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Siegel modular forms of small weight

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1 Introduction

Siegel modular forms are of fundamental interest in algebraic geometry and number theory. It is perhaps remarkable that except for degrees less than 4 very little is known about their dimension for small weights. Using the geometry of numbers, Siegel gave an upper bound for the dimension. His technique was refined by Eichler [E] and more recently by Poor and Yuen [PY2]. Concerning the non-existence of cusp forms of small weights, these techniques break down rather quickly.

In this paper, we apply the theory of L-functions to determine the dimension of Siegel modular forms of some small weights for the full Siegel modular group. The use of L-functions to improve upon the geometry of numbers techniques was introduced by Stark [St] and Odlyzko [O] in the 1970s, the latter in order to give lower bounds for the discriminants of number fields. The technique was used by Mestre [Me] in 1986 for giving lower bounds for conductors of algebraic varieties. Recently it has been applied in different ways by Fermigier [Fe] and Miller [Mi] in the context of the spectral theory of $GL(n)$.

It is well known that non-constant modular forms do not exist for weights less than or equal to 1. Christian [C] showed that they do not exist if the weight is 2. In this paper, we show that there are no cusp forms of weight less than or equal to 6 for any degree. Combining this with results of Böcherer, Freitag, Raghavan and Weissauer we are able to determine the dimension of modular forms for weights less than 7. We give less complete results for weights 7 and 8. In particular we give a different proof of the result given in [PY1] that the only cusp form of weight 8, degree 4 is the Schottky form. When the weight is 6, we use deep results of Weissauer on Hecke summation of Eisenstein series. In the process, we also complete the basis theorem for modular forms of weight 4 to all degrees by showing that every such modular form is a multiple of a theta series attached to E_8, the unique even unimodular lattice in 8 dimensions. Previously this wasn't known for degrees 6 and 7 (cf. [B2], [F3]).

It is worth mentioning that our results are unconditional. If one assumes various standard conjectures about the L-functions involved more information can be obtained. It is also possible to apply these techniques to vector valued modular forms transforming by higher degree representations.

2 The standard L-function and the explicit formula

We shall assume that the reader is familiar with the basic properties of Siegel modular forms and refer the reader to [F1] and [Kl].

A Siegel modular form of degree n and weight k of full level 1 is a holomorphic function f defined on $\mathscr{H}_n = \{Z = X + iY \mid Z^t = Z, Y > 0\}$ such that, for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n = Sp(n, \mathbf{Z})$

$$f(MZ) = f((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k f(Z)$$

and f is holomorphic at ∞ if $n = 1$.

We let M_n^k denote the vector space of Siegel modular forms of weight k and degree n and let S_n^k denote the space of Siegel cusp forms of weight k, the kernel of the Siegel's $\mathbf{C}$-linear map $\Phi\colon M_n^k \mapsto M_{n-1}^k$.

Any modular form $f \in M_n^k$ admits a Fourier expansion of the form

$$f(Z) = \sum_{T \geq 0} a(T) \exp(\pi tr TZ)$$

where T runs over positive semi-definite, half-integral $n \times n$ matrices. It is a fact that $f \in S_n^k$ precisely when $a(T) = 0$ for T with $\det T = 0$. On the other hand, a modular form is called *singular* if

$$a(T) \neq 0 \Longrightarrow \det T = 0.$$

It has been shown by Resnikoff [R1, R2] and Freitag [F2, F3] that $f \in M_n^k$ is singular if and only if $2k < n$. In particular there are no cusp forms if $2k < n$.

Classical examples of Siegel modular forms of level 1 are given by theta series attached to even unimodular lattices. For $m \equiv 0 \bmod 8$, let $\mathscr{S}_m$ be the set of all $m \times m$ symmetric positive definite even unimodular matrices. For positive integers m, n, let $H_\nu(m, n)$ be the space of polynomials $P\colon \mathbf{C}^{m,n} \mapsto \mathbf{C}$ so that for all $A \in GL(n, \mathbf{C})$ and $X = (x_{ij}) \in \mathbf{C}^{m,n}$, $P(XA) = (\det A)^\nu P(X)$ and P is harmonic in the sense that $\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_{ij}^2} P(X) = 0$. For $P \in H_\nu(m, n)$ and $S \in \mathscr{S}_m$ the theta function is defined for $Z \in \mathscr{H}_n$ by

$$\theta_{S,P}(Z) = \sum_{A\in \mathbf{Z}^{m,n}} P(S^{1/2}A)e(\frac{1}{2}trS[A]Z)$$

where $e(x) = e^{2\pi ix}$ and $S[A] = A^tSA$.

