#6: $\det(A-\lambda I)=\det\begin{pmatrix}-4-\lambda & 0 & 3\\ 3 & -1-\lambda & -3\\ -6 & 0 & 5-\lambda\end{pmatrix}=-(\lambda+1)\big((-4-\lambda)(5-\lambda)+18\big)=-(\lambda+1)(\lambda^2-\lambda-2)=-(\lambda+1)^2(\lambda-2)=0$

$\lambda=-1,-1,2$

$\lambda=2$: $\begin{pmatrix}-6 & 0 & 3\\ 3 & -3 & -3\\ -6 & 0 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad -6x_1+3x_3=0 \quad x_3=2x_1$ $\quad 3x_1-3x_2-3x_3=0 \quad x_2=x_1-x_3=-x_1$

$\vec{V}_1=\begin{pmatrix}1\\-1\\2\end{pmatrix}$

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$\lambda=-1$: $\begin{pmatrix}-3 & 0 & 3\\ 3 & 0 & -3\\ -6 & 0 & 6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad -3x_1+3x_3=0 \quad x_1=x_3$

$\vec{V}_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \vec{V}_3=\begin{pmatrix}1\\0\\1\end{pmatrix}$

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$$Y(t)=\begin{pmatrix}e^{2t} & 0 & e^{-t}\\ -e^{2t} & e^{-t} & 0\\ 2e^{2t} & 0 & e^{-t}\end{pmatrix} \qquad Y(0)^{-1}=\begin{pmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}-1 & 0 & 1\\ -1 & 1 & 1\\ 2 & 0 & -1\end{pmatrix}$$

$$e^{At}=\begin{pmatrix}e^{2t} & 0 & e^{-t}\\ -e^{2t} & e^{-t} & 0\\ 2e^{2t} & 0 & e^{-t}\end{pmatrix}\begin{pmatrix}-1 & 0 & 1\\ -1 & 1 & 1\\ 2 & 0 & -1\end{pmatrix}=\boxed{\begin{pmatrix}-e^{2t}+2e^{-t} & 0 & e^{2t}-e^{-t}\\ e^{2t}-e^{-t} & e^{-t} & -e^{2t}+e^{-t}\\ -2e^{2t}+2e^{-t} & 0 & 2e^{2t}-e^{-t}\end{pmatrix}}$$

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#7: $\det(A-\lambda I)=\det\begin{pmatrix}-2-\lambda & 1 & 0\\ 0 & -2-\lambda & 1\\ 0 & 0 & -2-\lambda\end{pmatrix}=(-2-\lambda)^3=0 \Longrightarrow \boxed{\lambda=-2}$ (algebraic multiplicity 3)

$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Longrightarrow \begin{matrix}x_2=0\\x_3=0\end{matrix}$ $\quad \boxed{\vec{V}_1=\begin{pmatrix}1\\0\\0\end{pmatrix} \text{ eigenvector (level 1)}}$

Solution: $\vec{y}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-2t}$

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$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix} \Longrightarrow \begin{matrix}x_2=1\\x_3=0\end{matrix}$ $\quad \boxed{\vec{V}_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \text{ generalized eigenvector (level 2)}}$

Solution: $\vec{y}_2=\left(\begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\0\end{pmatrix}\right)e^{-2t}=\begin{pmatrix}t\\1\\0\end{pmatrix}e^{-2t}$

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$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \Longrightarrow \begin{matrix}x_2=0\\x_3=1\end{matrix}$ $\quad \boxed{\vec{V}_3=\begin{pmatrix}0\\0\\1\end{pmatrix} \text{ generalized eigenvector (level 3)}}$

Solution: $\vec{y}_3=\left(\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}1\\0\\0\end{pmatrix}\right)e^{-2t}=\begin{pmatrix}\frac{1}{2}t^2\\t\\1\end{pmatrix}e^{-2t}$

$$\boxed{\text{General Solution: } \vec{y}=C_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-2t}+C_2\begin{pmatrix}t\\1\\0\end{pmatrix}e^{-2t}+C_3\begin{pmatrix}\frac{1}{2}t^2\\t\\1\end{pmatrix}e^{-2t}}$$

Note that $e^{At}=\begin{pmatrix}e^{-2t} & te^{-2t} & \frac{1}{2}t^2e^{-2t}\\ 0 & e^{-2t} & te^{-2t}\\ 0 & 0 & e^{-2t}\end{pmatrix}$

#8. $\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3 = 0 \Longrightarrow \boxed{\lambda = 3}$ (algebraic multiplicity 3)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow x_2 = 0 \qquad \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{eigenvectors (level 1)}}$$

Solutions: $\vec{y}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t}, \quad \vec{y}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}$

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The geometric multiplicity of the eigenvalue is 2, so we need 1 more generalized eigenvector:

Try $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$x_2 = 1$, $0 = 0$, $0 = 0$ OK! $\Longrightarrow \boxed{\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{Generalized eigenvector (level 2)}}$ $\qquad$ $x_2 = 0$, $0 = 0$, $0 = 1$ No solution!!!

