Math 33B - Midterm 1 Review Sheet

1. First-order differential equations

   - Separable equations: $y' = \frac{P(x)}{Q(y)}$
     Write as $Q(y)dy = P(x)dx$ and integrate both sides.
     See Section 2.2, problems 1 - 18
   - Linear equations: $y' + p(x)y = g(x)$
     Make sure equation is in "standard form" first! Multiply whole equation by integrating factor $u(x) = e^{\int p(x)dx}$ and integrate both sides. Left hand side becomes $u(x)y$.
     See Section 2.4, problems 1 - 21
   - Exact differential forms: $P(x, y)dx + Q(x, y)dy = 0$
     Try to find an integrating factor $u(x, y)$ to make the differential form exact. Then find a potential function $f(x, y)$, i.e. a function for which $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. The solution of the differential equation is then $f(x, y) = C$.
     - If the form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
     - The equation is separable if there's some integrating factor that makes P a function of x alone and Q a function of y alone.
     - The equation is linear if, after you divide by $Q(x, y)$, it has the form $(p(x)y + g(x))dx + dy = 0$. The integrating factor is then $u(x) = e^{\int p(x)dx}$.
     - If $h = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of x alone, then let $u(x) = e^{\int h(x)dx}$.
     - If $k = \frac{1}{P}(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$ is a function of y alone, then let $u(y) = e^{\int k(y)dy}$.

     See Section 2.6, problems 9 - 29
   - Qualitative analysis of autonomous equations: $y' = F(y)$
     - Find equilibrium points: set $F(y) = 0$ and solve.
     - Classify each equilibrium as stable/unstable.
     - Sketch solution curves, draw phase line, etc.

     See Section 2.9, problems 15 - 31

2. Second-order linear homogeneous differential equations

   - Given two linearly independent solutions y_1 and y_2, the general solution is $y = C_1y_1 + C_2y_2$.
   - Given one solution y_1, you can find another by *reduction of order*:
     - Let $y_2 = vy_1$ for some function $v(t)$.
     - Take first and second derivatives of y_2 and plug in. The v terms cancel, leaving only v' and v''.
     - Letting $u = v'$ gives a separable first-order equation, which you can solve to find u.

  - Since $u = v'$, you can integrate u to get v, then compute $y_2 = vy_1$.

  See Section 4.1, problems 26 - 30
- Constant coefficients: If the equation has the form $Ay'' + By' + Cy = 0$, solve the *characteristic polynomial* $Ar^2 + Br + C = 0$.
  - Two distinct real roots r_1, r_2: Let $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$.
  - Two complex conjugate roots $a \pm bi$: Let $y_1 = e^{at}\cos(bt)$, $y_2 = e^{at}\sin(bt)$.
  - One repeated real root r: Let $y_1 = e^{rt}$, $y_2 = te^{rt}$.

  See Section 4.3, problems 1 - 36

3. Theoretical considerations

- First-order equations with initial condition: $y' = F(t, y), \quad y(t_0) = y_0$
  - Existence Theorem: If F is continuous, a solution exists!
  - Uniqueness Theorem: If $\frac{\partial F}{\partial y}$ is continuous, the solution is unique!

  See Section 2.7, problems 1 - 10, 21, 22, 23
- Second-order linear equations: $y'' + p(t)y' + q(t)y = g(t)$
  - Existence and Uniqueness Theorem:
    If p, q, and g are continuous, then *one* and *only one* solution exists satisfying the initial conditions $y(t_0) = A$, $y'(t_0) = B$.
  - Linearity property (also called the *principle of superposition*):
    If the equation is *homogeneous*, i.e. $g = 0$, then if y_1 and y_2 are solutions, so is $C_1y_1 + C_2y_2$.

  See Section 4.1

4. Applications of differential equations (word problems)

- Exponential growth/decay: $y' = ky$.
  See Section 2.2
- Newton's Law of Cooling: $T' = -k(T - A)$.
  T = temp of object, A = temp of surroundings ("ambient temp"), k = positive constant
  See Section 2.2, problems 33, 34, 35
- Mixing problems:
  - Identify variables: Usually t = time, y = amount of substance in tank
  - Compute rate of substance flowing into tank
  - Compute rate of substance flowing out of tank. Remember concentration = $\frac{\text{amount of substance}}{\text{volume of solution}}$.
  - Write down differential equation: y' = rate in − rate out.

  See Section 2.5, problems 1 - 11