\Big(1 + \frac{b}{B}\Big)^{-4} \ll B\tau(h)\Big(1 + \frac{|h|}{AB}\Big)^{-4}.
\end{aligned}$$

Next, the integral $I(b_1, b_2)$ in $W(h)$ is bounded by

$$\Big(1 + \frac{b_1}{B}\Big)^{-4}\Big(1 + \frac{b_2}{B}\Big)^{-4}\int_{-\infty}^{\infty}\Big(1 + \frac{|x|}{b_2}\Big)^{-4}\Big(1 + \frac{|x-h|}{Ab_1}\Big)^{-4} dx.$$

Here we have

$$\Big(1 + \frac{b_1}{B}\Big)\Big(1 + \frac{b_2}{B}\Big)\Big(1 + \frac{|x|}{b_2}\Big)\Big(1 + \frac{|x-h|}{Ab_1}\Big)$$
$$\geqslant \Big(1 + \frac{|x|}{B}\Big)\Big(1 + \frac{|x-h|}{AB}\Big) \geqslant 1 + \frac{|h|}{AB}\,.$$

Hence the main term in (11.2) satisfies

$$W(h) \ll \Big(1+\frac{|h|}{AB}\Big)^{-2} \sum\sum_{(b_1,b_2)|h} \frac{(b_1,b_2)}{b_1}\Big(1+\frac{b_1}{B}\Big)^{-2}\Big(1+\frac{b_2}{B}\Big)^{-2}$$
$$\ll B\tau(h)\Big(1+\frac{|h|}{AB}\Big)^{-2} .$$

These estimates are absorbed by the error term in (16.6) showing that Theorem 16.1 is trivial if F is supported in $\mathcal{B}_1$. Similarly we see that Theorem 16.1 is trivial if F is supported in $\mathcal{B}_2$.

Now suppose F is supported in the positive box $\mathcal{B} = \mathcal{B}^{++} = [X_1, 4X_1]\times[X_2, 4X_2] \times [Y_1, 4Y_1] \times [Y_2, 4Y_2]$. First we estimate both $V(h)$ and $W(h)$ trivially using the bound $F \ll T^{-4}$ where

$$T = T(\mathcal{B}) = \Big(1+\frac{X_1}{A}\Big)\Big(1+\frac{X_2}{A}\Big)\Big(1+\frac{Y_1}{B}\Big)\Big(1+\frac{Y_2}{B}\Big) .$$

Hence

$$V(h) \ll T^{-4} \mid \{(a_1, a_2, b_1, b_2) \in \mathcal{B} \,; \quad a_1b_2 - a_2b_1 = h\} \mid$$
$$\ll T^{-4}\Big(1+\frac{|h|}{X_1Y_2+X_2Y_1}\Big)^{-2} \min(X_1Y_2,\ X_2Y_1)\ (X_1X_2Y_1Y_2)^{\varepsilon}$$

and, because $T\left(1+\frac{|h|}{X_1Y_2+X_2Y_1}\right) \geqslant 1+\frac{|h|}{AB}$, it follows that

$$V(h) \ll T^{-2}\left(1+\frac{|h|}{AB}\right)^{-2}(X_1X_2Y_1Y_2)^{\frac{1}{2}+\varepsilon} .$$

Similarly it follows that

$$W(h) \ll T^{-2}\left(1+\frac{|h|}{AB}\right)^{-2}(X_1X_2Y_1Y_2)^{\frac{1}{2}+\varepsilon} .$$

We apply these estimates for $V(h)$ and $W(h)$ when the box $\mathcal{B}$ does not satisfy

$$\begin{aligned} \Delta^2 A &< X_1, X_2 < \Delta^{-1}A \\ \Delta^2 B &< Y_1, Y_2 < \Delta^{-1}B \end{aligned} \tag{16.7}$$

where $\Delta > 0$ will be chosen later. We obtain

$$V(h) = W(h) + O\Big(\frac{\Delta}{T}\Big(1+\frac{|h|}{AB}\Big)^{-2}(AB)^{1+\varepsilon}\Big) . \tag{16.8}$$

Now we proceed to the essential part of the proof. We split the summation in (16.3) in accordance with the greatest common divisor d of b_1, b_2 getting

$$V(h) = \sum_{d|h}\ \sum\sum_{(b_1,b_2)=1} \gamma_{db_1}\delta_{db_2} \sum\sum_{a_1b_2-a_2b_1=h/d} F(a_1, a_2; db_1, db_2) .$$

To the inner double sum we apply Poisson's formula as follows

$$\sum_{a_1}\sum_{a_2} = \sum_{a_2\equiv -h\bar{b}_1/d(\mathrm{mod}\, b_2)} F\Big(\frac{h/d+a_2b_1}{b_2}\ , a_2; db_1, db_2\Big)$$
$$= \sum_r e\Big(r\frac{h}{d}\,\frac{\bar{b}_1}{b_2}\Big) I_r(b_1, b_2)$$

where

$$I_r(b_1,b_2) = \frac{1}{db_1b_2}\int F\Big(\frac{x+h}{db_2}\ ,\ \frac{x}{db_1}\ ; db_1, db_2\Big) e\Big(\frac{rx}{db_1b_2}\Big)dx\ . \tag{16.9}$$

From the zero frequency $r = 0$ we obtain the main term $W(h)$. We shall estimate the contribution of the other terms separately for each r using Proposition 16.2 with h given by $\frac{rh}{d}$ and X by $\frac{rx}{d}$. To this end we must estimate the integral I_r.

If $0 < |r| < R = \frac{BZ}{Ad}$ then the original expression (16.9) suffices for the estimation. For larger r we shall use the alternative expression, obtained by partial integration three times,

$$I_r(b_1,b_2) = -\Big(\frac{db_1b_2}{2\pi i r}\Big)^3 \frac{1}{db_1b_2}\int F(\quad)''' e\Big(\frac{rx}{db_1b_2}\Big)dx \tag{16.10}$$

where $F(\quad)'''$ is the third derivative with respect to x.

From the two expressions and (16.2) one obtains by trivial estimation

$$I_r(b_1,b_2) \ll \Big(1+\frac{|h|d}{AB}\Big)^{-4}\frac{Ad}{B}\min\Big\{1, \frac{R^3}{|r|^3}\Big\}\ . \tag{16.11}$$

Before applying this estimate we need to separate the variables b_1, b_2. This can be done using the Fourier transform at a cost to the resulting estimate of a factor Z^4.

