1. Let $T : V \to W$ and $S : W \to Z$ be linear transformations. Prove that $ST = T_0$ if and only if $R(T) \subseteq N(S)$. (Recall that T_0 denotes the "zero map": $T_0(x) = 0$ for all $x \in V$.)

$(\Longrightarrow)$ Assume $ST = T_0$, so $\forall x \in V$, $S(T(x)) = 0$.

Let $y \in R(T)$. So $y = T(x)$ for some $x \in V$. Thus

$S(y) = S(T(x)) = 0$, so $y \in N(S)$.

Hence $\forall y \in R(T)$, $y \in N(S)$. Thus $R(T) \subseteq N(S)$.

$(\Longleftarrow)$ Assume $R(T) \subseteq N(S)$.

Let $x \in V$. Then $T(x) \in R(T)$, so $T(x) \in N(S)$, so

$S(T(x)) = 0$, i.e. $ST(x) = 0$.

Hence $\forall x \in V$, $ST(x) = 0$. Thus $ST = T_0$.

2. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Prove that T is an isomorphism if and only if $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

($\Rightarrow$) Assume T is an isomorphism. Since T is 1-1 and $\{v_1, \ldots, v_n\}$ is a set of n distinct vectors, $\{T(v_1), \ldots, T(v_n)\}$ is also a set of n distinct vectors. Since $\dim(V) = n$, we will be done if we can show <u>either</u> that $\{T(v_1), \ldots, T(v_n)\}$ is linearly indep. <u>or</u> spans W.
Assume $a_1 T(v_1) + \ldots + a_n T(v_n) = 0$. Then

$$T(a_1 v_1 + \ldots + a_n v_n) = 0, \text{ so } a_1 v_1 + \ldots + a_n v_n \in N(T),$$

and since T is 1-1, $N(T) = \{0\}$. Thus $a_1 v_1 + \ldots + a_n v_n = 0$, so $a_1 = \ldots = a_n = 0$. Therefore $\{T(v_1), \ldots, T(v_n)\}$ is linearly indep, and contains n elements, so is a basis for W.

($\Leftarrow$) Assume $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W. We will show that T is onto. Let $y \in W$. Then for some scalars $a_1, \ldots, a_n$;

$$y = a_1 T(v_1) + \ldots + a_n T(v_n), \text{ so}$$

$$y = T(a_1 v_1 + \ldots + a_n v_n), \text{ so } y \in R(T).$$

Thus $W \subseteq R(T)$. But also obviously $W \supseteq R(T)$, so $R(T) = W$, and thus T is onto.
Now since $\dim(V) = \dim(W) = n$, by a corollary of the Dimension Theorem, T is also 1-1, and hence is an isomorphism.

3. (a) Write down a formula for a linear map $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ such that

$$T(X^2) = (3, 1, -2),$$
$$T(X^2 + X) = (1, -2, 1), \text{ and}$$
$$T(X^2 + X + 1) = (-3, 6, -3).$$

(Your answer should be in the form $T(a + bX + cX^2) = \ldots$.)

Let $a+bX+cX^2 \in P_2(\mathbb{R})$. Assume $a+bX+cX^2 = r(X^2)+s(X^2+X)+t(X^2+X+1)$. Then $t=a$, $s+t=b$, and $r+s+t=c$, so $s=b-a$ and $r=c-a-(b-a)=c-b$.

(In other words, $a+bX+cX^2 = (c-b)(X^2)+(b-a)(X^2+X)+a(X^2+X+1)$.)

Thus
$$\begin{aligned} T(a+bX+cX^2) &= (c-b)T(X^2)+(b-a)T(X^2+X)+aT(X^2+X+1) \\ &= (c-b)(3,1,-2)+(b-a)(1,-2,1)+a(-3,6,-3) \\ &= (3(c-b)+1(b-a)-3a,\ 1(c-b)-2(b-a)+6a,\ -2(c-b)+1(b-a)-3a) \\ &= \boxed{(-4a-2b+3c,\ 8a-3b+c,\ -4a+3b-2c)} \end{aligned}$$

(b) Compute rank(T) and nullity(T). Is T an isomorphism?

One method: $R(T) = \text{span}(\{T(X^2), T(X^2+X), T(X^2+X+1)\})$
$= \text{span}(\{(3,1,-2), (1,-2,1), (-3,6,-3)\})$.

This set generates $R(T)$, but is not lin. indep, because $(-3,6,-3)$ is a multiple of $(1,-2,1)$. But the subset $\{(3,1,-2),(1,-2,1)\}$ is lin. indep, because neither of these vectors is a multiple of the other. Thus $\{(3,1,-2),(1,-2,1)\}$ is a basis for $R(T)$, so rank$(T)=2$. Now by the Dimension Theorem, nullity$(T)=1$ (since $\dim(P_2(\mathbb{R}))=3$).

So T is neither 1-1 nor onto, so T is not an isomorphism.

Another method: Let $a+bX+cX^2 \in N(T)$, so $T(a+bX+cX^2)=(0,0,0)$,

so
$$\begin{cases} -4a-2b+3c=0 \\ 8a-3b+c=0 \\ -4a+3b-2c=0. \end{cases}$$

Solve this to get $b=c=4a$, i.e. solutions are of the form $(a, 4a, 4a)$.

Thus $N(T) = \text{span}\{(1,4,4)\}$, so nullity$(T)=1$.

Now as above, rank$(T)=2$, and T is not an isomorphism.

4. Define $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f) = f(1)\cdot X^2 + f'$. Let $\beta = \{X^2 - X - 3, X^2 + 2X + 1, 3X - 2\}$ and let $\gamma = \{X^2, X, 1\}$.

(a) Compute the matrix $[T]_\gamma$.

$T(X^2) = 1\cdot X^2 + 2X$, so $[T(X^2)]_\gamma = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$T(X) = 1\cdot X^2 + 1$, so $[T(X)]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$T(1) = 1\cdot X^2 + 0$, so $\to [T(1)]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Thus $[T]_\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) Let Q be the change of coordinate matrix that changes β-coordinates to γ-coordinates. Compute Q.

$Q = [1_{P_2(\mathbb{R})}]_\beta^\gamma$.

$[1_V(X^2-X-3)]_\gamma = [X^2-X-3]_\gamma = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$

$[1_V(X^2+2X+1)]_\gamma = [X^2+2X+1]_\gamma = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \implies Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ -3 & 1 & -2 \end{pmatrix}$

$[1_V(3X-2)]_\gamma = [3X-2]_\gamma = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$

(c) Using your answers from parts (a) and (b), write down an expression for the matrix $[T]_\beta$. (You do not need to multiply out the matrices.)

$[T]_\beta = Q^{-1}[T]_\gamma Q$

$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ -3 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ -3 & 1 & -2 \end{pmatrix}$