Solution: $\vec{y}_3 = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{3t} = \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} e^{3t}$

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$$\boxed{\text{General Solution:} \quad \vec{y} = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t} + C_2\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} e^{3t} + C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}}$$

Note that $e^{At} = \begin{pmatrix} e^{3t} & te^{3t} & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$

#9: $\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^4 = 0 \Longrightarrow \boxed{\lambda = 2}$ (algebraic multiplicity 4)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{matrix} x_3 = 0 \\ x_4 = 0 \end{matrix} \qquad \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{eigenvectors (level 1)}}$$

Solutions: $\vec{y}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t}, \ \vec{y}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$

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The geometric multiplicity of the eigenvalue is 2, so we need 2 more generalized eigenvectors:

Try $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$\begin{matrix} 0 = 1 \\ x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{matrix}$ No solution!!! $\qquad$ $\begin{matrix} 0 = 0 \\ x_3 = 1 \\ x_4 = 0 \\ 0 = 0 \end{matrix}$ OK! $\Rightarrow \boxed{\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{Generalized eigenvector (level 2)}}$

Solution: $\vec{y}_3 = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{2t} = \begin{pmatrix} 0 \\ t \\ 1 \\ 0 \end{pmatrix} e^{2t}$

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We still need 1 more generalized eigenvector:

Try $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Longrightarrow \begin{matrix} x_3 = 0 \\ x_4 = 1 \end{matrix}$ $\qquad \boxed{\vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{Generalized eigenvector (level 3)}}$

Solution: $\vec{y}_4 = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{2t} = \begin{pmatrix} 0 \\ \frac{1}{2}t^2 \\ t \\ 1 \end{pmatrix} e^{2t}$

$$\boxed{\text{General Solution:} \quad \vec{y} = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t} + C_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + C_3\begin{pmatrix} 0 \\ t \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_4\begin{pmatrix} 0 \\ \frac{1}{2}t^2 \\ t \\ 1 \end{pmatrix} e^{2t}}$$

Note that $e^{At} = \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{pmatrix}$

#10: $\det(A-\lambda I)=\det\begin{pmatrix}-5-\lambda & 1 & 0 & 0\\ 0 & -5-\lambda & 0 & 0\\ 0 & 0 & -5-\lambda & 1\\ 0 & 0 & 0 & -5-\lambda\end{pmatrix}=(-5-\lambda)^4=0 \Longrightarrow \boxed{\lambda=-5}$ (algebraic multiplicity 4)

$$\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Longrightarrow \begin{matrix}x_2=0\\x_4=0\end{matrix} \quad \boxed{\vec{v}_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \vec{v}_2=\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{eigenvectors (level 1)}}$$

Solutions: $\vec{y}_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-5t},\quad \vec{y}_2=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{-5t}$

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The geometric multiplicity is 2, so we need 2 more generalized eigenvectors:

Try $\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$\begin{matrix}x_2=1\\0=0\\x_4=0\\0=0\end{matrix}$ OK! $\qquad$ $\begin{matrix}x_2=0\\0=0\\x_4=1\\0=0\end{matrix}$ Also OK!

$$\boxed{\vec{v}_3=\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \vec{v}_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad \text{Generalized eigenvectors (level 2)}}$$

Solutions: $\vec{y}_3=\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)e^{-5t}=\begin{pmatrix}t\\1\\0\\0\end{pmatrix}e^{-5t}$ $\qquad$ $\vec{y}_4=\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right)e^{-5t}=\begin{pmatrix}0\\0\\t\\1\end{pmatrix}e^{-5t}$

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$$\boxed{\text{General Solution:} \quad \vec{y}=C_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-5t}+C_2\begin{pmatrix}t\\1\\0\\0\end{pmatrix}e^{-5t}+C_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{-5t}+C_4\begin{pmatrix}0\\0\\t\\1\end{pmatrix}e^{-5t}}$$

Note that $e^{At}=\begin{pmatrix}e^{-5t} & te^{-5t} & 0 & 0\\ 0 & e^{-5t} & 0 & 0\\ 0 & 0 & e^{-5t} & te^{-5t}\\ 0 & 0 & 0 & e^{-5t}\end{pmatrix}$