We obtain by Proposition 16.2,

$$\begin{aligned} V(h) - W(h) &\ll \sum_{d|h}\sum_{r\neq 0}\frac{Ad}{B}\min\Big\{1, \frac{R^3}{|r|^3}\Big\}\frac{B}{d}\Big(1+\frac{A|r|d}{B}\Big) \\ &\qquad \Big(|r|\frac{|h|}{d}+\frac{B^2}{d^2}\Big)^{\frac{3}{8}}\Big(\frac{B}{d}\Big)^{\frac{11}{48}}\Big(1+\frac{|h|d}{AB}\Big)^{-4}B^{\varepsilon} \\ &\ll \sum_{d|h}\frac{RAd}{B}\,\frac{B}{d}\Big(1+\frac{ARd}{B}\Big)\Big(\frac{R|h|}{d}+\frac{B^2}{d^2}\Big)^{\frac{3}{8}}\Big(\frac{B}{d}\Big)^{\frac{11}{48}}\Big(1+\frac{|h|d}{AB}\Big)^{-4}B^{\varepsilon} \\ &\ll Z^3\Big(1+\frac{|h|}{AB}\Big)^{-2}B^{\frac{95}{48}+\varepsilon}\ , \end{aligned} \tag{16.12}$$

valid for any positive box $\mathcal{B}$. Similarly one can show that (16.12) holds for any box of type $\mathcal{B}^\sigma$ with $\sigma = (\pm, \pm)$. Summing over the boxes $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}^\sigma$ we complete the proof of Theorem 16.1 since

$$\sum_{\mathcal{B}} T(\mathcal{B}) \ll (\log 2A)^2 (\log 2B)^2 \ll (AB)^\varepsilon \ .$$

For special coefficients we can successfully estimate the main term $W(h)$ defined in (16.4). We are interested in the coefficients given by

$$\begin{aligned} \gamma_{b_1} &= \chi_1(b_1/\ell_1) \quad \text{if } \ell_1 \mid b_1\ , \ \ \gamma_{b_1} = 0 \quad \text{if } \ell_1 \nmid b_1\ , \\ \delta_{b_2} &= \chi_2(b_2/\ell_2) \quad \text{if } \ell_2 \mid b_2\ , \ \ \delta_{b_2} = 0 \quad \text{if } \ell_2 \nmid b_2\ , \end{aligned} \tag{16.13}$$

where $\chi_1 \pmod{D_1}$, $\chi_2 \pmod{D_2}$ are non-trivial Dirichlet characters. In this case we write

$$W(h) = \sum_{\delta d \mid h} \mu(\delta) d \sum_{\substack{b_1 \equiv 0([\delta d, \ell_1]) \\ b_2 \equiv 0([\delta d, \ell_2])}}\sum \chi_1(b_1)\chi_2(b_2) \frac{I(b_1, b_2)}{b_1 b_2}.$$

Trivially $I(b_1, b_2) \ll A \min(b_1, b_2) \leqslant A\sqrt{b_1 b_2}$, but the condition (16.2) implies that

$$\begin{aligned} & b_1^{\beta_1} b_2^{\beta_2} \frac{\partial^{\beta_1+\beta_2}}{\partial b_1^{\beta_1} \partial b_2^{\beta_2}} I(b_1, b_2) \\ & \ll Z^{\beta_1+\beta_2} A\sqrt{b_1 b_2}\Big(1 + \frac{b_1}{B}\Big)^{-2}\Big(1 + \frac{b_2}{B}\Big)^{-2}\Big(1 + \frac{|h|}{AB}\Big)^{-2} . \end{aligned}$$

Hence, applying Burgess's estimate (see [B2])

$$\sum_{b \leqslant B} \chi(b) b^{-\frac{1}{2}} \ll D^{\frac{3}{16}+\varepsilon} ,$$

which holds for any non-trivial character $\chi \pmod{D}$, one derives

Proposition 16.3. *If the coefficients are given by (16.13) then*

$$W(h) \ll \tau(h)\Big(1 + \frac{|h|}{AB}\Big)^{-2} Z^2 A (D_1 D_2)^{\frac{3}{16}+\varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending on ε.

17. Estimation of $V(h)$

In the previous section we gave a general treatment of sums of type $V(h)$. In this section we apply these results to our specific sum (15.4). Opening $\alpha(m)$ by (15.1) and $\sigma_{\mathfrak{G}}(n,\chi)$ by (10.1) we write $V(h)$ in the form

$$V(h) = \sum\sum\sum\sum_{a_1b_2-a_2b_1=h} \overline{H}(a_1,b_2)\bar{\delta}(b_2)\mathfrak{G}(a_2,b_1)\overline{\chi}(b_1)\ . \tag{17.1}$$

From now on we specialize by choosing

$$\delta(b_2) = \overline{\chi}(b_2)\ . \tag{17.2}$$

This sum is of the type considered in Sect. 16 with

$$F(a_1,a_2;b_1,b_2) = \overline{H}(a_1,b_2)\mathfrak{G}(a_2,b_1) \tag{17.3}$$

and the coefficients given by (16.13) with

$$\chi_1 = \chi_2 = \overline{\chi}\ ,\quad D_1 = D_2 = D\ ,\quad \ell_1 = 1,\ \ell_2 = \ell\ .$$

It remains to verify the condition (16.2) for our choice of F once it has been suitably normalized. Recall that

$$\mathfrak{G}(a_2,b_1) = \iint F(cx_1,cx_2)\mathcal{I}\left(4\pi\sqrt{a_1b_2x_1x_2}\right) \\ e\left(-x_1b_1 - x_2a_2\right)dx_1dx_2\ . \tag{17.4}$$

In order to estimate the partial derivatives of $F(a_1,a_2;b_1,b_2)$ we need those of $H(a_1,b_2)$ and of $\mathfrak{G}(a_2,b_1)$. For the first of these we already have (15.2). For the latter we use (11.1) for $F(x_1,x_2)$ and we still need bounds for the derivatives of $\mathcal{I}(x)$. These are given by

Proposition 17.1. *For $\nu \geqslant 0$ we have*

$$\mathcal{I}^{(\nu)}(x) \ll x^{A+1-\nu} \quad \textit{if}\ 0 < x < 1 \tag{17.5}$$

and

$$\mathcal{I}^{(\nu)}(x) \ll x \quad \textit{if}\ x \geqslant 1\ , \tag{17.6}$$

the implied constant depending on ν.

We remark that (17.6) can be strengthened but we do not need this. The exponent A is the constant fixed in our choice of $q(r)$, see (14.2).

