Math 115A
Homework 5
Due Friday, October 29, 2010

- Do the following problems from the book:
  Section 2.1: 11, 17, 27*, 28, 30, 32, 35
  ~~Section 2.2: 2(a,c,e,g), 5(d,f,g), 11, 12, 16~~† *(Postponed to homework 6.)*

- Do the following additional problems:

  1. Prove the following theorem, which *should* be stated in class on Wednesday, 10/27:

     Theorem (Theorem 2.10, more general version)**.** *Let V, W, and Z be vector spaces (over the same field F).*

     (a) For $T_1, T_2 \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, Z)$, $S(T_1 + T_2) = ST_1 + ST_2$.
     Likewise, for $T \in \mathcal{L}(V, W)$ and $S_1, S_2 \in \mathcal{L}(W, Z)$, $(S_1 + S_2)T = S_1T + S_2T$.

     (b) For $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, Z)$ and $a \in F$, $a(ST) = (aS)T = S(aT)$.

  2. Let V be a vector space, and let W_1 and W_2 be subspaces such that $V = W_1 \oplus W_2$. Let $T : V \to V$ be the projection on W_1 along W_2.

     (a) Prove that T is a linear transformation.

     (b) Prove that $W_1 = \{x \in V \mid T(x) = x\}$.

     (c) Prove that $R(T) = W_1$ and $N(T) = W_2$. (Thus $V = R(T) \oplus N(T)$.)

     (d) Prove that $T^2 = T$. (That is, prove that $T(T(x)) = T(x)$ for all $x \in V$.) *(Hint: For $x \in V$, let $y = T(x)$. Then $y \in R(T) = W_1$ by part (c). Now apply part (b).)*

  3. Let V be a vector space, and let $T : V \to V$ be a linear operator on V such that $T^2 = T$. (Note: Do not assume in this problem that V is finite-dimensional.)

     (a) Prove that $V = R(T) + N(T)$. *(Hint: For $x \in V$, write $x = T(x) + (x - T(x))$, and show that $x - T(x) \in N(T)$.)*

     (b) Prove that $R(T) \cap N(T) = \{0\}$. Conclude that $V = R(T) \oplus N(T)$.

     (c) Prove that T is the projection on $R(T)$ along $N(T)$.

  Note that, as a result of the last two problems, you have proved the following theorem:

  Theorem. *Let V be a vector space, and let T be a linear operator on V. Then T is a projection if and only $T^2 = T$.*

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*Hint for #27: Refer to problem 34 from Section 1.6. (This was part of homework 3.)

†Hint for #16: Start with a basis $\{u_1, \ldots, u_n\}$ for $N(T)$ (where $n = \text{nullity}(T)$), and extend this to a basis $\{u_1, \ldots, u_n, v_1, \ldots, v_r\}$ for V (where $r = \text{rank}(T)$). Show that $\{T(v_1), \ldots, T(v_r)\}$ is linearly independent, and extend this to a basis for W. Be careful to order the two bases appropriately in order to make the matrix diagonal.