Math 115A
Homework 3
Due Friday, October 15, 2010

- Do the following problems from the book:
  Section 1.6: 5, 7, 14, 21, 23, 25, 26, 29, 33, 34

- Do the following four additional problems:

  1. Let V be a vector space (over any field), and let L be a maximal linearly independent subset of V. (That is, assume that L is a linearly independent set, but for any $v \in V \setminus L$, the set $L \cup \{v\}$ is linearly dependent.) Prove that L is a basis for V. *(Hint: Use Theorem 1.7.)*
  2. Let V be a vector space, and let β be a basis for V. Prove that β is a maximal linearly independent subset of V. That is, show that for any $v \in V \setminus \beta$, the set $\beta \cup \{v\}$ is linearly dependent. *(Hint: Use Theorem 1.7 again.)*
  3. Let V be a vector space, and let G be a minimal generating set for V. (That is, assume that G generates V, but for any $v \in G$, the set $G \setminus \{v\}$ does not generate V.) Prove that G is a basis for V. *(Hint: Prove this by contradiction, and use the fact (which I proved in class) that a set S is linearly dependent iff there exists $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$.)*
  4. Let V be a vector space, and let β be a basis for V. Prove that β is a minimal generating set for V. That is, show that for any $v \in \beta$, the set $\beta \setminus \{v\}$ does not generate V. *(Same hint as for the previous problem.)*

  Note that, as a result of these last four problems, you have proved the following theorem:

  Theorem. *Let V be a vector space, and let $\beta \subseteq V$. The following are equivalent:*

  1. *β is a basis for V.*
  2. *β is a maximal linearly independent subset of V.*
  3. *β is a minimal generating set for V.*