Before emarking on the proof of this result we shall establish an integral representation for $I(x) = I(x,r)$ in terms of the J–Bessel function. We begin by writing the power series expansions of a number of Bessel functions. We have

$$K_{2ir}(z) = \frac{\pi i}{2\,\mathrm{sh}\,2\pi r}\left(I_{2ir}(z) - I_{-2ir}(z)\right)\ ,$$

with

$$I_{2ir}(z) = \sum_{\ell=0}^{\infty} \frac{1}{\ell! \Gamma(\ell+1+2ir)} \left(\frac{z}{2}\right)^{2\ell+2ir},$$

by (8.445) of [GR] and

$$J_{2ir}(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell! \Gamma(\ell+1+2ir)} \left(\frac{z}{2}\right)^{2\ell+2ir},$$

by (8.440) of [GR]. Now using

$$\int_{-i}^{i} (-i\zeta)^{k-1} \zeta^{2\ell+2ir} d\zeta = \frac{(-1)^{k-1} 2 i^{2\ell+1}}{k+2\ell+2ir} \left\{ \begin{matrix} \operatorname{sh} \pi r \\ \operatorname{ch} \pi r \end{matrix} \right\}$$

we obtain

$$I(2x) = 2\pi x \left\{ \begin{matrix} 1/\operatorname{ch} \pi r \\ 1/\operatorname{sh} \pi r \end{matrix} \right\} \sum_{\ell=0}^{\infty} \frac{(ix)^{2\ell}}{\ell!}$$

$$\left\{ \frac{(-1)^{k-1} x^{2ir}}{(k+2\ell+2ir)\Gamma(\ell+1+2ir)} - \frac{x^{-2ir}}{(k+2\ell-2ir)\Gamma(\ell+1-2ir)} \right\}.$$

Differentiating we obtain

$$\begin{aligned} \left(x^{k-1} I(2x)\right)' = 2\pi x^{k-1} &\left\{ \begin{matrix} 1/\operatorname{ch} \pi r \\ 1/\operatorname{sh} \pi r \end{matrix} \right\} \\ &\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} x^{2\ell}}{\ell!} \left\{ \frac{(-1)^{k-1} x^{2ir}}{\Gamma(\ell+1+2ir)} - \frac{x^{-2ir}}{\Gamma(\ell+1-2ir)} \right\} \\ = 2\pi x^{k-1} &\left\{ \begin{matrix} 1/\operatorname{ch} \pi r \\ 1/\operatorname{sh} \pi r \end{matrix} \right\} \left\{ (-1)^{k-1} J_{2ir}(2x) - J_{-2ir}(2x) \right\} \end{aligned}$$

and from this we shall derive

Lemma 17.2. *For k even we have*

$$I(x) = -\frac{\pi x^{1-k}}{\operatorname{ch} \pi r} \int_0^x \left(J_{2ir}(y) + J_{-2ir}(y)\right\} y^{k-1} dy \tag{17.7}$$

and for k odd

$$I(x) = \frac{\pi x^{1-k}}{\operatorname{sh} \pi r} \int_0^x \left(J_{2ir}(y) - J_{-2ir}(y)\right\} y^{k-1} dy\,. \tag{17.8}$$

Proof. If $k \geqslant 1$ these formulae are obtained up to an additive constant on integrating from 0 to x, then checking the constant is zero by letting x approach zero, the integration being justified by absolute convergence.

In case $k = 0$ this convergence in (17.7) would not be absolute so instead we integrate from x to ∞ then check that the additive constant is zero by letting x approach infinity. Thus we have

$$I(x) = \frac{\pi x}{\operatorname{ch} \pi r} \int_x^\infty \big(J_{2ir}(y) + J_{-2ir}(y)\big) \frac{dy}{y} \,. \tag{17.9}$$

The formula (6.561.14), p. 684 of [GR] in our case reduces to

$$\int_0^\infty J_{2ir}(y) \frac{dy}{y} = \frac{1}{2ir}$$

for $r \neq 0$. Adding this for r and for $-r$ we find that the complete integral vanishes and (17.9) becomes (17.7).

Now we are ready to prove the proposition.

Proof of Proposition 17.1. Recall the definition

$$\mathcal{I}(x) = \int_{-\infty}^{\infty} q(r) I(x, r) dr.$$

where $I(x) = I(x, r)$ is given by (17.7), (17.8) and $q(r)$ by (14.2). Note that $q(r)I(x, r)$ is holomorphic in the strip $|\operatorname{Im} r| < A$. Using (17.7), (17.8) to write $\mathcal{I}(x)$ as the sum of two integrals we move the first to $\operatorname{Im} r = -\frac{A}{2}$ and the second to $\operatorname{Im} r = \frac{A}{2}$. Since $q(r)$ is even the two integrals are equal and we get

$$\mathcal{I}(x) = 2\pi \int_0^x \Big(\frac{-y}{x}\Big)^{k-1} \int_{\mathcal{C}} \begin{Bmatrix} 1/\operatorname{ch} \pi r \\ 1/\operatorname{sh} \pi r \end{Bmatrix} q(r) J_{2ir}(y) dr\, dy \tag{17.10}$$

where the inner integral is over the horizontal line $\operatorname{Im} r = -\frac{A}{2}$.

We now can re-introduce the power series expansion for $J_{2ir}(y)$ getting

$$\mathcal{I}(x) = 8\pi \sum_{\ell=0}^{\infty} \frac{(-1)^{k+\ell-1}}{\ell!} \int_{\mathcal{C}} \Big(\frac{x}{2}\Big)^{2\ell+1+2ir} \frac{r}{(r^2 + A^2)^2}$$
$$\Big(\frac{\operatorname{ch} \pi r}{2A}\Big)^{-4A} \frac{\Gamma(\ell + 1 + 2ir)^{-1}}{k + 2\ell + 2ir} \begin{Bmatrix} \operatorname{sh} \pi r \\ \operatorname{ch} \pi r \end{Bmatrix} dr \,.$$

By using Stirling's formula we get the lower bound

$$\Big|\Gamma(\ell + 1 + 2ir)\Big| \gg e^{-\pi |r|},$$

and hence (17.5) follows.

To prove (17.6) we return to (17.10) and move the integration back to the real line. As a result $\mathcal{I}(x)$ becomes

$$4\pi \int_0^x \Big(\frac{-y}{x}\Big)^{k-1} \int_{-\infty}^{\infty} \frac{rJ_{2ir}(y)}{(r^2+A^2)^2}\Big(\frac{\operatorname{ch}\pi r}{2A}\Big)^{-4A} \begin{Bmatrix} \operatorname{sh}\pi r \\ \operatorname{ch}\pi r \end{Bmatrix} dr\, dy$$

and, using the bound

$$J_{2ir}^{(\nu)}(y) \ll (|r|+1)^{\nu} \operatorname{ch}\pi r\ ,$$

we obtain (17.6).

