Math 33B
Final Exam

Tuesday, December 11, 2007

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Problem	Max	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (10 pts) Find the solution of the initial value problem

$$ty' + 2(t^2 + 1)y = 1, \qquad y(1) = 2.$$

2. (10 pts) A tank initially contains 400 liters of pure water to be used to cool a small nuclear reactor. To keep the water cold, fresh water from two nearby streams is fed in through two pipes: pipe A at 3 L/s, and pipe B at 5 L/s. To keep the volume of water in the tank constant, a drain is opened that allows water to leave at a rate of 8 L/s. If the water coming in through pipe B contains a corrosive chemical at a concentration of 24 g/L, find the quantity of this chemical in the tank at any time t. (Assume that the water in the tank is perfectly mixed before it leaves through the drain.)

3. (10 pts) Consider the autonomous differential equation

$$y' = y(2+y)(1-y)^2 e^y.$$

Find the equilibria, and classify each one as stable or unstable. Also draw the phase line.

4. (10 pts) Find the general solution of the differential equation

$$2y'' + y' + -3y = 5te^t.$$

5. (10 pts) Consider the inhomogeneous linear differential equation

$$t^2y'' - 2(t^2 + 2t)y' + 2(3 + 2t)y = -4t^5 \qquad (t > 0).$$

(a) (2 pts) Show that $y_1 = t^2$ is a solution of the associated homogeneous equation.

(b) (4 pts) Find another solution y_2 of the associated homogeneous equation such that y_1 and y_2 are linearly independent.

(c) (4 pts) Find the general solution of the inhomogeneous equation.

6. (10 pts) Consider the inhomogeneous system of differential equations

$$\mathbf{y}' = \begin{pmatrix} t^{-1} & 1 \\ -1 & t^{-1} \end{pmatrix} \mathbf{y} + \begin{pmatrix} \cos t + t \\ -\sin t \end{pmatrix} \qquad (t > 0).$$

(a) (2 pts) Show that $\mathbf{y}_1 = \begin{pmatrix} t\cos t \\ -t\sin t \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} t\sin t \\ t\cos t \end{pmatrix}$ are solutions of the associated homogeneous system.

(b) (8 pts) Find the general solution of the inhomogeneous system.

7. (10 pts) Compute the general solution of the system of differential equations

$$\mathbf{y}' = \begin{pmatrix} -8 & 3 & 9 \\ 0 & -2 & 0 \\ -6 & 2 & 7 \end{pmatrix} \mathbf{y}.$$

(Hint: To find the characteristic polynomial, it will be easiest to expand along the middle row.)

8. (a) (8 pts) Let $A = \begin{pmatrix} 2 & 6 \\ -3 & -4 \end{pmatrix}$. Compute e^{At}.

(b) (2 pts) Use your answer from part (a) to solve

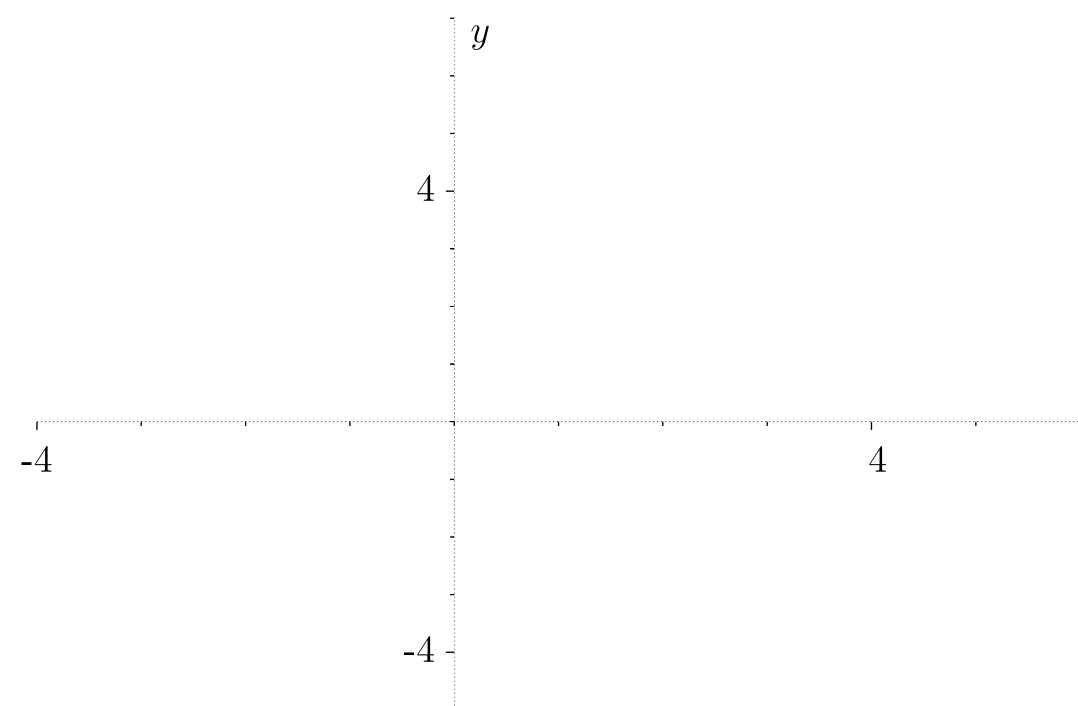
$$\mathbf{y}' = \begin{pmatrix} 2 & 6 \\ -3 & -4 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

Classify the type of equilibrium of this system.