It is known that (see [F1]) $\theta_{S,P}(Z) \in M_n^{m/2+\nu}$. Let $B_n^k(m)$ be the $\mathbf{C}$-span of theta series $\theta_{S,P}$ with $P \in H_{k-m/2}(m,n)$ and $S \in \mathscr{S}_m$. $B_n^k(m)$ is a Hecke invariant subspace of M_n^k.

As in the classical case $n = 1$ Hecke operators can be defined and S_n^k and M_n^k have a basis consisting of simultaneous eigenforms (see [F1], [A]). For a Siegel cusp form f of weight k and degree n which is a Hecke eigenform, the standard L-function associated to f (see [L], [A], [B1]) is

$$L(s,f,St) = \prod_p \{(1-p^{-s})\prod_{j=1}^{n}(1-\alpha_j(p)p^{-s})(1-\alpha_j(p)^{-1}p^{-s})\}^{-1} \qquad (1)$$

where p runs over all primes and $\alpha_j(p)$, $(1 \le j \le n)$ are the Satake parameters of f.

Let

$$\Lambda(s,f,St) = (2\pi)^{-ns}\pi^{-s/2}\Gamma(\frac{s+\epsilon}{2})\prod_{j=1}^{n}\Gamma(s+k-j)L(s,f,St) \qquad (2)$$

where $\epsilon = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$

Andrianov-Kalinin [AK], Böcherer [B1], and Piatetski-Shapiro, Rallis [PSR] have shown that $\Lambda(s,f,St)$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation $\Lambda(s,f,St) = \Lambda(1-s,f,St)$.

The following theorems about the analytic properties of $\Lambda(s,f)$ can be found in [M].

Theorem 2.1. *Let $f \in S_n^k$ be an eigenform with positive integers k and n. If $k \ge n$, then $\Lambda(s,f,St)$ is holomorphic except for simple poles at $s = 0$ and $s = 1$; it has a pole at $s = 0$ (or equivalently, at $s = 1$) if and only if $f \in B_n^k(2n) \cap S_n^k$.*

Theorem 2.2. *Let $f \in S_n^k$ be an eigenform with $k < n$. Let $\nu = 0$ or 1 according as k is even or odd. Then $\Lambda(s,f,St)$ has at most simple poles at $s = n-k+\nu, n-k+\nu+1, 1-n+k-\nu, k-\nu-n$, at most double poles at $2+k-\nu-n \le s \le n-k+\nu-1$, $s \in \mathbf{Z}$, and is holomorphic elsewhere.*

The location of the poles of $\Lambda(s,f,St)$ in Theorems 2.1 and 2.2 are due to Mizumoto [M] and the second assertion in Theorem 2.1 is due to Weissauer [W1] and Böcherer [B2].

We will need the following bounds for the Satake parameters which are due to Duke, Howe and Li [DHL].

Theorem 2.3. *Let $f \in S_n^k$ be a Hecke eigenform with Satake parameters $\alpha_j(p)$. Then for $n = 2^m$, $|\alpha_j(p)^l + \alpha_j(p)^{-l}| \leq p^{nl/2} + p^{-nl/2}$ and for other n, $|\alpha_j(p)^l + \alpha_j(p)^{-l}| \leq p^{2nl/3} + p^{-2nl/3}$.*

We will also need the following result on the region of absolute convergence.

Theorem 2.4. *Let $f \in S_n^k$ be a Hecke eigenform. Then the Euler product in $\Lambda(s, f, St)$ (eqn.(2)) is absolutely convergent for $Re(s) > n/2 + 1$.*

For $n = 2^m$ this follows from Theorem 2.3. For general n, it follows from a recent paper of Shimura [S].

