Math 115A
Final Exam

Friday, December 10, 2010

Name: ____________________

Student ID: ____________________

Signature: ____________________

Problem	Max	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	20	
8	10	
Total	90	

1. (10 pts) Prove that the set $\{3X^2 + X + 1, X^2 - 2X - 2, 2X^2 - 3\}$ is a basis of $P_2(\mathbb{R})$.

2. (10 pts) Let $T : V \to W$ and $S : W \to Z$ be linear transformations. Prove that ST is one-to-one if and only if T is one-to-one and $R(T) \cap N(S) = \{0\}$.

3. (10 pts) Let $T : V \to W$ be a linear transformation. Prove that T is one-to-one if and only if, for any linearly independent subset $\{v_1, \ldots, v_n\}$ of V, the set $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent in W.
(Hint: For the ($\Longleftarrow$) direction, try proof by contrapositive/contradiction.)

4. (10 pts) Let $T : \mathbb{R}^3 \to P_3(\mathbb{R})$ be a linear transformation. Let

$$\begin{aligned} \beta &= \{(1,0,0),(1,1,0),(1,1,1)\} \quad \text{and} \\ \gamma &= \{2X^3+X-1, X^3-2X^2, 3X^2+5, X^2-X+3\}. \end{aligned}$$

Suppose that

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 3 \\ 1 & -1 & -2 \\ 3 & 0 & -4 \end{pmatrix}.$$

Compute $T(3,1,2)$.

5. (10 pts) Define a linear operator on $P_3(\mathbb{R})$ by

$$T(f) = f(-2)X^3 + f'.$$

Compute $\det(T)$.

6. (10 pts) Let V be a finite-dimensional vector space, and let T be an invertible linear operator on V.

   (a) (5 pts) If x is a nonzero vector in V and λ a nonzero scalar, prove that λ is an eigenvalue of T corresponding to the eigenvector x if and only if λ^{-1} is an eigenvalue of T^{-1} corresponding to the eigenvector x.

   (b) (5 pts) Use part (a) to prove that if T is diagonalizable, so is T^{-1}.

7. (20 pts) Let $\langle \cdot, \cdot \rangle$ be the standard inner product (the dot product) on $\mathbb{R}^3$. Let $y = (2, 0, 1)$. Define a linear operator T on $\mathbb{R}^3$ by

$$T(x) = \langle x, y \rangle y + 2x \quad \text{for any } x \in \mathbb{R}^3.$$

(a) (5 pts) Let γ be the standard ordered basis for $\mathbb{R}^3$. Compute $[T]_\gamma$.

(b) (6 pts) Find the eigenvalues of T. State the algebraic multiplicity of each eigenvalue.

(c) (6 pts) For each eigenvalue λ of T, find a basis of the eigenspace E_λ of T, and state the geometric multiplicity of λ.

(d) (3 pts) Is T diagonalizable? If so, write down a basis β of $\mathbb{R}^3$ such that $[T]_\beta$ is diagonal, and write down $[T]_\beta$. If T is not diagonalizable, explain why not.

8. (10 pts) Let V be an inner product space. Let S be a subset of V, and let

$$W = \{v \in V \mid \langle v, x \rangle = 0 \quad \forall x \in S\}.$$

(In other words, W is the set of all vectors $v \in V$ that are perpendicular to *every* vector in S.)

(a) (6 pts) Prove that W is a subspace of V.

(b) (4 pts) Prove that if S is a subspace of V, then $S \cap W = \{0\}$.