Chapter 2

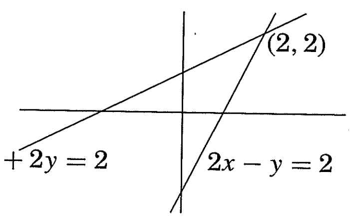
Language and Proofs

Understanding mathematical reasoning requires familiarity with the precise meaning of words like "every", "some", "not", "and", "or", etc.; these arise often in analyzing mathematical problems. Relevant aspects of language include word order, quantifiers, logical statements, and logical symbols. With these, we can discuss elementary techniques of proof.

TWO THEOREMS ABOUT EQUATIONS

We begin with two problems that illustrate both the need for careful use of language and the variety of techniques in proofs.

2.1. Definition. A **linear equation** in two variables x and y is an equation $ax + by = r$, where the coefficients a, b and the constant r are real numbers. A **line** in $\mathbb{R}^2$ is the set of pairs (x, y) satisfying a linear equation whose coefficients a and b are not both 0.



Geometric intuition suggests three possibilities for a pair of linear equations in two variables. If each equation describes a line, then the lines may intersect in one point, may be parallel, or may be identical. The equations then have one, none, or infinitely many common solutions, respectively. We can analyze this without relying on geometric intuition, because we have defined "line" using only arithmetic of real numbers.

2.2. Theorem. Let $ax + by = r$ and $cx + dy = s$ be linear equations in two variables x and y. If $ad - bc \neq 0$, then there is a unique common solution. If $ad - bc = 0$, then there is no common solution or there are infinitely many, depending on the values of r and s.

Proof: If all four coefficients are zero, then there is no solution unless $r = s = 0$, in which case all pairs (x, y) are solutions. Otherwise, at least one coefficient is nonzero. By interchanging the equations and/or interchanging the roles of x and y, we may assume that $d \neq 0$. We can now solve the second equation for y, obtaining $y = (s - cx)/d$. By substituting this expression for y into the first equation and simplifying, we obtain $(a - \frac{bc}{d})x + \frac{bs}{d} = r$. Multiplying by d yields $(ad - bc)x + bs = rd$.

When $ad - bc \neq 0$, we may divide by $ad - bc$ to obtain $x = \frac{rd-bs}{ad-bc}$. Substituting this into the equation for y yields the unique solution

$$(x, y) = \left(\frac{rd - bs}{ad - bc}, \frac{as - rc}{ad - bc}\right).$$

When $ad - bc = 0$, the equation for x becomes $bs = rd$. If $bs \neq rd$, then there is no solution. If $bs = rd$, then for each x we obtain the solution $(x, y) = (x, (s - cx)/d)$; here there are infinitely many solutions. ■

When $ad - bc \neq 0$, the equations define lines with one common point. When $ad - bc = 0$ and both equations describe lines, there may be no solution (parallel lines) or infinitely many solutions (the lines coincide). An equation does not describe a line if both its coefficients are 0; here there is no solution unless the equation is $0x + 0y = 0$, in which case the common solutions are the solutions to the other equation in the pair.

In the proof, avoiding division by 0 leads us to consider cases. No single solution formula holds for all pairs of linear equations; the form of the solution changes when $ad - bc = 0$. The solution statement itself requires careful attention to language.

Our next argument uses the fundamental method of *proof by contradiction*; we suppose that the desired conclusion is false and then derive a contradiction from this hypothesis. The method is particularly useful for proving statements of nonexistence. Here we combine the method of proof by contradiction with an understanding of rational numbers and several elementary observations about odd and even numbers.

2.3. Theorem. If a, b, c are odd integers, then $ax^2 + bx + c = 0$ has no solution in the set of rational numbers.

Proof: Suppose that there *is* a rational solution x. We write this as p/q for integers p, q. We may assume that p/q expresses x "in lowest terms", meaning that p and q have no common integer factor larger than 1. From $ax^2 + bx + c = 0$ we obtain $ap^2 + bpq + cq^2 = 0$ after multiplying by q^2.

