1. (10 pts) Prove that the set $\{3X^2 + X + 1, X^2 - 2X - 2, 2X^2 - 3\}$ is a basis of $P_2(\mathbb{R})$.

Assume $a(3X^2+X+1)+b(X^2-2X-2)+c(2X^2-3)=0.$

WTS $a=b=c=0.$

$$(3a+b+2c)X^2+(a-2b)X+(a-2b-3c)=0$$

so $\begin{cases} 3a+b+2c=0 \\ a-2b=0 \\ a-2b-3c=0 \end{cases}$

Subtracting the last two gives $3c=0$, so $c=0.$

Now $3a+b=0$. and
$a-2b=0.$

Multiplying the first of these by 2 and adding to the second gives $7a=0$, so $a=0.$ Plugging this into the first equation gives $b=0.$

Therefore the set is linearly independent.

Now since $\dim(P_2(\mathbb{R}))=3$, and we have a linearly independent set with exactly 3 vectors, it is a basis. $\square$

2. (10 pts) Let $T : V \to W$ and $S : W \to Z$ be linear transformations. Prove that ST is one-to-one if and only if T is one-to-one and $R(T) \cap N(S) = \{0\}$.

($\Longrightarrow$) Assume ST is one-to-one.

If $T(x_1) = T(x_2)$ for some $x_1, x_2 \in V$, then $S(T(x_1)) = S(T(x_2))$, so $ST(x_1) = ST(x_2)$, so $x_1 = x_2$ since ST is one-to-one.

Thus T is one-to-one.

Now let $y \in R(T) \cap N(S)$. Since $y \in R(T)$, $y = T(x)$ for some $x \in V$. But since $y \in N(S)$, $S(y) = 0$, so $S(T(x)) = 0$, so $ST(x) = 0$. Thus $x \in N(ST)$, but $N(ST) = \{0\}$ since ST is one-to-one. So $x = 0$, and thus $y = T(x) = T(0) = 0$.

Thus $R(T) \cap N(S) \subseteq \{0\}$. Obviously $R(T) \cap N(S) \supseteq \{0\}$, since both $R(T)$ and $N(S)$ are subspaces of W.

So $R(T) \cap N(S) = \{0\}$.

($\Longleftarrow$) Assume that T is one-to-one and $R(T) \cap N(S) = \{0\}$.

Let $x \in N(ST)$. Then $ST(x) = 0$, so $S(T(x)) = 0$.

Let $y = T(x)$. So $S(y) = S(T(x)) = 0$, so $y \in N(S)$.

But since $y = T(x)$, $y \in R(T)$ also. Thus $y \in R(T) \cap N(S)$.

Therefore $y = 0$.

Now we have $T(x) = 0$, so $x \in N(T)$, but since T is one-to-one, $N(T) = \{0\}$, so $x = 0$.

Thus $N(ST) \subseteq \{0\}$. Obviously $N(ST) \supseteq \{0\}$ since $N(ST)$ is a subspace of V. So $N(ST) = \{0\}$, and therefore ST is one-to-one.

$\square$

3. (10 pts) Let $T : V \to W$ be a linear transformation. Prove that T is one-to-one if and only if, for any linearly independent subset $\{v_1, \ldots, v_n\}$ of V, the set $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent in W.
(Hint: For the ($\Longleftarrow$) direction, try proof by contrapositive/contradiction.)

($\Longrightarrow$) Assume T is one-to-one.
Let $\{v_1, \ldots, v_n\}$ be a linearly independent subset of V.
Suppose $\sum_{i=1}^{n} a_i T(v_i) = 0$ for some scalars $a_1, \ldots, a_n$.
Since T is linear, $T(\sum_{i=1}^{n} a_i v_i) = 0$, so $\sum_{i=1}^{n} a_i v_i \in N(T)$.
But since T is one-to-one, $N(T) = \{0\}$, so $\sum_{i=1}^{n} a_i v_i = 0$.
But now since $\{v_1, \ldots, v_n\}$ is linearly independent, $a_1 = a_2 = \ldots = a_n = 0$.
Thus $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent.
We have proved that if $\{v_1, \ldots, v_n\}$ is any lin. indep. subset of V, then $\{T(v_1), \ldots, T(v_n)\}$ is a lin. indep. subset of W.

