Math 115A
Homework 2
Due Friday, October 8, 2010

1. (Sec. 1.3, #10) Prove that $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1 + a_2 + \cdots + a_n = 1\}$ is not a subspace of F^n.

2. (Sec. 1.3, #11) Let n be a positive integer and let F be a field, and recall that $P(F)$ denotes the vector space of all polynomials with coefficients in F. Is the set $W = \{f \in P(F) \mid f = 0 \text{ or } \deg(f) = n\}$ a subspace of $P(F)$?

3. (Sec. 1.3, #13) Let X be a nonempty set and let F be a field, and recall that $\mathcal{F}(X, F)$ denotes the vector space of all functions $f : X \to F$. Let $x_0 \in X$. Let $W = \{f \in \mathcal{F}(X, F) \mid f(x_0) = 0\}$. Prove that W is a subspace of $\mathcal{F}(X, F)$.

4. (Sec. 1.3, #17) Let V be a vector space over a field F, and let $W \subseteq V$. Prove that W is a subspace of V if and only if W is nonempty, closed under addition, and closed under scalar multiplication.

5. (Sec. 1.3, #22) Let F and K be fields. A function $g \in \mathcal{F}(K, F)$ is called an **even** function if $g(-x) = g(x)$ for all $x \in K$, and is called an **odd** function if $g(-x) = -g(x)$ for all $x \in K$. Let W_1 be the set of all odd functions in $\mathcal{F}(K, F)$, and W_2 be the set of all even functions in $\mathcal{F}(K, F)$. Prove that both W_1 and W_2 are subspaces of $\mathcal{F}(K, F)$.

Several of the remaining exercises (and many that you will encounter later in the course) make use of the following definitions and notation.

Definition. Let V be a vector space, and let X and Y be nonempty subsets of V. Then the **sum** of X and Y, denoted $X + Y$, is the set $\{x + y \in V \mid x \in X \text{ and } y \in Y\}$.

Definition. Let V be a vector space, and let W_1 and W_2 be subspaces of V. We say that V is the **direct sum** of W_1 and W_2 if (1) $V = W_1 + W_2$ and (2) $W_1 \cap W_2 = \{0\}$. If V is the direct sum of W_1 and W_2, we write $V = W_1 \oplus W_2$.

6. (Sec. 1.3, #23) Let V be a vector space, and let W_1 and W_2 be subspaces of V.
   (a) Prove that $W_1 + W_2$ is a subspace of V.
   (b) Let X be a subspace of V such that $W_1 \subseteq X$ and $W_2 \subseteq X$. Prove that $W_1 + W_2 \subseteq X$.

7. (Sec. 1.3, #25) Let F be a field. Let W_1 be the subset of $P(F)$ consisting of polynomials p of the form

$$p(X) = a_1 X + a_3 X^3 + a_5 X^5 + \cdots + a_n X^n$$

for some odd integer n, and let W_2 be the subset of V consisting of polynomials p of the form

$$p(X) = a_0 + a_2X^2 + a_4X^4 + \cdots + a_nX^n$$

for some even integer n.

(a) Prove that W_1 and W_2 are subspaces of $P(F)$.
(b) Prove that $P(F) = W_1 \oplus W_2$.

8. (Not in book) Let V be a vector space, and let W_1 and W_2 be subspaces of V. Prove that $V = W_1 \oplus W_2$ if and only if for all $v \in V$ there exist *unique* $w_1 \in W_1$ and *unique* $w_2 \in W_2$ such that $v = w_1 + w_2$.

9. (Not in book) Let K be any field, and let $V = \mathcal{F}(K, \mathbb{R})$. As in problem 5 above, let W_1 be the subspace of all odd functions in V, and W_2 be the subspace of all even functions in V. Prove that $V = W_1 \oplus W_2$.

10. (Sec. 1.4, #6) Let F be any field in which $1 + 1 \neq 0$. Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ generate F^3.

11. (Sec. 1.4, #12) Let V be a vector space, and let $W \subseteq V$. Prove that W is a subspace of V if and only if $\text{span}(W) = W$.

12. (Sec. 1.4, #13) Let V be a vector space, and let S_1 and S_2 be subsets of V such that $S_1 \subseteq S_2$. Prove that $\text{span}(S_1) \subseteq \text{span}(S_2)$. Deduce that if $\text{span}(S_1) = V$ then $\text{span}(S_2) = V$.

13. (Sec. 1.4, #14) Let V be a vector space, and let S_1 and S_2 be subsets of V. Prove that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

14. (Sec. 1.4, #15) Let V be a vector space, and let S_1 and S_2 be subsets of V.

    (a) Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.
    (b) Give an example in which $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$.
    (c) Give an example in which $\text{span}(S_1 \cap S_2) \subsetneq \text{span}(S_1) \cap \text{span}(S_2)$.

15. (Sec. 1.5, #9) Let V be a vector space, and let u and v be distinct vectors in V. Prove that the set $\{u, v\}$ is linearly dependent if and only if one of u or v is a scalar multiple of the other.

16. (Sec. 1.5, #10) Give an example of a set of three vectors in $\mathbb{R}^3$ that is linearly dependent, but none of the three vectors is a scalar multiple of either of the other ones.

17. (Sec. 1.5, #14) Let V be a vector space, and let $S \subseteq V$. Prove that S is linearly dependent if and only if either $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \ldots, u_n \in S$ such that v is equal to a linear combination of $u_1, u_2, \ldots, u_n$.

18. (Sec. 1.5, #16) Let V be a vector space, and let $S \subseteq V$. Prove that S is linearly independent if and only if every finite subset of S is linearly independent.