We obtain a contradiction by showing that $ap^2 + bpq + cq^2$ cannot equal 0. We do this by proving the stronger statement that it is odd.

Because we expressed x as a rational number in lowest terms, p and q cannot both be even. If both are odd, then the three terms in the sum are all odd, since the product of odd numbers is odd. Since the sum of three odd numbers is odd, we have the desired contradiction in this case. If p is odd and q is even (or vice versa), then we have the sum of two even numbers and an odd number, which again is odd. In each case, the assumption of a rational solution leads to a contradiction. ■

QUANTIFIERS AND LOGICAL STATEMENTS

Understanding a subject and writing clearly about it go together. We next discuss the use of well-chosen words and symbols to express mathematical ideas precisely. The language of mathematical statements will become familiar as we use it in later chapters to solve problems.

Using proof by contradiction requires understanding what it means for a statement to be false. Consider the sentence "Every classroom has a chair that is not broken". Without using words of negation, can we write a sentence with the opposite meaning? This will be easy once we learn how logical operations are expressed in English.

2.4. Example. *Negation of simple sentences.* What is the negation of "All students are male"? Some would say, incorrectly, "All students are not male". The correct negation is "At least one student is not male". Similarly, the negation of "all integers are odd" is *not* "all integers are not odd"; the correct negation is "at least one integer is even". ■

Common English permits ambiguities; the listener can obtain the intended meaning from context. Mathematics must avoid ambiguities.

2.5. Example. *Word order and context.* Consider the sentence "There is a real number y such that $x = y^3$ for every real number x". This seems to say that some number y is the cube root of all numbers, which is false. To say that every number has a cube root, we write "For every real number x, there is a real number y such that $x = y^3$".

In both English and mathematics, meaning depends on word order. Compare "Mary made Jane eat the food", "Eat, Mary; Jane made the food", and "Eat the food Mary Jane made". Meaning can also depend on context, as in "The bartender served two aces". This may have different meanings, depending on whether we are watching tennis or relaxing in a bar on an airbase. Mathematics can present similar difficulties; words such as "square" and "cycle" have several mathematical meanings. ■

The fundamental issue in mathematics is whether mathematical statements are true or false. Before discussing proofs, we must agree on what to accept as mathematical statements. We first require correct grammar for both words and mathematical symbols. Grammar eliminates both "food Mary Jane" and "1+ =".

The sentences "$1 + 1 = 3$" and "$1 + 1 < 3$" are mathematical statements, even though the first is false. Similarly, "$(1+1)^{4\cdot 3}$ is 96 more than 4000" is acceptable. We accept grammtically correct assertions where performing the indicated computations determines truth or falsity. This computational criterion extends to more complicated operations and to objects defined using sets and numbers.

We also consider general assertions about many numbers or objects, such as "the square of each odd integer is one more than a multiple of 8". This statement is closely related to the list of statements "$1^2 = 1 + 0 \cdot 8$", "$3^2 = 1 + 1 \cdot 8$", "$5^2 = 1 + 3 \cdot 8$", $\cdots$. We can describe many related mathematical statements by introducing a **variable**. If $P(x)$ is a mathematical statement when the variable x takes a specific value in a set S, then we accept as mathematical statements the sentences below. They have different meanings when S has more than one element.

"For all x in S, the assertion $P(x)$ is true."
"There exists an x in S such that the assertion $P(x)$ is true."

2.6. Example. The sentence "$x^2 - 1 = 0$" by itself is not a mathematical statement, but it becomes one when we specify a value for x. Consider

"For all $x \in \{1, -1\}$, $x^2 - 1 = 0$."
"For all $x \in \{1, 0\}$, $x^2 - 1 = 0$."
"There exists $x \in \{1, 0\}$ such that $x^2 - 1 = 0$."

All three are mathematical statements. The first is true; there are two values of x to check, and each satisfies the conclusion. The second statement is false, and the third is true. ∎

If it is not possible to assign "True" or "False" to an assertion, then it is not a mathematical statement. Consider the sentence "This statement is false"; call it P. If the words "this statement" in P refer to another sentence Q, then P has a truth value. If "this sentence" refers to P itself, then P must be false if it is true, and true if it is false! In this case, P has no truth value and is not a mathematical statement.