($\Longleftarrow$) Assume T is <u>not</u> one-to-one.
Then $N(T) \neq \{0\}$. Choose some $x \in N(T)$ with $x \neq 0$.
Then the set $\{x\}$ <u>is</u> linearly independent in V, but the set $\{T(x)\}$ is just $\{0\}$, which is <u>not</u> linearly independent. (Any set containing 0 is linearly dependent.)
Therefore it is <u>not</u> true that for any lin. indep. set $\{v_1, \ldots, v_n\}$ of V, $\{T(v_1), \ldots, T(v_n)\}$ is lin. indep. in W.
Now by contrapositive, if the above statement <u>is</u> true, then T must be one-to-one.

$\square$

4. (10 pts) Let $T : \mathbb{R}^3 \to P_3(\mathbb{R})$ be a linear transformation. Let

$$\beta = \{(1,0,0),(1,1,0),(1,1,1)\} \quad \text{and}$$
$$\gamma = \{2X^3+X-1, X^3-2X^2, 3X^2+5, X^2-X+3\}.$$

Suppose that

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 3 \\ 1 & -1 & -2 \\ 3 & 0 & -4 \end{pmatrix}.$$

Compute $T(3,1,2)$.

Let $x = (3,1,2)$.

$(3,1,2) = a_1(1,0,0) + a_2(1,1,0) + a_3(1,1,1) = (a_1+a_2+a_3, a_2+a_3, a_3)$

$$\begin{cases} a_1+a_2+a_3 = 3 \\ a_2+a_3 = 1 \\ a_3 = 2 \end{cases} \Longrightarrow \begin{matrix} a_1 = 2 \\ a_2 = -1 \\ a_3 = 2 \end{matrix}$$

So $[x]_\beta = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 3 \\ 1 & -1 & -2 \\ 3 & 0 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \\ -2 \end{pmatrix}$$

So $T(x) = 3(2X^3+X-1) + 2(X^3-2X^2) - 1(3X^2+5) - 2(X^2-X+3)$

$= \boxed{8X^3 - 9X^2 + 5X - 14}$

5. (10 pts) Define a linear operator on $P_3(\mathbb{R})$ by

$$T(f) = f(-2)X^3 + f'.$$

Compute $\det(T)$.

Let $\beta = \{1, X, X^2, X^3\}$. We'll first compute $[T]_\beta$.

$T(1) = X^3$ $\quad [T(1)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$T(X) = -2X^3 + 1$ $\quad [T(X)]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}$

$T(X^2) = 4X^3 + 2X$ $\quad [T(X^2)]_\beta = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}$

$T(X^3) = -8X^3 + 3X^2$ $\quad [T(X^3)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -8 \end{pmatrix}$

So $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & -2 & 4 & -8 \end{pmatrix}$.

To compute $\det(T) = \det([T]_\beta)$, we expand along the first column, to get

$$\det(T) = 0 \cdot \det(\cdots) - 0 \cdot \det(\cdots) + 0 \cdot \det(\cdots) - 1 \cdot \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
$$= -1 \cdot 6 = \boxed{-6}$$

6. (10 pts) Let V be a finite-dimensional vector space, and let T be an invertible linear operator on V.

(a) (5 pts) If x is a nonzero vector in V and λ a nonzero scalar, prove that λ is an eigenvalue of T corresponding to the eigenvector x if and only if λ^{-1} is an eigenvalue of T^{-1} corresponding to the eigenvector x.

λ is an eigenvalue of T corresponding to the eigenvector x

$\Leftrightarrow T(x) = \lambda x$

$\Leftrightarrow T^{-1}(T(x)) = T^{-1}(\lambda x)$

$\Leftrightarrow x = T^{-1}(\lambda x) = \lambda T^{-1}(x)$

$\Leftrightarrow \lambda^{-1} x = T^{-1}(x)$

$\Leftrightarrow \lambda^{-1}$ is an eigenvalue of T^{-1} corresponding to the eigenvector x.

$\square$

(b) (5 pts) Use part (a) to prove that if T is diagonalizable, so is T^{-1}.

Assume T is diagonalizable. By a theorem from class (Theorem 5.1) there exists a basis $\beta = \{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of T. By part (a), any eigenvector of T is also an eigenvector of T^{-1}. Thus β is a basis of V consisting of eigenvectors of T^{-1}. So by the same theorem mentioned above, T^{-1} is diagonalizable. $\square$

7. (20 pts) Let $\langle \cdot, \cdot \rangle$ be the standard inner product (the dot product) on $\mathbb{R}^3$. Let $y = (2, 0, 1)$. Define a linear operator T on $\mathbb{R}^3$ by

$$T(x) = \langle x, y \rangle y + 2x \quad \text{for any } x \in \mathbb{R}^3.$$

(a) (5 pts) Let γ be the standard ordered basis for $\mathbb{R}^3$. Compute $[T]_\gamma$.

