1. Let $V = P_2(\mathbb{Q})$, the vector space of polynomials of degree at most 2 with rational coefficients, and let $\beta = \{1 - X + X^2, 1 + X^2, X - X^2\}$.

   (a) (6 pts) Prove that β is a basis for V.

   Assume $a(1-X+X^2) + b(1+X^2) + c(X-X^2) = 0.$

   Then $(a+b) + (c-a)X + (a+b-c)X^2 = 0$, so

   $a+b=0$, $c-a=0$, and $a+b-c=0$.

   But then $b=-a$, $c=a$, so $a+b-c = a+(-a)-a = -a = 0$.

   Thus $a=0$, $b=0$, and $c=0$. This shows that β is linearly independent.

   Now since $\dim(V)=3$, and β is a linearly independent set containing 3 elements, β is a basis for V.

   (b) (4 pts) Write $3 - 2X + 5X^2$ as a linear combination of elements of β.

   $$3-2X+5X^2 = a(1-X+X^2)+b(1+X^2)+c(X-X^2)$$
   $$= (a+b)+(c-a)X+(a+b-c)X^2$$

   $$\begin{cases} a+b=3 \\ c-a=-2 \\ a+b-c=5 \end{cases} \Longrightarrow \text{(1) - (3) gives } c=-2$$

   Then (2) gives $a=0$.
   Then (1) gives $b=3$.

   $$\boxed{3-2X+5X^2 = 0(1-X+X^2)+3(1+X^2)-2(X-X^2)}$$

2. (10 pts) Let $V = M_{2\times 2}(\mathbb{R})$, the vector space of all 2×2 matrices with real entries. Let

$$W_1 = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in V \;\middle|\; a, b, c \in \mathbb{R} \right\},$$
$$W_2 = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in V \;\middle|\; b \in \mathbb{R} \right\}.$$

Then W_1 and W_2 are subspaces of V. (You don't need to prove this.) Prove that $V = W_1 \oplus W_2$.

Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W_1 \cap W_2$. Then since $m \in W_2$, $a = 0$ and $d = 0$, and $c = -b$. But since $m \in W_1$, $c = b$, so $b = -b$, so $2b = 0$, so $b = 0$, and thus also $c = 0$. Thus $m = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
Therefore $W_1 \cap W_2 \subseteq \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} = \{0_v\}$. Obviously $\{0_v\} \subseteq W_1 \cap W_2$, so $W_1 \cap W_2 = \{0_v\}$.

Now let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$. We want to show $m \in W_1 + W_2$. If this is true, we must have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix} + \begin{pmatrix} 0 & c' \\ -c' & 0 \end{pmatrix}$ for some $a', b', c', d' \in \mathbb{R}$. Solving this gives $a' = a$, $d' = d$, $b' + c' = b$, and $b' - c' = c$. Thus $b' = \frac{b+c}{2}$ and $c' = \frac{b-c}{2}$.

So let $w_1 = \begin{pmatrix} a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & d \end{pmatrix} \in W_1$ and $w_2 = \begin{pmatrix} 0 & \frac{1}{2}(b-c) \\ -\frac{1}{2}(b-c) & 0 \end{pmatrix} \in W_2$.

Then $m = w_1 + w_2$, so $m \in W_1 + W_2$, and thus $V \subseteq W_1 + W_2$. Clearly $W_1 + W_2 \subseteq V$, so we have $V = W_1 + W_2$.

Now, since $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0_v\}$, we have $V = W_1 \oplus W_2$.

3. (10 pts) Let V be a vector space, and let W_1 and W_2 be subspaces of V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

($\Leftarrow$) Assume that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

If $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$, which is a subspace of V.
If $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$, which is a subspace of V.
Thus, either way, $W_1 \cup W_2$ is a subspace of V.

($\Rightarrow$) Now assume that $W_1 \cup W_2$ is a subspace of V.
Assume that $W_1 \not\subseteq W_2$. In other words, it is not true that $\forall x \in W_1$, $x \in W_2$. Thus $\exists x \in W_1$ such that $x \notin W_2$.

We want to show that $W_2 \subseteq W_1$. So let $y \in W_2$.
Since $x \in W_1$ and $y \in W_2$, both x and y are in $W_1 \cup W_2$. Since $W_1 \cup W_2$ is a subspace of V (and hence is closed under addition), $x + y \in W_1 \cup W_2$. Thus either $x+y \in W_1$ or $x+y \in W_2$.

Assume first that $x+y \in W_2$. Then since $y \in W_2$ also, $-y \in W_2$, and $(x+y)+(-y) \in W_2$, so $x \in W_2$.
But this contradicts our assumption about x, so we must have $x+y \notin W_2$.

Therefore we must have $x+y \in W_1$. But since $x \in W_1$ also, we have $-x \in W_1$, and thus $(-x)+(x+y) \in W_1$, so $y \in W_1$. Thus $W_2 \subseteq W_1$.

We have proved that if $W_1 \not\subseteq W_2$, then $W_2 \subseteq W_1$.
In other words, either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

4. (10 pts) Let V be a vector space over $\mathbb{C}$ of dimension n. Recall (from a homework exercise) that V is also then automatically a vector space over $\mathbb{R}$. Prove that, as a vector space over $\mathbb{R}$, the dimension of V is $2n$.

Let $\{u_1, \dots, u_n\}$ be a basis of V (as a vector space over $\mathbb{C}$). Let $\beta = \{u_1, \dots, u_n, iu_1, \dots, iu_n\}$. We will show that β is a basis for V as a vector space over $\mathbb{R}$.

Let $v \in V$. Then $v = c_1 u_1 + \dots + c_n u_n$ for some $c_1, \dots, c_n \in \mathbb{C}$. For each k, we can write $c_k = a_k + ib_k$ for some $a_k, b_k \in \mathbb{R}$. Thus $v = (a_1 + ib_1)u_1 + \dots + (a_n + ib_n)u_n = a_1 u_1 + \dots + a_n u_n + b_1 iu_1 + \dots + b_n iu_n$, so $v \in \operatorname{span}(\beta)$.

Therefore $V \subseteq \operatorname{span}(\beta)$. Obviously $\operatorname{span}(\beta) \subseteq V$, so $V = \operatorname{span}(\beta)$.

Now suppose $a_1 u_1 + \dots + a_n u_n + b_1 iu_1 + \dots + b_n iu_n = 0$ for some $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$. Then

$$(a_1 + b_1 i)u_1 + \dots + (a_n + b_n i)u_n = 0.$$

Letting $c_k = a_k + ib_k$ for each k, this becomes

$$c_1 u_1 + \dots + c_n u_n = 0,$$ where the c_k's are in $\mathbb{C}$.

Since $\{u_1, \dots, u_n\}$ is linearly independent <u>over $\mathbb{C}$</u>, we must have $c_k = 0$ for each k. Thus $a_k + ib_k = 0$ in $\mathbb{C}$ for each k, so $a_k = 0$ and $b_k = 0$ for each k.

Thus β is linearly independent over $\mathbb{R}$.

Therefore β is a basis for V, as a vector space over $\mathbb{R}$. Since β contains $2n$ elements, the dimension of V as a vector space over $\mathbb{R}$ is $2n$.