2.7. Definition. We use uppercase "$P, Q, R \cdots$" to denote mathematical statements. The truth or falsity of a statement is its **truth value**. Negating a statement reverses its truth value. We use $\neg$ to indicate **negation**, so "$\neg P$" means "not P". If P is false, then $\neg P$ is true.

In the statement "For all x in S, $P(x)$ is true", the variable x is **universally quantified**. We write this as $(\forall x \in S)P(x)$ and say that $\forall$ is a **universal quantifier**. In "There exists an x in S such that $P(x)$ is true", the variable x is **existentially quantified**. We write this as $(\exists x \in S)P(x)$ and say that $\exists$ is an **existential quantifier**. The set of allowed values for a variable is its **universe**. ∎

2.8. Remark. *English words that express quantification.* Typically, "every" and "for all" represent universal quantifiers, while "some" and "there is" represent existential quantifiers. We can also express universal quantification by referring to an arbitrary element of the universe, as in "Let x be an integer," or "A student failing the exam will fail the course". Below we list common indicators of quantification.

Universal ($\forall$)	(helpers)	Existential ($\exists$)	(helpers)
for [all], for every		for some	
if	then	there exists	such that
whenever, for, given		at least one	for which
every, any	satisfies	some	satisfies
a, arbitrary	must, is	has a	such that
let	be		

The "helpers" may be absent. Consider "The square of a real number is nonnegative." This means $x^2 \geq 0$ for *every* $x \in \mathbb{R}$; it is not a statement about one real number and cannot be verified by an example. ∎

In conversation, a quantifier may appear after the expression it quantifies. "I drink whenever I eat" differs from "Whenever I eat, I drink" only in what is emphasized. Similarly, we easily understand "The AGM Inequality states that $(a+b)/2 \geq \sqrt{ab}$ for every pair a, b of positive real numbers" and "The value of $x^2 - 1$ is 0 for some x between 0 and 2". These quantifiers appear at the end for smoother reading. Error is unlikely in sentences with only one quantifier, but the order of quantification matters when there is more than one.

2.9. Remark. *Order of quantifiers.* We adopt a convention to avoid ambiguity. Consider "If n is even, then n is the sum of two odd numbers". Letting E and O be the sets of even and odd integers, and letting $P(n, x, y)$ be "$n = x + y$", the sentence becomes

$$(\forall n \in E)(\exists x, y \in O)P(n, x, y).$$

In this format, the value chosen for a quantified variable remains unchanged for later expressions but can be chosen in terms of variables quantified earlier. When we reach $(\exists x, y \in O)P(n, x, y)$, we treat "$n$" as a constant, already chosen. We use the same convention when writing mathematics in English: quantifiers appear in order at the beginning of the sentence so that the value of each variable is chosen independently of subsequently quantified variables. ∎

2.10. Example. *Parameters and implicit quantifiers.* Consider the exercise "Let a and b be real numbers. Prove that the equation $ax^2 + bx = a$ has a real solution." Using quantifiers, this becomes $(\forall a, b \in \mathbb{R})(\exists x \in \mathbb{R})(ax^2 + bx = a)$. In solving the problem, we treat a and b as *parameters*. Although these are variables and we must find a solution for each choice of these variables, the scope of the quantification is that we treat a and b as constants when we study x.

We find a suitable x in terms of a and b. When $a = 0$, $x = 0$ works for all b. When $a \neq 0$, the quadratic formula tells us that $x = (-b+\sqrt{b^2+4a^2})/2a$ works. This is real (since positive real numbers have square roots), and it satisfies the equation.

The negative square root also yields a solution. We do not need it, because the statement asked only for the existence of a solution. ■

2.11. Example. *Order of quantifiers.* Compare the statements below.

$$(\forall x \in A)(\exists y \in B)P(x, y) \qquad (\exists y \in B)(\forall x \in A)P(x, y)$$

Regardless of the meanings of A, B, P, the second statement always implies the first. The first statement is true if for each x we can pick a y that "works". For the second statement to be true, there must be a single y that will always work, no matter which x is chosen.