Finally, we need the explicit formula for the standard L-function. For F a real valued even function on $\mathbf{R}$ with $F(0) = 1$ and satisfying certain growth conditions, one has the following formula for $\Lambda(s, f, St)$ (cf. [Me], p.212).

$$\begin{aligned}\sum_\rho \Phi(\rho) - \sum_\mu \Phi(\mu) + 2I(1/2, \epsilon/2) + 2\sum_{j=1}^{n} I(1, k-j) \\ = (-2n)\log(2\pi) - \log(\pi) - 2\sum_{p,l>1} F(l\log p)(\log p)p^{-l/2} \\ -2\sum_{j=1}^{n}\sum_{p,l>1}(\alpha_j(p)^l + \alpha_j(p)^{-l})F(l\log p)(\log p)p^{-l/2} \qquad (3)\end{aligned}$$

where $I(a,b) = a\int_0^\infty F(ax)\frac{e^{-(a/2+b)x}}{(1-e^{-x})} - \frac{e^{-x}}{x}dx$ and $\Phi(s) = \int_{-\infty}^{\infty} F(x)e^{(s-1/2)x}dx$.

Here ρ (resp. μ) runs over the zeros (resp. poles) of $\Lambda(s, f, St)$ with real parts between $-c$ and $c+1$ where c is such that the Euler product in equation (2) converges absolutely for $Re(s) > c+1$.

3 The results

The vector spaces M_n^k and S_n^k are finite dimensional. For $n \leq 3$ the structure of the graded rings $M_n = \bigoplus_k M_n^k$ and $S_n = \bigoplus_k S_n^k$ are completely understood (see [I], [T]). For $n = 3$, the explicit structure is already quite complicated and for $n \geq 4$ much less is known. Poor and Yuen [PY1] proved the following theorem.

Theorem 3.1. $\dim S_4^6 = 0$; $\dim S_4^8 = 1$; $\dim M_4^8 = 2$; $\dim S_4^{12} = 2$; *and* $\dim M_4^{12} = 6$.

They also gave explicit generators for these spaces of cusp forms and in particular have shown that S_4^8 is generated by the Schottky modular form.

For $n > 4$ very little is known about the dimension of the space of Siegel modular forms of small weight. The main results of this paper are the following theorems.

Theorem 1. *For any n, and for $k \leq 6$,* $\dim S_n^k = 0$.
$\dim S_4^7 = \dim S_6^7 = 0$.
$\dim S_8^8 = 0$.

Theorem 2. $\dim M_n^3 = \dim M_n^5 = 0$.
$\dim M_n^4 = 1$ *and* $M_n^4 =< \theta_{E_8} >_{\mathbf{C}}$ *is the* **C***-linear span of the theta series attached to the unique 8 dimensional even unimodular lattice* E_8.

$$\dim M_n^6 = \begin{cases} 1, & \text{if } n < 9; \\ 0, & \text{otherwise.} \end{cases}$$

Remark: We also give a new proof that $\dim S_4^8 = 1$. If we assume the better bound of Proposition 2.3 holds for all n then we may in addition deduce that $\dim S_n^8 = 0$ for $5 \leq n \leq 8$.

Proof of Theorem 1. To prove Theorem 1 we use the analytic properties of the L-function $\Lambda(s, f, St)$ and the explicit formula to conclude that for small weights such an L-function cannot exist. Roughly speaking, the idea is to apply the explicit formula with a specially chosen test function to deduce an inequality and numerically contradict this inequality to deduce the non-existence of cusp forms.

For the test function $F(x)$ in the explicit formula (equation (3)), we will take the function of Odlyzko.

For $\lambda > 0$, let $F(x) = F_{n,\lambda}(x) = g_\lambda(x)/\cosh(\frac{x(n+1)}{2\lambda})$ where

$$g_\lambda(x) = \begin{cases} (1 - |x|/\lambda)\cos(\pi|x|/\lambda) + (1/\pi)(\sin(\pi|x|/\lambda), & \text{if } |x| \leq \lambda\ ; \\ 0, & \text{otherwise.} \end{cases}$$

Using the region of absolute convergence given in Theorem 2.4 along with this choice of F we have that $\Phi(\rho) \geq 0$ for $-n/2 \leq Re(\rho) \leq n/2+1$, (cf. Poitou [P]) and hence by equation (3),

$$\sum_\mu \Phi(\mu) - 2I(1/2, \epsilon/2) - 2\sum_{j=1}^{n} I(1, k-j) \geq (2n)\log(2\pi) + \log(\pi)$$

$$+2\sum_{p,l>1} F(l\log p)(\log p)p^{-l/2} + 2\sum_{j=1}^{n}\sum_{p,l>1}(\alpha_j(p)^l + \alpha_j(p)^{-l})F(l\log p)(\log p)p^{-l/2}. \tag{4}$$

In particular, if we choose $\lambda = \log 2$, we get

$$\sum_\mu \Phi(\mu) - 2I(1/2, \epsilon/2) - 2\sum_{j=1}^{n} I(1, k-j) \geq (2n)\log(2\pi) + \log(\pi) \tag{5}$$

We can prove this inequality wrong numerically for small weights.

