Math 115A
Homework 8
Due Friday, November 19, 2010

Do the following problems from the book:

- Section 5.1: 3(a,b), 10, 11, 15, 16

Do the following additional problems:

1. Let V be a finite-dimensional vector space over a field F, and let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V. Let $Q \in M_{n\times n}(F)$ be invertible, and for each j $(1 \le j \le n)$ define
$$w_j = \sum_{i=1}^{n} Q_{ij} v_i.$$
Let $\gamma = \{w_1, \ldots, w_n\}$.
   (a) Prove that γ is a basis for V.
   (b) Prove that the change of coordinate matrix that changes γ-coordinates to β-coordinates is Q.

2. Let V be a finite-dimensional vector space over a field F, let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V, and let $T : V \to V$ be linear. Recall (from a corollary to Theorem 2.23) that if γ is another basis for V, then $[T]_\gamma$ is similar to $[T]_\beta$. Prove the following converse of this fact: If A is an $n \times n$ matrix that is similar to $[T]_\beta$, then there exists a basis γ for V such that $A = [T]_\gamma$. *(Hint: Use the previous problem.)*

3. (Problem 6 from Section 4.4) Let k and m be positive integers, and let $n = k + m$. Consider $n \times n$ matrices of the form
$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$
where A is a $k \times k$ matrix, D is an $m \times m$ matrix, and B is a $k \times m$ matrix, or in other words,
$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & b_{11} & b_{12} & \cdots & b_{1m} \\ a_{21} & a_{22} & \cdots & a_{2k} & b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & b_{k1} & b_{k2} & \cdots & b_{km} \\ 0 & 0 & \cdots & 0 & d_{11} & d_{12} & \cdots & d_{1m} \\ 0 & 0 & \cdots & 0 & d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & d_{m1} & d_{m2} & \cdots & d_{mm} \end{pmatrix}.$$
Prove that for any such matrix M, $\det(M) = \det(A)\det(D)$. *(Hint: Prove this by induction on k, by expanding along the first column. It may be helpful to use the notation $\widetilde{M_{ij}}$ (and $\widetilde{A_{ij}}$), which I introduced in class on 11/10 (see p. 232 in the book).)*