Math 115A
Homework 7
Due Friday, November 12, 2010

Do the following problems from the book:

- Section 2.5: 2(a,b,c), 3(a,c,e), 4, 5, 8, 11
- Section 4.4: 1, 4(a,b,c)

Recall that for a function $f : X \to Y$, we say that $g : Y \to X$ is an **inverse** of f if both $f \circ g = 1_Y$ and $g \circ f = 1_X$. We also know that a function has an inverse if and only if it is both one-to-one and onto, and if it has an inverse, this inverse is unique. We will now weaken this definition and explore the consequences. Let X and Y be any sets, and let $f : X \to Y$ and $g : Y \to X$ be any functions. We will say that g is a **right inverse** of f if $f \circ g = 1_Y$, and g is a **left inverse** of f if $g \circ f = 1_X$. Thus an inverse of f is merely a function g that is both a right inverse and a left inverse simultaneously.

1. Prove that f has a left inverse if and only if f is injective (one-to-one). (One direction of this is easy; the other is slightly tricky.)

2. Prove that if f has a right inverse, then f is surjective (onto).*

3. (Problem 10(c) from Section 2.4) Let V and W be finite-dimensional vector spaces with $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear.

   (a) Prove that if $S : W \to V$ is a right inverse of T, then T is invertible and S is the inverse of T.
   (b) Prove that if $S : W \to V$ is a left inverse of T, then T is invertible and S is the inverse of T.

4. (a) Give an example of a linear transformation $T : V \to W$ that has a right inverse, but does not have a left inverse.
   (b) For the function T you chose in part (a), give two *different* linear transformations S_1 and S_2 that are right inverses of T. This shows that, in general, right inverses are *not* unique.

5. (a) Give an example of a linear transformation $T : V \to W$ that has a left inverse, but does not have a right inverse.
   (b) For the function T you chose in part (a), give two *different* linear transformations S_1 and S_2 that are left inverses of T. This shows that, in general, left inverses are *not* unique.

*The converse of this statement is also true, but the proof involves (in a somewhat subtle way) a set-theoretic concept called the Axiom of Choice.

Just like above, we can also define left and right inverses for matrices. (In what follows, for any positive integer n, I_n will denote the $n \times n$ identity matrix.) Recall that for a matrix $A \in M_{m\times n}(F)$, an inverse of A is a matrix $B \in M_{n\times m}(F)$ such that both $AB = I_m$ and $BA = I_n$. Now if $A \in M_{m\times n}(F)$ and $B \in M_{n\times m}(F)$, we will say that B is a **right inverse** of A if $AB = I_m$, and B is a **left inverse** of A if $BA = I_n$. Thus an inverse of A is merely a matrix B that is both a right inverse and a left inverse simultaneously. (Note that, contrary to the definition of inverse matrices that you may be used to, this appears to allow us to deal with inverses of $m \times n$ matrices when $m \neq n$, i.e., non-square matrices. However, you will prove in the following problems that, even though this definition appears to allow that possibility, it can never happen. In other words, only a square matrix can have an inverse. You will also prove that, for a square matrix, a left or right inverse is automatically an inverse.)

6. (a) Let $A \in M_{m\times n}(F)$ with $m < n$. Show that A cannot have a left inverse. *(Hint: Consider L_A, and apply problem 1 above.)*
   (b) Give an example of a matrix $A \in M_{m\times n}(\mathbb{R})$ with $m < n$ such that A has a right inverse.
   (c) For the matrix A you chose in part (b), give two *different* matrices $B_1, B_2 \in M_{n\times m}(F)$ that are right inverses of A. This shows that, in general, right inverses are *not* unique.

7. (a) Let $A \in M_{m\times n}(F)$ with $m > n$. Show that A cannot have a right inverse. *(Hint: Consider L_A, and apply problem 2 above.)*
   (b) Give an example of a matrix $A \in M_{m\times n}(\mathbb{R})$ with $m > n$ such that A has a left inverse.
   (c) For the matrix A you chose in part (b), give two *different* matrices $B_1, B_2 \in M_{n\times m}(F)$ that are left inverses of A. This shows that, in general, left inverses are *not* unique.

8. (Problem 10(a/b) from Section 2.4) Let $A \in M_{n\times n}(F)$.

   (a) Prove that if A has a left inverse B, then A is invertible and $B = A^{-1}$. *(Hint: Consider L_A, and apply problem 3 above.)*
   (b) Prove that if A has a right inverse B, then A is invertible and $B = A^{-1}$.

9. (Problem 9 from Section 2.4)

   (a) Let $A, B \in M_{n\times n}(F)$, and assume AB is invertible. Prove *(using the previous problem)* that both A and B are invertible.
   (b) Give an example of non-square (and hence non-invertible) matrices A and B such that AB is invertible. *(Hint: You can just reuse one of your examples from problem 6 or 7 above.)*

10. Define a linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(f) = f' + f(1)X^2$. Compute $\det(T)$.