$\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = \langle (1,0,0), (2,0,1) \rangle (2,0,1) + 2(1,0,0) = (4,0,2) + (2,0,0) = (6,0,2)$

$[T(1,0,0)]_\gamma = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}$

$T(0,1,0) = \langle (0,1,0), (2,0,1) \rangle (2,0,1) + 2(0,1,0) = (0,2,0)$

$[T(0,1,0)]_\gamma = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$T(0,0,1) = \langle (0,0,1), (2,0,1) \rangle (2,0,1) + 2(0,0,1) = (2,0,3)$

$[T(0,0,1)]_\gamma = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$

$$[T]_\gamma = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

(b) (6 pts) Find the eigenvalues of T. State the algebraic multiplicity of each eigenvalue.

$$\det \begin{pmatrix} 6-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{pmatrix} = (2-\lambda) \det \begin{pmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)\big((6-\lambda)(3-\lambda) - 4\big)$$

$$= (2-\lambda)(\lambda^2 - 9\lambda + 14)$$

$$= (2-\lambda)(\lambda-2)(\lambda-7)$$

$$= -(\lambda-2)^2(\lambda-7)$$

Eigenvalues are 2 (with alg. multiplicity 2)
and 7 (with alg. multiplicity 1)

(c) (6 pts) For each eigenvalue λ of T, find a basis of the eigenspace E_λ of T, and state the geometric multiplicity of λ.

$\lambda = 2$: $\begin{pmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow x_3 = -2x_1$

Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for example.

$\boxed{\begin{array}{c}\text{Basis for } E_2: \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \\ \text{Geometric multiplicity } 2\end{array}}$

$\lambda = 7$: $\begin{pmatrix} -1 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{array}{l} x_1 = 2x_3 \\ x_2 = 0 \end{array}$

Eigenvector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ for example

$\boxed{\begin{array}{c}\text{Basis for } E_7: \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\} \\ \text{Geometric multiplicity } 1\end{array}}$

(d) (3 pts) Is T diagonalizable? If so, write down a basis β of $\mathbb{R}^3$ such that $[T]_\beta$ is diagonal, and write down $[T]_\beta$. If T is not diagonalizable, explain why not.

Yes.

If $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ then $[T]_\beta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

8. (10 pts) Let V be an inner product space. Let S be a subset of V, and let

$$W = \{v \in V \mid \langle v, x\rangle = 0 \quad \forall x \in S\}.$$

(In other words, W is the set of all vectors $v \in V$ that are perpendicular to *every* vector in S.)

(a) (6 pts) Prove that W is a subspace of V.

1. $\langle 0, x\rangle = 0 \ \forall x \in S$, so $0 \in W$
2. Let $v_1, v_2 \in W$, so $\langle v_1, x\rangle = 0$ and $\langle v_2, x\rangle = 0 \ \forall x \in S$. Then $\langle v_1 + v_2, x\rangle = \langle v_1, x\rangle + \langle v_2, x\rangle = 0 + 0 = 0 \ \forall x \in S$, so $v_1 + v_2 \in W$. Thus W is closed under addition.
3. Let $v \in W$ and let a be a scalar. Since $v \in W$, $\langle v, x\rangle = 0 \ \forall x \in S$, so $\langle av, x\rangle = a\langle v, x\rangle = a \cdot 0 = 0 \ \forall x \in S$. Thus $av \in W$. Hence W is closed under scalar multiplication.

Since W contains 0 and is closed under addition and scalar multiplication, W is a subspace of V. $\square$

(b) (4 pts) Prove that if S is a subspace of V, then $S \cap W = \{0\}$.

Assume S is a subspace of V. Then obviously $0 \in S$ and $0 \in W$ (by part a), so $0 \in S \cap W$. Thus $S \cap W \supseteq \{0\}$.

Now let $v \in S \cap W$. Since $v \in W$, $\langle v, x\rangle = 0 \ \forall x \in S$. But since $v \in S$ as well, this means in particular that $\langle v, v\rangle = 0$, so $\|v\|^2 = 0$, so $\|v\| = 0$, so $v = 0$.

Thus $S \cap W \subseteq \{0\}$, so $S \cap W = \{0\}$. $\square$