Now we can explain how to check that the condition (16.2) applies to $F(a_1, a_2; b_1, b_2)$. Put $\theta = c^{-1}\sqrt{X_1X_2Y_1Y_2}$. First, by a trivial estimation of the integral (17.4) and (17.6) we get the upper bound

$$F(a_1, a_2; b_1, b_2) \ll \theta \min\big\{1, \theta^{A'}\big\} c^{-2} X_1 X_2 = \|F\|, \text{ say.}$$

Here A' is the exponent A from (14.2) and (17.5) where we have changed the name slightly to avoid possible confusion with the A restricting the support of $F(a_1, a_2; b_1, b_2)$. Integrating repeatedly by parts $\mathfrak{G}(a_2, b_1)$ in (17.4) we obtain for any $\nu \geqslant 0$

$$\mathfrak{G}(a_2, b_1) \ll \|F\|\Big(1+\frac{|a_2|}{A_2}\Big)^{-\nu}\Big(1+\frac{|b_1|}{B_1}\Big)^{-\nu} P^{2\nu}$$

where $A_2 = \sqrt{Y_1Y_2X_1/X_2}$ and $B_1 = \sqrt{Y_1Y_2X_2/X_1}$ and P comes from the property (11.1) of $F(x_1, x_2)$. By the support of $H(a_1, b_2)$ we also know that $a_1 \sim Y_1$, $b_2 \sim Y_2$ so we have

$$F(a_1, a_2; b_1, b_2) \ll \|F\|\Big(1+\frac{a_1}{Y_1}\Big)^{-\nu}\Big(1+\frac{b_2}{Y_2}\Big)^{-\nu} \Big(1+\frac{a_2}{A_2}\Big)^{-\nu}\Big(1+\frac{b_1}{B_1}\Big)^{-\nu} P^{2\nu}\ .$$

Applying the differential operator as on the left side of (16.2) we see that (16.2) holds up to the factor $P^8\|F\|$ with the following choices:

$$A = Y_1 + \sqrt{Y_1Y_2X_1/X_2}\ , \quad B = Y_2 + \sqrt{Y_1Y_2X_2/X_1}$$

and $Z = (\theta+1)P = c^{-1}\big(c + \sqrt{X_1X_2Y_1Y_2}\big)$.

By Propositions 16.1 and 16.2 we can now conclude that:

Proposition 17.3. *We have for any $h \neq 0$,*

$$V(h) \ll \tau(h)\Big(1+\frac{|h|}{AB}\Big)^{-2} \Big\{Z^2AD^{\frac{3}{8}+\varepsilon} + \Big(Z^8A^{-1}B^{\frac{47}{48}}\Big)^{\frac{1}{24}}(AB)^{1+\varepsilon}\Big\} P^8\|F\|\ . \tag{17.11}$$

18. Completion of Proof of Proposition 15.1

We need to estimate the sum of Ramanujan sums on the right side of (15.5). Using the trivial bound $|S(h,0;c)| \leqslant (h,c)$ and (17.11) we get

$$\sum_{c\equiv 0(D)} \sum_{h\neq 0} S(h,0;c)V(h)$$

$$\ll \sum_{c\equiv 0(D)} \left\{ Z^2 A D^{\frac{3}{8}} + \left(Z^8 A^{-1} B^{\frac{47}{48}} \right)^{\frac{1}{24}} AB \right\} ABP^8 \|F\| (cABD)^{\varepsilon} .$$

Note that $\|F\|$ is very small if c is slightly larger than $\sqrt{X_1X_2Y_1Y_2}$. Hence the above sum is bounded by

$$\left\{ AD^{\frac{3}{8}} + \left(A^{-1} B^{\frac{47}{48}} \right)^{\frac{1}{24}} AB \right\} ABP^{10} D^{-5+\varepsilon} X_1X_2 (X_1X_2Y_1Y_2)^{\frac{3}{2}+\varepsilon} .$$

Here we have

$$\begin{aligned} AD^{\frac{3}{8}} + \left(A^{-1} B^{\frac{47}{48}} \right)^{\frac{1}{24}} AB &\ll (A+B)\left\{ D^{\frac{3}{8}} + (A+B)^{1-\frac{1}{24\cdot 48}} \right\} \\ &\ll \left(\frac{X_1}{X_2} + \frac{X_2}{X_1} \right) \left\{ D^{\frac{3}{8}} + (Y_1+Y_2)^{1-\frac{1}{24\cdot 48}} \right\} (Y_1+Y_2) \end{aligned}$$

and

$$AB \ll \left(\frac{X_1}{X_2} + \frac{X_2}{X_1} \right)^{\frac{1}{2}} \left(\frac{Y_1}{Y_2} + \frac{Y_2}{Y_1} \right)^{\frac{1}{2}} Y_1Y_2 .$$

Hence the above sum is bounded by

$$\begin{gathered} (Y_1+Y_2)\left\{ D^{\frac{3}{8}} + (Y_1+Y_2)^{1-\frac{1}{24\cdot 48}} \right\} \\ \left(\frac{X_1}{X_2} + \frac{X_2}{X_1} \right)^{\frac{3}{2}} \left(\frac{Y_1}{Y_2} + \frac{Y_2}{Y_1} \right)^{\frac{1}{2}} (X_1X_2Y_1Y_2)^{\frac{5}{2}} D^{-5} P^{10} \left(X_1X_2Y_1Y_2D \right)^{\varepsilon} . \end{gathered}$$

This together with (15.5) completes the proof of Proposition 15.1.

19. Estimates for the coefficients of a cusp form

To make proper use of our spectral summation formulae we need good lower bounds for the normalizing factors ν_j and $\nu_{\mathfrak{a}}(t)$ given in (6.22) and (6.23). For the latter we have an explicit expression (7.14) in terms of Dirichlet L–functions from which the bound (7.15) follows and this is sufficient for the Eisenstein case. In the case of the cusp forms the corresponding expression for ν_j involves the associated symmetric square L–function so we need to study the properties of these before we can prove the corresponding bound (7.16). The same ideas provide also a proof of another ingredient, Proposition 19.6, which will be needed in Sect. 21.

We begin with the Rankin-Selberg L–function. Let $u_j(z)$ be the Hecke-Maass cusp form whose Fourier-Whittaker expansion is given by (5.1) normalized by $\|u_j\|^2 = \langle u_j, u_j \rangle = 1$. To this we associate the Rankin-Selberg L–function

$$L_j^{|2|}(s) = \sum_{n=1}^{\infty} |\lambda_j(n)|^2 n^{-s} . \tag{19.1}$$

Recall that the Hecke-Maass eigenvalues $\lambda_j(n)$ are related to the Fourier coefficients $\rho_j(n)$ by (6.14).

We give an integral representation for this L–function which allows us to derive the analytic continuation and the functional equation from those of the Eisenstein series. This method, the unfolding method of Rankin-Selberg, is well-known. However, references in the case of the non-holomorphic cusp forms are not easy to find so we provide complete details.

The starting point is the integral

$$I_j(s) = \int_{\Gamma\backslash\mathbb{H}} |u_j(z)|^2 E(z,s) d\mu z$$

where

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}\gamma z)^s$$

is the Eisenstein series of weight zero for the cusp ∞ of the group $\Gamma = \Gamma_0(D)$. Unfolding the fundamental domain $\Gamma \backslash \mathbb{H}$ into the vertical strip $\Gamma_\infty \backslash \mathbb{H}$ we obtain by the modularity (4.34) of $u_j(z)$ that

$$I_j(s) = \int_{\Gamma_\infty\backslash\mathbb{H}} |u_j(z)|^2 y^s d\mu z .$$

Introducing the Fourier series (5.1) we obtain by Parseval (since W is real)

$$\begin{aligned} I_j(s) &= \sum_{n\neq 0} |\rho_j(n)|^2 \int_0^\infty W^2_{\frac{kn}{2|n|},it_j}(4\pi|n|y) y^{s-2} dy \\ &= (4\pi)^{1-s} \left\{ w_j^+(s)|\rho_j(1)|^2 + w_j^-(s)|\rho_j(-1)|^2 \right\} L_j^{|2|}(s) \end{aligned}$$

where

$$w_j^\pm(s) = \int_0^\infty W^2_{\pm\frac{k}{2},it_j}(y) y^{s-2} dy \tag{19.2}$$

and by (4.70) we arrive at the desired integral representation

$$w_j(s)\nu_j L_j^{|2|}(s) = 4\pi^2 \int_{\Gamma\backslash\mathbb{H}} |u_j(z)|^2 E(z,s) d\mu z \tag{19.3}$$

where

$$w_j(s) = (4\pi)^{1-s}\left\{w_j^+(s) + w_j^-(s)\ \left|\ \frac{\Gamma\left(it_j + \frac{k}{2}\right)}{\Gamma\left(it_j - \frac{k}{2}\right)}\ \right|^2\right\} \tag{19.4}$$

and ν_j is defined by (6.22).