9. (10 pts) For the nonlinear system of differential equations given below, find and plot the x- and y-nullclines, and draw the arrows along them in the correct directions. Compute the equilibria and mark them on your graph, and classify each one (using the Jacobian if necessary). Finally, sketch arrows to fill out the phase portrait. You do not need to sketch the actual solution curves.

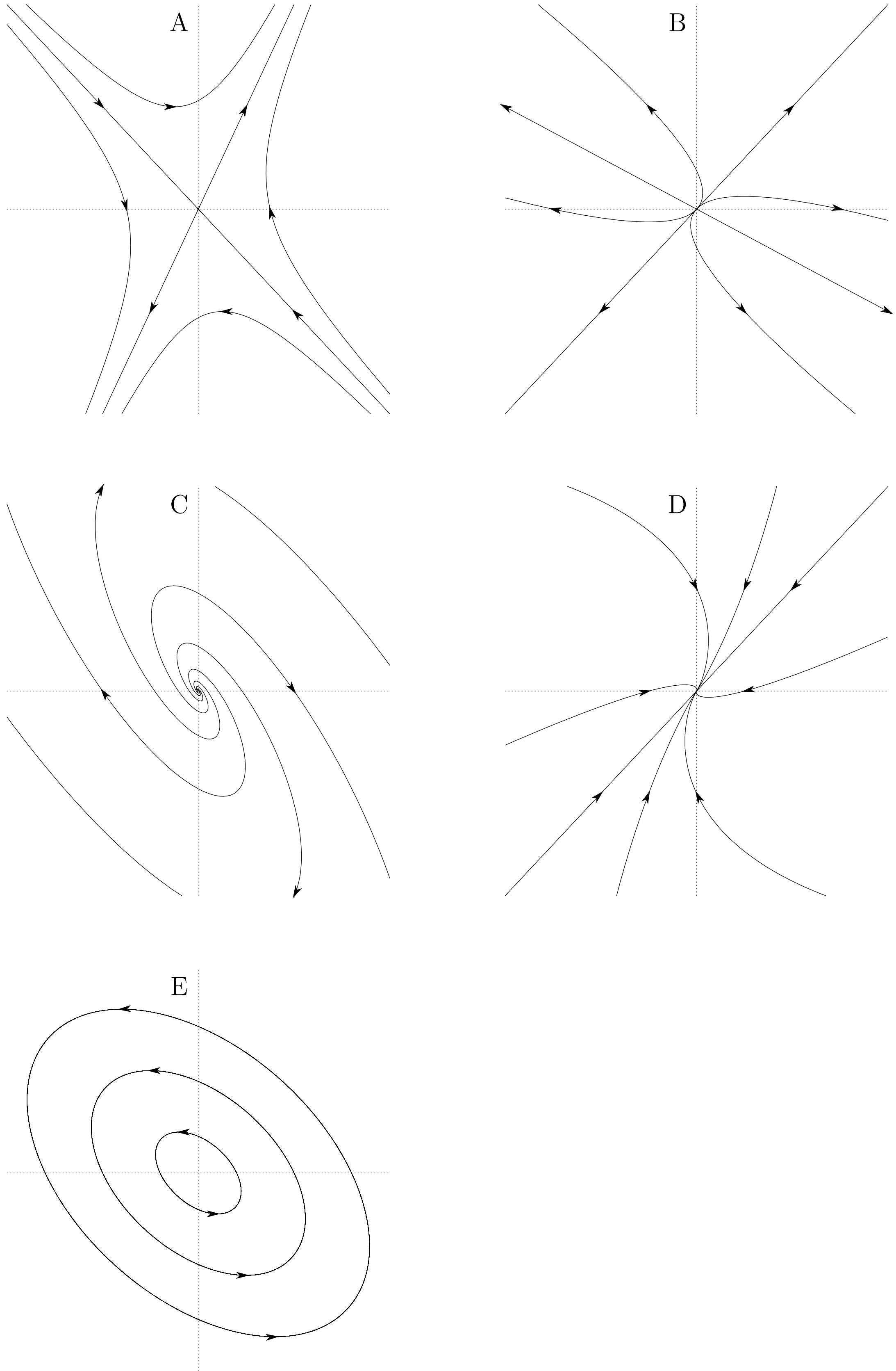
$$\begin{cases} x' = (x^2 - 1)(x^2 - 2x - y - 3) \\ y' = xy - 5x \end{cases}$$

Equilibrium	Type



10. (10 pts) Match the following five linear systems with the phase plane plots on the next page, and classify the equilibrium at the origin for each one.

Linear System	Plot	Type of Equilibrium
$\mathbf{y}' = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \mathbf{y}$	________	________________
$\mathbf{y}' = \begin{pmatrix} -2 & -5 \\ 4 & 2 \end{pmatrix} \mathbf{y}$	________	________________
$\mathbf{y}' = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix} \mathbf{y}$	________	________________
$\mathbf{y}' = \begin{pmatrix} -4 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{y}$	________	________________
$\mathbf{y}' = \begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix} \mathbf{y}$	________	________________


A
B
C
D
E