Math 33B - Final Exam Review Sheet

1. Classification of the Equilibria of $\mathbf{y}' = A\mathbf{y}$
   where A is a 2×2 matrix with trace T and determinant D
   Generic Cases:

   - Real eigenvalues, one positive, one negative ($D < 0$): **Saddle**
   - Distinct real e-values, both negative ($D > 0$, $T^2 - 4D > 0$, $T < 0$): **Nodal Sink**
   - Distinct real e-values, both positive ($D > 0$, $T^2 - 4D > 0$, $T > 0$): **Nodal Source**
   - Complex e-values with negative real part ($T^2 - 4D < 0$, $T < 0$): **Spiral Sink**
   - Complex e-values with positive real part ($T^2 - 4D < 0$, $T > 0$): **Spiral Source**

   Degenerate Cases:

   - Purely imaginary eigenvalues ($D > 0$, $T = 0$): **Center**
   - Negative e-value of multiplicity two ($T^2 - 4D = 0$, $T < 0$): **Degenerate Sink**
     (Like a nodal sink, but a solution may curve significantly as it approaches the origin, like it's "almost" a spiral.)
   - Positive e-value of multiplicity two ($T^2 - 4D = 0$, $T > 0$): **Degenerate Source**
     (Like a nodal source, but a solution may curve significantly as it leaves the origin, like it's "almost" a spiral.)
   - One of the eigenvalues is 0 ($D = 0$): No Catchy Name
     In this case there are infinitely many equilibria, arranged along a line in the plane. If $T < 0$ they are all stable. If $T > 0$, they are all unstable. If $T = 0$ they are neither. All other solutions are straight lines.

2. Qualitative Analysis of Autonomous Nonlinear Systems: $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

   - The x**-nullcline** is the set of points (x, y) where $f(x, y) = 0$, i.e. where the arrows all point straight up or down. The y**-nullcline** is the set of points (x, y) where $g(x, y) = 0$, i.e. where the arrows all point straight left or right.
   - A point (x_0, y_0) is an **equilibrium** if $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$, i.e. if it is on *both* nullclines.
   - The **Jacobian** of the system is the matrix $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$
   - The **linearization** of the system at an equilibrium (x_0, y_0) is the homogeneous linear system $\mathbf{y}' = J(x_0, y_0)\mathbf{y}$.
   - If the linearization at (x_0, y_0) is one of the five generic types (saddle, nodal sink, nodal source, spiral sink, spiral source), then the equilibrium at (x_0, y_0) is of the same type.

3. Fundamental Matrices

   - Definition: A **fundamental matrix** for $A(t)$ is a matrix $Y(t)$ that is invertible and satisfies $Y'(t) = A(t)Y(t)$.
   - To compute a fundamental matrix for $A(t)$, find n linearly independent solutions of the $n \times n$ linear system $\mathbf{y}'(t) = A(t)\mathbf{y}(t)$, and create a matrix with those solutions as its columns.
   - Given a fundamental matrix $Y(t)$ for $A(t)$, if M is any invertible constant matrix, then $Y(t)M$ is also a fundamental matrix for $A(t)$, and furthermore all fundamental matrices for $A(t)$ arise in this way.

4. Variation of Parameters for Inhomogeneous Linear Systems
   To solve $\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{g}(t)$, first find a fundamental matrix $Y(t)$ for $A(t)$. Then a solution is given by
   $$\mathbf{y}(t) = Y(t)\int Y(t)^{-1}\mathbf{g}(t)dt.$$

5. The Matrix Exponential e^{At}

   - Definition: $e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots$
   - e^{At} is the unique fundamental matrix $Y(t)$ for A satisfying $Y(0) = I$. Similarly, $e^{A(t-t_0)}$ is the unique fundamental matrix $Y(t)$ for A satisfying $Y(t_0) = I$.
   - If $Y(t)$ is any fundamental matrix for A, then $e^{At} = Y(t)Y(0)^{-1}$.
   - The solution of the initial value problem $\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{y}_0$ is
   $$\mathbf{y} = e^{A(t-t_0)}\mathbf{y}_0.$$

6. Repeated Eigenvalues/Generalized Eigenvectors
   If you can't find enough solutions to $\mathbf{y}' = A\mathbf{y}$ because of a repeated eigenvalue, you must use generalized eigenvectors:

   - A vector $\mathbf{v}$ is called a **generalized eigenvector** for the eigenvalue λ if $(A - \lambda I)^m\mathbf{v} = \mathbf{0}$ for some m.
   - If $(A - \lambda I)^m\mathbf{v} = \mathbf{0}$, then the following is a solution:
   $$\mathbf{y} = \left(\mathbf{v} + t(A-\lambda I)\mathbf{v} + \frac{t^2}{2}(A-\lambda I)^2\mathbf{v} + \dots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\mathbf{v}\right)e^{\lambda t}$$
   - To find generalized eigenvectors and the corresponding solutions, try to solve the following equations in order:
   $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}$ gives the solution $\mathbf{y} = \mathbf{v}_1 e^{\lambda t}$
   $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$ gives the solution $\mathbf{y} = (\mathbf{v}_2 + t\mathbf{v}_1)e^{\lambda t}$
   $(A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$ gives the solution $\mathbf{y} = \left(\mathbf{v}_3 + t\mathbf{v}_2 + \frac{t^2}{2}\mathbf{v}_1\right)e^{\lambda t}$
   $(A - \lambda I)\mathbf{v}_4 = \mathbf{v}_3$ gives the solution $\mathbf{y} = \left(\mathbf{v}_4 + t\mathbf{v}_3 + \frac{t^2}{2}\mathbf{v}_2 + \frac{t^3}{6}\mathbf{v}_1\right)e^{\lambda t}$
   etc...