In general the integrals (19.2) seem difficult to compute but for the special value $s = 1$ we may express it in terms of the gamma function and we shall need this result. Specifically, from the Fourier integral representation (cf. [GR], p. 321)

$$\begin{aligned} W_{\kappa,\nu}(y) = \pi^{-1}(\pi y)^{\frac{1}{2}-\nu}\Gamma\Big(\nu + \frac{\kappa+1}{2}\Big) \\ \int_{-\infty}^{\infty}\Big(\frac{1+ix}{1-ix}\Big)^{\kappa}(1+x^2)^{-\frac{1}{2}-\nu}e(-xy)dx. \end{aligned} \tag{19.5}$$

Applying Plancherel (see (14.9) of [DFI4]),

$$\int_0^{\infty} W^2(4\pi y)y^{-1}dy = \frac{1}{8\pi}\Big|\Gamma\Big(\nu + \frac{\kappa+1}{2}\Big)\Big|^2. \tag{19.6}$$

Hence, by (19.4), we have

$$w_j(1) = \frac{1}{8\pi}\Big|\Gamma\Big(s_j + \frac{k}{2}\Big)\Big|^2\ . \tag{19.7}$$

Next we need to recall some basic facts about the Rankin-Selberg L–function $L_j^{|2|}(s)$. In the first place $L_j^{|2|}(s)$ is holomorphic in the plane apart from a simple pole at $s = 1$. More precisely

$$L_j^{|2|}(s) = \frac{\zeta(s)}{\zeta_D(s)}\frac{L_j^{(2)}(s)}{\zeta(2s)}$$

where $\zeta(s)$ is the Riemann zeta function, $\zeta_D(s) = \prod_{p|D}\left(1 - p^{-s}\right)^{-1}$, and $L_j^{(2)}(s)$ is the L–function associated to the symmetric square representation which is given by the following series:

$$L_j^{(2)}(s) = \zeta(2s)\sum_{\ell=1}^{\infty}\lambda_j(\ell^2)\ell^{-s}\ . \tag{19.8}$$

It is known, essentially due to Shimura [S], that $L_j^{(2)}(s)$ has analytic continuation to the whole s–plane. Hence

$$\operatorname*{res}_{s=1} L_j^{|2|}(s) = \frac{6}{\pi^2\zeta_D(1)}L_j^{(2)}(1)\ .$$

On the other hand

$$\operatorname*{res}_{s=1} E(z,s) = \frac{1}{\mathrm{vol}(\Gamma \setminus \mathbb{H})} = \frac{3}{\pi \nu(D)} .$$

Now, comparing the residues in (19.3) and using (19.6), we arrive at

Proposition 19.1. *Let $u_j(z)$ be a Hecke–Maass cusp form. We have*

$$\nu_j = \frac{(2\pi)^4 \zeta_D(2)}{D \left| \Gamma\left(s_j + \frac{k}{2}\right) \right|^2 L_j^{(2)}(1)} . \tag{19.9}$$

Note the similarity of this formula with that in Proposition 7.1. Just as that result was used to derive the bound (7.15) we shall employ this one to obtain (7.16). Now however the Dirichlet L-functions are replaced by the symmetric–square L-function for Hecke–Maass cusp forms so more work is required. We begin with an estimate for an integral of the Whittaker function.

Lemma 19.2. *Let $W = W_{\kappa,\nu}(y)$ be the Whittaker function with $\kappa \geqslant 0$ and $\nu = it$, $t \geqslant 1$. For some positive absolute constants α, β we have*

$$\int_\alpha^{\beta t(\kappa+1)} W^2(4\pi y) y^{-1} dy \geqslant \frac{1}{9\pi} \left| \Gamma\left(\nu + \frac{\kappa+1}{2}\right) \right|^2 . \tag{19.10}$$

Proof. It follows from (19.5) that if $f(y)$ and $\hat{f}(x)$ are both non-negative, then

$$\int_0^\infty W^2(4\pi y) f(y) \frac{dy}{y} \leqslant \frac{1}{\pi} \left| \Gamma\left(\nu + \frac{\kappa+1}{2}\right) \right|^2 \int_0^\infty W_{00}^2(4\pi y) f(y) \frac{dy}{y}.$$

But

$$W_{00}(4\pi y) = \sqrt{y} \int_{-\infty}^{\infty} (1+x^2)^{-\frac{1}{2}} e(-xy) dx \ll \sqrt{y} \log\left(2 + \frac{1}{y}\right) .$$

Hence, by a judicious choice of f we derive

$$\int_0^\alpha W^2(4\pi y) y^{-1} dy \leqslant \frac{1}{200\pi} \left| \Gamma\left(\nu + \frac{\kappa+1}{2}\right) \right|^2 \tag{19.11}$$

for some small positive absolute constant α. Moreover, it follows from (19.5) after integration by parts that

$$W(4\pi y) \ll \left| \Gamma\left(\nu + \frac{\kappa+1}{2}\right) \right| |\nu| (\kappa+1) y^{-\frac{1}{2}},$$

provided $y \geqslant 2|\nu|$ since there is then no stationary point. Hence

$$\int_{\beta|\nu|(\kappa+1)}^\infty W^2(4\pi y) y^{-1} dy < \frac{1}{200\pi} \left| \Gamma\left(\nu + \frac{\kappa+1}{2}\right) \right|^2 \tag{19.12}$$

if β is sufficiently large. Finally, subtracting (19.11) and (19.12) from (19.5) we obtain the lemma.

Lemma 19.3. *Suppose* $\|u_j\| = 1$. *Then*

$$\sum_{0<n\leqslant N} n|\rho_j(n)|^2 \ll_k \Big(\frac{N}{D}+1\Big)|s_j|e^{\pi|s_j|} \tag{19.13}$$

where the implied constant depends on k.