First note that for $n > 2k$ all modular forms are singular hence there are no cusp forms of weight k and degree n in that case and we do not need to check the inequality for $n > 2k$.

More specifically, to show that there are no cusp forms of weight 3 we only need to look at the degrees $n \leq 6$. Since it is already known that (cf. Freitag [F1],p 50) for $k < 5$ and $n \leq 5$, $\dim S_n^k = 0$, we only check the case $n = 6$. If $n = 6$, the left hand side of the inequality (5) is at most 20.3025 where as the right hand side is 23.1993. Hence there are no cusp forms of weight 3 for any degree n.

In this and all other numerical calculations we use Theorem 2.1 and Theorem 2.2 to calculate the contribution of the poles. (The explanation of the numerical methods used are given in the Appendix.)

For $k = 4$, it is known that there are no cusp forms of weight 4 for $n \leq 5$ (cf. [F1]) and we obtain the following numerical values for the remaining cases $6 \leq n \leq 8$.

n	LHS(5)	RHS(5)
6	19.1154	23.1993
7	22.8228	26.8750
8	25.9474	30.5508

Note that in the above table, for each n, the left hand side is less than the right hand side contradicting the inequality (5). Hence there are no cusp forms of weight 4.

For $k = 5$, we only need to check the even degrees $n = 4, 6, 8, 10$ and in this case numerical results are as follows.

n	LHS(5)	RHS(5)
4	13.5709	15.8477
6	21.3004	23.1993
8	28.9227	30.5508
10	35.8420	37.9023

For $k = 6$, there are no cusp forms for $n = 1, 2, 3$ and for $4 \leq n \leq 12$ we use the following numerical results to conclude that there are no cusp forms of weight 6.

n	LHS(5)	RHS(5)
4	14.8223	15.8477
5	17.2036	19.5235
6	19.9490	23.1993
7	25.2466	26.8750
8	28.4122	30.5508
9	32.6461	34.2265
10	36.0132	37.9023
11	39.9510	41.5780
12	43.2640	45.2538

Next we show that there are no cusp forms of weight 7 for $n = 4$ and $n = 6$. To this end, we first obtain the following numerical result.

n	LHS(5)	RHS(5)
4	15.9410	15.8477
6	21.6814	23.1993

Note that for $n = 4$ we cannot obtain a contradiction. To obtain the desired contradiction, we let $\lambda = \log 3$ and use the inequality (4). Here we make use of the bounds for the Satake parameters given in the Theorem 2.3 and obtain

n	LHS(4)	RHS(4)
4	14.6731	14.8398

Hence there are no cusp forms of weight 7 for $n = 4$.

Finally, for $n = 8$ and $k = 8$, LHS(4) = 26.4961 and RHS(4) = 28.4174 and hence $\dim S_8^8 = 0$.

Proof of Theorem 2. Since $\Phi\colon M_n^k \mapsto M_{n-1}^k$ is a linear map with $ker\Phi = S_n^k$,

$$\dim M_n^k \leq \dim S_n^k + \dim M_{n-1}^k. \tag{6}$$

By Theorem 1, for $k \leq 6$, $\dim S_n^k = 0$. This and the inequality (6) implies that for $k \leq 6$, $\dim M_n^k \leq \dim M_1^k$.

For $k = 3, 5$, $\dim M_1^k = 0$ and therefore $\dim M_n^3 = \dim M_n^5 = 0$.

For $k = 4$, $\dim M_1^4 = 1$ and $\dim M_n^4 \leq 1$. Since the theta series θ_{E_8} of degree n attached to the unique even unimodular lattice E_8 of dimension 8 is a non-zero Siegel modular form of weight 4, $\dim M_n^4 = 1$ and $M_n^4 =< \theta_{E_8} >$.

For $k = 6$, $\dim M_1^6 = 1$ and $\dim M_n^6 \leq 1$. Since singular modular forms exist only for weights $k \equiv 0 \bmod 4$ (cf. Freitag [F3]), $\dim M_n^6 = 0$ for $n > 12$.

