Math 33B - Midterm 2 Review Sheet

1. Second-order linear inhomogeneous differential equations

   - To solve $y'' + p(t)y' + q(t)y = g(t)$:
     Find two linearly independent solutions y_1 and y_2 of the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$. Then use *undetermined coefficients* or *variation of parameters* (see below) to find one solution y_p of the inhomogeneous equation. The general solution is then $y = C_1 y_1 + C_2 y_2 + y_p$.
   - Method of Undetermined Coefficients: (See Section 4.5)
     Make an educated guess about the form of y_p, with unknown constants $a, b, c, \ldots$ in front of each term. Plug in to the equation and solve for the unknown constants. Guidelines for the educated guess:
     - Include terms from $g(t)$ and all its derivatives.
     - For any term in your guess that is a solution of the homogeneous equation, multiply the whole guess by t.
     - If $g(t)$ has several different terms added together, it's often easier to do this procedure for each term separately, then add the results.

     This method only works for equations with constant coefficients and for which the function $g(t)$ is fairly simple.
   - Method of Variation of Parameters: (See Section 4.6)
     Let W be the Wronskian of y_1 and y_2. Then
     $$y_p = y_1 \int \frac{-y_2 g}{W} + y_2 \int \frac{y_1 g}{W}$$
     This method always works. Note that the differential equation *must* be in the form above. (In particular, the coefficient of y'' must be 1.)

2. Harmonic Motion

   - Unforced: $y'' + 2cy' + \omega_0^2 y = 0$ (See Section 4.4)
     ω_0 = natural frequency, c = damping constant (both positive)
     - Case 1: $c = 0$ (undamped)
       Solution: $y = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A\cos(\omega_0 t - \phi)$
       A = amplitude, ϕ = phase angle
       The solution oscillates forever ("simple harmonic oscillator")
       To find A and ϕ from C_1 and C_2, think of (C_1, C_2) as a point in Cartesian coordinates, and convert to polar coordinates (A, ϕ).
     - Case 2: $c < \omega_0$ (underdamped)
       Solution: $y = Ae^{-ct}\cos(\omega t - \phi)$
       $\omega = \sqrt{\omega_0^2 - c^2} < \omega_0$
       The solution oscillates, but the oscillations decrease to 0 because of the e^{-ct} term.

- Case 3: $c = \omega_0$ (critically damped)
  Solution: $y = (C_1 + C_2 t)e^{-ct}$
  The solution curve may have a "half oscillation" at first, but then goes to 0 quickly.
- Case 4: $c > \omega_0$ (overdamped)
  Solution: $y = C_1 e^{-at} + C_2 e^{-bt}$
  The solution exponentially decays to 0.

- Forced: $y'' + 2cy' + \omega_0^2 y = A\cos(\omega t)$ (See Section 4.7)
  A = amplitude of forcing function
  ω = frequency of forcing function ("driving frequency")
  - Case 1: $c \neq 0$ (damped, the typical case)
    Solution: $y = y_h + B\cos(\omega t - \varphi)$
    y_h is called the *transient response* — it goes to 0 quickly (see cases 2, 3, 4 above). The other part of the solution is called the *steady-state response*, since it keeps oscillating without decaying to 0.
  - Case 2: $c = 0, \omega \neq \omega_0$ (interference, i.e. "beats")
    Solution: $y = C\cos(\omega_0 t - \phi) + B\cos(\omega t)$
    The two oscillators with different frequencies interfere with each other, causing a pattern of "beats": a higher-frequency oscillation whose amplitude oscillates at a lower frequency.
  - Case 3: $c = 0, \omega = \omega_0$ (resonance)
    Solution: $y = C\cos(\omega_0 t - \phi) + Bt\sin(\omega_0 t)$
    The last term of the solution is an oscillator whose amplitude grows linearly!

3. Linear systems of differential equations: (See Sections 9.1 and 9.2)

$$\mathbf{y}' = A\mathbf{y} \qquad \text{A an } n \times n \text{ matrix with constant entries}$$

- General idea: Find n linearly independent solutions $y_1, \ldots, y_n$. The general solution is then $y = C_1 y_1 + \ldots + C_n y_n$.
- To find solutions, find the *eigenvalues* of A by solving $\det(A - \lambda I) = 0$. Then for each eigenvalue λ, find its corresponding *eigenvectors* by solving $(A - \lambda I)\mathbf{v} = 0$.
  - For a distinct eigenvalue λ and its corresponding eigenvector $\mathbf{v}$, form the solution $y = \mathbf{v}e^{\lambda t}$.
  - For a pair of complex conjugate eigenvalues $\lambda = a \pm bi$, pick *one* and compute its eigenvector $\mathbf{v}$, then form the solution $y = \mathbf{v}e^{\lambda t}$ as before, and simplify using Euler's formula. Then the real part of this gives one solution, and the imaginary part gives another.
  - For a repeated eigenvalue λ with two linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, just form the two solutions $y_1 = \mathbf{v}_1 e^{\lambda t}$, $y_2 = \mathbf{v}_2 e^{\lambda t}$.
  - For a repeated eigenvalue λ with only one linearly independent eigenvector $\mathbf{v}$, find a *generalized eigenvector* $\mathbf{w}$ by solving $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then form the solution $y = (\mathbf{w} + \mathbf{v}t)e^{\lambda t}$.