Do the following problems from the book:

- Section 2.5: 2(a,b,c), 3(a,c,e), 4, 5, 8, 11
- Section 4.4: 1, 4(a,b,c)

Recall that for a function $f : X \rightarrow Y$, we say that $g : Y \rightarrow X$ is an inverse of $f$ if both $f \circ g = 1_Y$ and $g \circ f = 1_X$. We also know that a function has an inverse if and only if it is both one-to-one and onto, and if it has an inverse, this inverse is unique. We will now weaken this definition and explore the consequences. Let $X$ and $Y$ be any sets, and let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be any functions. We will say that $g$ is a right inverse of $f$ if $f \circ g = 1_Y$, and $g$ is a left inverse of $f$ if $g \circ f = 1_X$. Thus an inverse of $f$ is merely a function $g$ that is both a right inverse and a left inverse simultaneously.

1. Prove that $f$ has a left inverse if and only if $f$ is injective (one-to-one). (One direction of this is easy; the other is slightly tricky.)

2. Prove that if $f$ has a right inverse, then $f$ is surjective (onto).

3. (Problem 10(c) from Section 2.4) Let $V$ and $W$ be finite-dimensional vector spaces with $\dim(V) = \dim(W)$, and let $T : V \rightarrow W$ be linear.

   (a) Prove that if $S : W \rightarrow V$ is a right inverse of $T$, then $T$ is invertible and $S$ is the inverse of $T$.
   (b) Prove that if $S : W \rightarrow V$ is a left inverse of $T$, then $T$ is invertible and $S$ is the inverse of $T$.

4. (a) Give an example of a linear transformation $T : V \rightarrow W$ that has a right inverse, but does not have a left inverse.
   (b) For the function $T$ you chose in part (a), give two different linear transformations $S_1$ and $S_2$ that are right inverses of $T$. This shows that, in general, right inverses are not unique.

5. (a) Give an example of a linear transformation $T : V \rightarrow W$ that has a left inverse, but does not have a right inverse.
   (b) For the function $T$ you chose in part (a), give two different linear transformations $S_1$ and $S_2$ that are left inverses of $T$. This shows that, in general, left inverses are not unique.

*The converse of this statement is also true, but the proof involves (in a somewhat subtle way) a set-theoretic concept called the Axiom of Choice.
Just like above, we can also define left and right inverses for matrices. (In what follows, for any positive integer \( n \), \( I_n \) will denote the \( n \times n \) identity matrix.) Recall that for a matrix \( A \in M_{m\times n}(F) \), an inverse of \( A \) is a matrix \( B \in M_{n\times m}(F) \) such that both \( AB = I_m \) and \( BA = I_n \). Now if \( A \in M_{m\times n}(F) \) and \( B \in M_{n\times m}(F) \), we will say that \( B \) is a right inverse of \( A \) if \( AB = I_m \), and \( B \) is a left inverse of \( A \) if \( BA = I_n \). Thus an inverse of \( A \) is merely a matrix \( B \) that is both a right inverse and a left inverse simultaneously. (Note that, contrary to the definition of inverse matrices that you may be used to, this appears to allow us to deal with inverses of \( m \times n \) matrices when \( m \neq n \), i.e., non-square matrices. However, you will prove in the following problems that, even though this definition appears to allow that possibility, it can never happen. In other words, only a square matrix can have an inverse. You will also prove that, for a square matrix, a left or right inverse is automatically an inverse.)

6. (a) Let \( A \in M_{m\times n}(F) \) with \( m < n \). Show that \( A \) cannot have a left inverse. \( \text{Hint: Consider} \ L_A, \text{and apply problem 1 above.} \)

(b) Give an example of a matrix \( A \in M_{m\times n}({\mathbb R}) \) with \( m < n \) such that \( A \) has a right inverse.

(c) For the matrix \( A \) you chose in part (b), give two different matrices \( B_1, B_2 \in M_{n\times m}(F) \) that are right inverses of \( A \). This shows that, in general, right inverses are not unique.

7. (a) Let \( A \in M_{m\times n}(F) \) with \( m > n \). Show that \( A \) cannot have a right inverse. \( \text{Hint: Consider} \ L_A, \text{and apply problem 2 above.} \)

(b) Give an example of a matrix \( A \in M_{m\times n}({\mathbb R}) \) with \( m > n \) such that \( A \) has a left inverse.

(c) For the matrix \( A \) you chose in part (b), give two different matrices \( B_1, B_2 \in M_{n\times m}(F) \) that are left inverses of \( A \). This shows that, in general, left inverses are not unique.

8. (Problem 10(a/b) from Section 2.4) Let \( A \in M_{n\times n}(F) \).

(a) Prove that if \( A \) has a left inverse \( B \), then \( A \) is invertible and \( B = A^{-1} \). \( \text{Hint: Consider} \ L_A, \text{and apply problem 3 above.} \)

(b) Prove that if \( A \) has a right inverse \( B \), then \( A \) is invertible and \( B = A^{-1} \).

9. (Problem 9 from Section 2.4)

(a) Let \( A, B \in M_{n\times n}(F) \), and assume \( AB \) is invertible. Prove \( \text{using the previous problem} \) that both \( A \) and \( B \) are invertible.

(b) Give an example of non-square (and hence non-invertible) matrices \( A \) and \( B \) such that \( AB \) is invertible. \( \text{Hint: You can just reuse one of your examples from problem 6 or 7 above.} \)

10. Define a linear transformation \( T : P_2({\mathbb R}) \to P_2({\mathbb R}) \) by \( T(f) = f' + f(1)X^2 \). Compute \( \det(T) \).