Proof. We begin with the equation

$$\int_0^1 |u_j(z)|^2 dx = \sum_{n\neq 0} |\rho_j(n)|^2 W^2(4\pi|n|y)$$

which holds by Parseval. Integrating over y we obtain

$$\begin{aligned} 1+\frac{10}{DY} &\geqslant \int_Y^\infty \int_0^1 |u_j(z)|^2 d\mu z \\ &= \sum_{n\neq 0} |\rho_j(n)|^2 \int_Y^\infty W^2(4\pi|n|y)y^{-2}dy \end{aligned}$$

because, by Lemma 2.10 of [I2], every orbit $\{\gamma z;\, z\in\Gamma\}$ has no more than $1+\frac{10}{DY}$ points in the half strip $0\leqslant x\leqslant 1$, $y>Y$. By Lemma 19.2 and Stirling's formula we have

$$\int_\alpha^\infty W^2(4\pi y)y^{-2}dy \gg |s_j|^{-1}e^{-\pi|s_j|} \tag{19.14}$$

where $\alpha>0$ is an absolute constant, the implied constant depending on k. Hence

$$\int_Y^\infty W^2(4\pi|n|y)y^{-2}dy \gg |ns_j^{-1}|e^{-\pi|s_j|}$$

if $nY\leqslant\alpha$. Setting $Y=\alpha N^{-1}$ we derive Lemma 19.3.

Let

$$\eta_j = \nu_j^{-1}|s_j|e^{\pi|s_j|}$$

and note that $\eta_j \gg 1$ by Lemma 19.3 with $N=1$.

Corollary 19.4. *We have*

$$\sum_{1\leqslant n\leqslant N} |\lambda_j(n)|^2 \ll \Big(\frac{N}{D}+1\Big)\eta_j. \tag{19.15}$$

This estimate is quite good for large N. We shall improve it for small N by employing the multiplicativity of $\lambda_j(n)$.

Let $S(x) = \sum_{n \leqslant x} |\lambda_j(n)|^2$. By the multiplicativity (6.6) and by induction on r we have

$$|\lambda_j(n_1) \cdots \lambda_j(n_r)| \leqslant \sum_{d^2 | n_1 \dots n_r} \tau_r(d) \left| \lambda_j\Big(\frac{n_1 \cdots n_r}{d^2}\Big) \right| .$$

Hence, by Cauchy's inequality

$$|\lambda_j(n_1) \cdots \lambda_j(n_r)|^2 \leqslant \Big(\sum_{d^2 | n_1 \dots n_r} \tau_r^2(d) \Big) \Big(\sum_{d^2 | n_1 \dots n_r} \left| \lambda_j\Big(\frac{n_1 \dots n_r}{d^2}\Big) \right|^2 \Big) .$$

We obtain

$$S(x)^r \leqslant \sum_{m \leqslant x^r} \tau_r(m)^2 \Big(\sum_{d^2 | m} \tau_r(d)^2 \Big) \sum_{d^2 | m} \left| \lambda_j\Big(\frac{m}{d^2}\Big) \right|^2$$

and, estimating the divisor function by x^ε, we get $S(x)^r \ll x^\varepsilon S(x^r)$ for any $\varepsilon > 0$, $r \geqslant 1$, the implied constant depending only on ε and r. By Corollary 19.4 this yields

$$S(x)^r \ll x^{r+\varepsilon} \eta_j \ . \tag{19.16}$$

We have established the following bound.

Corollary 19.5. *For any $\varepsilon > 0$, $x \geqslant 1$,*

$$S(x) \ll x(x\eta_j)^\varepsilon$$

where the implied constant depends only on ε and k.

Using Corollary 19.5 we derive an upper bound for $L_j^{(2)}(1)$. To this end we truncate the series for $L_j^{(2)}(1)$ getting

$$\zeta(2) L_j^{(2)}(1) = \sum_{\ell \leqslant L} \frac{\lambda_j(\ell^2)}{\ell} + O\Big(\frac{(D|s_j|)^{10}}{L} \Big). \tag{19.17}$$

By (6.5)

$$\lambda_j(\ell) = \sum_{d | \ell} \lambda_j\Big(\Big(\frac{\ell}{d}\Big)^2 \Big) \chi(d)$$

so, by Möbius inversion,

$$\lambda_j(\ell^2) = \sum_{d | \ell} \mu(d) \chi(d) \lambda_j^2\Big(\frac{\ell}{d}\Big)$$

and, inserting this in (19.17), we deduce

$$L_j^{(2)}(1) \ll \sum_{\ell \leqslant L} \frac{|\lambda_j(\ell)|^2}{\ell} + \frac{(D|s_j|)^{10}}{L} \ . \tag{19.18}$$

An application of Corollary 19.5 gives

$$L_j^{(2)}(1) \ll \left(D|s_j|\eta_j\right)^{\varepsilon} \ . \tag{19.19}$$

This bound allows us to complete the proofs. Note that by Stirling's formula, the identity (19.12), and (19.19) we obtain

$$|s_j|^k \nu_j D e^{-\pi|s_j|} \left(D|s_j| \frac{e^{\pi|s_j|}}{\nu_j}\right)^{\varepsilon} \gg 1.$$

This implies

$$\nu_j \gg D^{-1}|s_j|^{-k} e^{\pi|s_j|}(D|s_j|)^{-\varepsilon} \ .$$

and proves (7.16). It also proves that $\eta_j \ll D|s_j|^{k+1}(D|s_j|)^{\varepsilon}$ which allows us to drop η_j from the estimate (19.19) and Corollary 19.5 now gives the following result.

Proposition 19.6. *For any $x \geqslant 1$ and $\varepsilon > 0$*

$$\sum_{n \leqslant x} |\lambda_j(n)|^2 \ll x(xD|s_j|)^{\varepsilon} \ .$$

20. Preparing for amplification

Actually we shall need, rather than Proposition 15.1, a corollary which holds when the coefficients have a special shape as will occur in the application.

Proposition 20.1. *Let $\mathcal{N}_j$ be given by (10.2) with the bit function satisfying (11.1). Let $\mathfrak{H}(t)$ be as in (14.4) with $q(r)$ given by (14.2). Then, for every $\ell \geqslant 1$ and $X_1, X_2 \geqslant 1$ we have*

$$\sum_j \mathfrak{H}(t_j)\nu_j\lambda_j(\ell)|\mathcal{N}_j|^2$$

$$\ll X_1X_2P^{10}\Big[\big\{\ell^{-\frac{1}{2}} + (X_1+X_2+\sqrt{D})^{\frac{1}{2}}(X_1X_2)^{\frac{1}{2}}D^{-\frac{13}{16}}\big\}$$

$$+\frac{\ell^4}{D^5}(X_1+X_2)^3(X_1X_2)^2\big\{D^{\frac{3}{8}} + (X_1+X_2)^{1-\theta}\big\}\Big](\ell D X_1 X_2)^{\varepsilon}$$

where $\theta = \frac{1}{1152}$.