To see if and when modular forms of weight 6 exist for $n \leq 12$, consider the Eisenstein series

$$E_n^k(Z, s) = \sum_{C,D} \det(CZ + D)^{-k} \frac{\det(ImZ)^s}{|\det(CZ + D)|^{2s}}$$

If $E_n^k(Z, s)$ is regular at $s = 0$ for all Z then $E_n^k(Z, 0) = E_n^k(Z)$ is said to have Hecke summation. Weissauer [W1],[W2] has shown that the method of Hecke summation produces holomorphic modular forms except may be in two irregular cases $k = (n + 2)/2$ and $k = (n + 3)/2$. The defined modular form $E_n^k(Z)$ does not vanish if $k > (n + 3)/2$ (cf. [W2], Proposition 2.2). Since for $n < 9$, E_n^6 is a non-zero holomorphic modular form it follows that $\dim M_n^6 = 1$ for $n < 9$. For $n = 9$, E_9^6 is non-holomorphic and in this case it follows from Theorem 13, (p.123) of [W1] that $\dim M_9^6 = 0$. Since there are no cusp forms of weight 6 for any degree the inequality (6) implies that $\dim M_n^6 = 0$ for $n \geq 9$.

Finally, we give a new proof of the result that $\dim S_4^8 = 1$. When $n = 4$ and $k = 8$ the right hand side of equation (4) is 14.8398 whereas the left hand side *without* the contribution of the poles is 14.4571 and with the contribution of the poles it is 15.9123. These numerical results imply that the L-function of any cusp form of weight 8 degree 4 must have a pole. By second part of Theorem 2.1, this can only happen if the cusp form is in $B_4^8(8)$, i.e. it is a theta series. On the other hand by a result of Weissauer (cf. [W1], p. 20), the dimension of cusp forms in $B_4^8(8)$ is at most one. Since the Schottky form is in $B_4^8(8)$, S_4^8 is one dimensional and spanned by the Schottky's modular form.

4 Appendix

We have used Mathematica in our numerical calculations. One can either use the numerical integration packages of Mathematica or as in [Me] use the series expansion of $e^{ux}/(e^x - 1)$ in terms of the Bernoulli polynomials to rewrite the integral $I(a, b)$ as an infinite series. We will demonstrate this second method with an example. For $n = 4$ and $k = 8$, one of the terms on the left hand side of equation (5) is

$$I(1,4) = \int_0^\infty F(x)\frac{e^{(1/2-4)x}}{(e^x - 1)} - \frac{e^{-x}}{x}dx$$

where $F(x) = F_{4,\lambda}(x) = g_\lambda(x)/\cosh(5x/2\lambda)$.

For $|x| \leq 2\pi$, $\dfrac{e^{ux}}{e^x - 1} = \sum_{m=0}^\infty \dfrac{B_m(u)x^{m-1}}{m!}$, where $B_m(u)$ are the Bernoulli polynomials.

Using this we write $I(1, 4) = J(\lambda) + K(\lambda)$ where

$$J(\lambda) = \int_0^\lambda \frac{F(x) - e^{-x}}{x}dx - \int_\lambda^\infty \frac{e^{-x}}{x}dx$$

and

$$K(\lambda) = \sum_{m=1}^\infty \frac{B_m(-7/2)}{m!}\int_0^\lambda F(x)x^{m-1}dx$$

$J(\lambda)$ can be handled as in [Me], p.231. For $K(\lambda)$, we note that $B_m(u + 1) - B_m(u) = mu^{m-1}$, and $B_m(1/2) = (2^{1-m} - 1)B(m)$ where $B(m)$ are the Bernoulli numbers. Hence $K(\lambda) = \sum_{m=1}^\infty c(m)$ where

$$\begin{aligned} c(m) \quad = \quad & \left(\frac{1}{m!}\right)\Bigg[(2^{1-m} - 1)B(m) \\ & -m(-1)^{m-1}\left(\left(\frac{1}{2}\right)^{m-1} + \left(\frac{3}{2}\right)^{m-1} + \left(\frac{5}{2}\right)^{m-1} + \left(\frac{7}{2}\right)^{m-1}\right)\Bigg] \\ & \times \int_0^\lambda F(x)x^{m-1}dx. \end{aligned}$$

Note that $K(\lambda)$ can be written as a sum of two alternating series and can for example be estimated using the first 10 terms as -0.707514 with an error less than 4×10^{-7}.

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