Proof. By the multiplicativity formula (6.5) we get

$$\begin{aligned}
\lambda_j(\ell)\mathcal{N}_j &= \sum_n \lambda_j(\ell)\lambda_j(n)\sigma_F(n,\chi) \\
&= \sum_{d\mid\ell}\chi(d)\sum_{n_1n_2\equiv 0(d)}\lambda_j\Big(\frac{\ell n_1n_2}{d^2}\Big)F(n_1,n_2)\overline{\chi}(n_2) \\
&= \sum_{d\mid\ell}\chi(d)\sum_{\delta\mid d}\sum_{\left(n_1,\frac{d}{\delta}\right)=1}\sum_{n_2}\lambda_j\Big(\frac{\ell}{d}n_1n_2\Big)F\Big(\delta n_1,\frac{d}{\delta}n_2\Big)\overline{\chi}\Big(\frac{d}{\delta}n_2\Big) \\
&= \sum_{d\mid\ell}\chi(d)\sum_{\delta\mid d}\sum_{\nu\mid\frac{d}{\delta}}\mu(\nu)\,S_{d\delta\nu}
\end{aligned}$$

where

$$\begin{aligned}
S_{d\delta\nu} &= \sum_{m_1}\sum_{n_2}\lambda_j\Big(\frac{\ell}{d}\nu m_1n_2\Big)F\Big(\delta\nu m_1,\frac{d}{\delta}n_2\Big)\overline{\chi}\Big(\frac{d}{\delta}n_2\Big) \\
&= \sum_m\lambda_j(m)\sum_{\substack{m_1m_2=m\\ m_2\equiv 0\left(\frac{\ell\nu}{d}\right)}} F\Big(\delta\nu m_1\frac{d^2}{\ell\nu\delta}m_2\Big)\overline{\chi}\Big(\frac{d^2}{\ell\nu\delta}m_2\Big).
\end{aligned}$$

We see that, for given d,δ,ν, Proposition 15.1 is applicable with the bit function $H(y_1,y_2)=F(\delta\nu y_1,\frac{d^2}{\ell\nu\delta}y_2)$ and the coefficients $\delta(\mu)=\overline{\chi}\left(\frac{d}{\delta}\mu\right)$. Thus $Y_1=X_1/\delta\nu$, $Y_2=\ell\nu\delta X_2/d^2$ and ℓ is replaced by $\ell\nu/d$. Hence one completes the proof of Proposition 20.1.

To simplify Proposition 20.1 we write

$$X = X_1 + X_2 + \sqrt{D} \tag{20.1}$$

and this gives

$$\sum_j \mathfrak{H}(t_j)\nu_j\lambda_j(\ell)|\mathcal{N}_j|^2 \ll \frac{X_1X_2}{\sqrt{\ell}}P^{10}\big\{1+\ell^4X^{10-\theta}D^{-5}\big\}(\ell X)^{\varepsilon}\,. \tag{20.2}$$

Finally, by (14.7) and (7.16) we obtain

$$\mathfrak{H}(t_j)\nu_j \gg (|t_j|+1)^{-17}D^{-1-\varepsilon} \tag{20.3}$$

and inserting this we complete the proof of Theorem 2.1. Actually the above argument carries an extra factor ℓ^{ε} which is not needed since it can be absorbed unless $\ell > X^{2001}$ and in this case the theorem is trivial. For similar reasons we have removed L^{ε} in several places below.

21. Amplification

Let c_ℓ be complex numbers for $\ell \leqslant L$ with $(\ell, D) = 1$ and let $\|\mathbf{c}\|$ be the ℓ_2-norm, that is

$$\|\mathbf{c}\|^2 = \sum_{\ell \leqslant L} |c_\ell|^2.$$

To each cusp form in our basis we asssociate the amplifier

$$\mathfrak{C}_j = \sum_{\ell \leqslant L} c_\ell \lambda_j(\ell) \,. \tag{21.1}$$

Using again (20.3), Theorem 2.2 follows at once from the following result.

Proposition 21.1. *Under the same assumptions as Proposition 20.1 and with X defined by (20.1), we have*

$$\sum_j \mathfrak{H}(t_j)\nu_j |\mathfrak{C}_j|^2 \, |\mathcal{N}_j|^2 \ll X_1 X_2 P^{10} \left(1 + L^4 D^{-5} X^{10-\theta}\right) \|\mathbf{c}\|^2 X^\varepsilon \,. \tag{21.2}$$

Proof. By the multiplicativity (6.5) and since $\bar{\lambda}_j(\ell) = \overline{\chi}(\ell)\lambda_j(\ell)$ for $(\ell, D) = 1$, see (6.6), we get

$$|\mathfrak{C}_j|^2 = \sum_{(d,D)=1} \sum_{\ell_1} \sum_{\ell_2} \lambda_j(\ell_1\ell_2) c_{d\ell_1} \bar{c}_{d\ell_2} \overline{\chi}(\ell_2) \,.$$

Applying (20.2) with $\ell = \ell_1\ell_2$ and Cauchy's inequality we obtain (21.2).

Note that $\mathfrak{H}(t_j) > 0$ and $\nu_j > 0$. Therefore, dropping every term in (21.2) but the one with eigenvalue $\lambda_j = s_j(1 - s_j)$, $s_j = \frac{1}{2} + it_j$, we obtain

$$\mathfrak{H}(t_j)\nu_j |\mathfrak{C}_j \mathcal{N}_j|^2 \ll X_1 X_2 P^{10}\big(1 + L^4 D^{-5} X^{10-\theta}\big)\|\mathbf{c}\|^2 X^\varepsilon \,. \tag{21.3}$$

We want $\mathfrak{C}_j$ to be large. We choose c_ℓ as in (1.17) of [DFI7], that is

$$c_\ell = \begin{cases} \lambda_j(p)\overline{\chi}(p) & \text{if } \ell = p,\ \frac{1}{2}\sqrt{L} < p \leqslant \sqrt{L} \\ -\overline{\chi}(p) & \text{if } \ell = p^2,\ \frac{1}{2}\sqrt{L} < p \leqslant \sqrt{L} \,. \end{cases} \tag{21.4}$$

Then using the relation, see (6.5),

$$\lambda_j^2(p) - \lambda_j(p^2) = \chi(p) \tag{21.5}$$

we find that

$$\mathfrak{C}_j \asymp \sqrt{L}(\log L)^{-1} \tag{21.6}$$

provided that $L \gg (\log D)^2$.

From Proposition 19.6 we obtain

$$\|\mathbf{c}\| \ll L^{\frac{1}{4}}\left(|\lambda_j| + k + D + L\right)^{\varepsilon}. \tag{21.7}$$

We insert (21.6) in the left side of (21.3) and (21.7) in the right side to deduce

$$\mathfrak{H}(t_j)\nu_j|\mathcal{N}_j|^2 \ll L^{-\frac{1}{2}} X_1 X_2 P^{10}\left(1 + L^4 D^{-5} X^{10-\theta}\right)\left(k + |\lambda_j| + X\right)^{\varepsilon}.$$

To simplify the computation we replace L^4 by L^5, then we choose $L = DX^{-2+\frac{\theta}{5}} + (\log D)^2$, getting by (20.1)

$$\mathfrak{H}(t_j)\nu_j|\mathcal{N}_j|^2 \ll X_1 X_2 P^{10}\left((k + |\lambda_j| + D + X_1 + X_2\right)^{\varepsilon} \left\{D^{-\frac{1}{2}}(X_1 + X_2)^{1-\frac{\theta}{10}} + D^{-5}(X_1 + X_2)^{10-\theta} + D^{-\frac{\theta}{20}}\right\}.$$

Finally, inserting (20.3) we complete the proof of the upper bound for $\mathcal{N}_j$ stated in Theorem 2.3.

We apply Theorem 2.3 for $\mathcal{N}_j$ the inner sum of (9.8) so that F is given by (9.9) and P is given by (9.11). In this case we obtain

$$\mathcal{N}_j \ll \left(|t_j| + |s|\right)^{14}\left(\frac{DN^2}{\delta^2 d}\right)^{\frac{1}{2}+\varepsilon}\left\{D^{-\frac{1}{2}}N^{1-\frac{\theta}{10}} + D^{-5}N^{10-\theta} + D^{-\frac{\theta}{20}}\right\}^{\frac{1}{2}}.$$

Introducing this in (9.8) and summing over d and δ we get

$$G_j(N) \ll \left(|t_j| + |s|\right)^{7}(DN^2)^{\frac{1}{4}+\varepsilon}\left\{D^{-\frac{1}{2}}N^{1-\frac{\theta}{10}} + D^{-5}N^{10-\theta} + D^{-\frac{\theta}{20}}\right\}^{\frac{1}{4}}.$$

Inserting this in (9.7) and summing over $N = 2^{\nu/2}$ we obtain

$$L_j(s) \ll \left(|t_j| + |s|\right)^{\frac{19}{2}} D^{\frac{1}{4}-\frac{\theta}{20}+\varepsilon},$$

giving Theorem 2.4 with a little to spare in both exponents.

22. Applications to the class group

In this section we prove Theorems 2.7 and 2.8. Recall that K is a quadratic field with discriminant d, that $\mathcal{C}l(K)$ is the (narrow) class group of K and ψ is a character of $\mathcal{C}l(K)$. Let $f(x)$ be a C^∞ function supported in $[1, 2]$ which satisfies $\int_0^\infty f(x)dx = 1$. We detect ideals with small norm and with various needed properties using the sum

$$S(x, \psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) f(N(\mathfrak{a})/x).$$

A standard application of Mellin inversion gives

$$S(x,\psi)=\frac{1}{2\pi i}\int_{(2)}\tilde{f}(s)L_K(s,\psi)x^s\,ds$$

where

$$\tilde{f}(s)=\int_0^\infty f(x)x^{s-1}dx$$

is entire and of rapid decay in vertical strips. Theorem 2.6 yields, after a contour shift to $\mathrm{Re}(s)=\frac{1}{2}$, our starting point

$$S(x,\psi)=\delta_\psi L(1,\chi_d)x+O\big(x^{\frac{1}{2}}|d|^{\frac{1}{4}-\frac{1}{23041}}\big) \tag{22.1}$$

where δ_ψ is 1 if ψ is trivial and 0 otherwise. Here we explicitly use the polynomial dependence on $|s|$ of our bound in Theorem 2.6.

To prove Theorem 2.7, let $H\subset \mathcal{C}l(K)$ be a subgroup for which $G=\mathcal{C}l(K)/H$. For $g\in G$ a coset in $\mathcal{C}l(K)$ let

$$S_g(x)=\sum_{\mathfrak{a}\in g}f(N(\mathfrak{a})/x)$$

and let $S_g^*(x)$ be the same sum restricted to primitive $\mathfrak{a}\in g$. We have

$$S_g(x)=|G|^{-1}\sum_{\psi(H)=1}\bar{\psi}(g)S(x,\psi)$$

so by Möbius inversion we get

$$S_g^*(x)=\sum_{m\geqslant 1}\mu(m)S_g(x/m^2)=\sum_{1\leqslant m\leqslant\sqrt{2x}}\mu(m)S_g(x/m^2).$$

Using (22.1) we derive the uniform asymptotic formula

$$S_g^*(x)=\frac{6}{\pi^2}\frac{L(1,\chi_d)}{|G|}x+O\Big(x^{\frac{1}{2}}(\log x)|d|^{\frac{1}{4}-\frac{1}{23041}}\Big). \tag{22.2}$$

By Siegel's theorem $L(1,\chi_d)\gg |d|^{-\varepsilon}$ so we deduce Theorem 2.7, but with an ineffective implied constant.

We turn next to the proof of Theorem 2.8. Let H be a cyclic subgroup of $\mathcal{C}l(K)$ of order h and index k in $\mathcal{C}l(K)$, so $hk=|\mathcal{C}l(K)|$. If $\delta_H(\mathfrak{a})$ is the characteristic function of the generators of H then for any fixed generator $\mathfrak{g}$ of H we have the formula

$$\delta_H(\mathfrak{a})=\frac{1}{hk}\sum_{\psi}\bar{\psi}(\mathfrak{a})\sum_{\substack{\ell (\mathrm{mod}\, h)\\ (\ell,h)=1}}\psi(\mathfrak{g}^\ell) \tag{22.3}$$

which is easily proven using the orthogonality relations and the fact that $\mathfrak{a}$ generates H if and only if $\mathfrak{a} = \mathfrak{g}^\ell$ for some ℓ with $(\ell, h) = 1$. We then derive the formula

$$\delta_H(\mathfrak{a}) = \frac{1}{k} \sum_{m \mid h} \frac{\mu(m)}{m} \sum_{\psi(\mathfrak{g})^m = 1} \psi(\mathfrak{a}) \tag{22.4}$$

by applying Möbius inversion to the second sum in (22.3):

$$\sum_{(\ell,h)=1} \psi(\mathfrak{g}^\ell) = \sum_{m \mid h} \mu(m) \sum_{\ell (\operatorname{mod} h/m)} \psi(\mathfrak{g}^{m\ell}) = h \sum_{\substack{m \mid h \\ \psi(\mathfrak{g})^m = 1}} \frac{\mu(m)}{m}.$$

Letting

$$S_H(x) = \frac{h}{\varphi(h)} \sum_{\mathfrak{a}} \delta_H(\mathfrak{a}) f(N(\mathfrak{a})/x)$$

where $\varphi(\,)$ is the Euler function, we derive from (22.4) that

$$S_H(x) = \frac{h}{k\varphi(h)} \sum_{m \mid h} \frac{\mu(m)}{m} \sum_{\psi(H)^m = 1} S(x, \psi).$$

Using (22.1) we derive the asymptotic formula

$$S_H(x) = \frac{1}{k} L(1, \chi_d) x + O(x^{\frac{1}{2}} |d|^{\frac{1}{4} - \frac{1}{23041.5}}). \tag{22.5}$$

Theorem 2.8 now follows.

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