Simple examples clarify the distinction. Let A be the set of children, let B be the set of parents, and let $P(x, y)$ be "y is the parent of x". The first statement is true, but the second statement is too strong and is not true. Another example occurred in Example 2.5, with $A = B = \mathbb{R}$, and with $P(x, y)$ being "$x = y^3$". Consider also the statement in Remark 2.9.

Sometimes both statements are true. For example, let $A = B = \mathbb{R}$, and let $P(x, y)$ be "$xy = 0$". ■

2.12. Remark. *Negation of quantified statements.* After placing a statement involving quantifiers in the conventional order, negating the statement is easy. If it is false that $P(x)$ is true for every value of x, then there must be some value of x such that $P(x)$ is false, and vice versa. Similarly, if it is false that $P(x)$ is true for some value of x, then $P(x)$ is false for every value of x. Thus in notation,

$\neg[(\forall x)P(x)]$ has the same meaning as $(\exists x)(\neg P(x))$.
$\neg[(\exists x)P(x)]$ has the same meaning as $(\forall x)(\neg P(x))$.

Note that when using logical symbols, we may add matched parentheses or brackets to clarify grouping. ■

Understanding negation of quantified statements by passing the negation through the quantifier and changing the type of quantifier is imperative for understanding the mathematics in this book.

When negating quantified statements with specified universes, one must not change the universe of potential values. Also, when negating $(\forall x)P(x)$ or $(\exists x)P(x)$, it may be that $P(x)$ itself is a quantified statement.

2.13. Example. *Negation involving universes.* The negation of "Every Good Boy Does Fine" (a mnemonic for reading music) is "some good boy does not do fine"; it says nothing about bad boys. The negation of "Every chair in this room is broken" is "Some chair in this room is not broken"; it says nothing about chairs outside this room.

Similarly, the negation of the statement $(\forall n \in \mathbb{N})(\exists x \in A)(nx < 1)$ is $(\exists n \in \mathbb{N})(\forall x \in A)(nx \geq 1)$. The negated sentence means that the set A has a lower bound that is the reciprocal of an integer. It does not mention values of n outside $\mathbb{N}$ or values of x outside A. ■

2.14. Example. Let us rephrase "It is false that every classroom has a chair that is not broken". The quantifiers make it improper to cancel the "double negative"; the sentence "every classroom has a chair that is broken" has a different meaning.

The original statement has a universal quantifier ("every") and an existential quantifier ("has a"). By successively negating these quantifiers, we obtain first "There is a classroom that has no chair that is not broken" and then "There is a classroom in which every chair is broken".

We can also express this manipulation symbolically. Let R denote the set of classrooms. Given a room r, let $C(r)$ denote the set of chairs in r. For a chair c, let $B(c)$ be the statement that c is broken. The successive statements (all having the same meaning) now become

$$\neg\big[(\forall r \in R)(\exists c \in C(r))(\neg B(c))\big]$$
$$(\exists r \in R)\big(\neg\big[(\exists c \in C(r))(\neg B(c))\big]\big)$$
$$(\exists r \in R)(\forall c \in C(r))B(c).$$

■

2.15. Example. In Definition 1.31, we defined bounded function. We negate this to obtain "f is *unbounded* if for every real number M, some real number x satisfies $|f(x)| > M$." In notation, the two conditions are

$$\text{bounded}: \quad (\exists M \in \mathbb{R})(\forall x \in \mathbb{R})(|f(x)| \leq M)$$
$$\text{unbounded}: \quad (\forall M \in \mathbb{R})(\exists x \in \mathbb{R})(|f(x)| > M).$$

Thus unboundedness implies $(\forall n \in \mathbb{N})(\exists x_n \in \mathbb{R})(|f(x_n)| > n)$. ■

COMPOUND STATEMENTS

The negation of a logical statement is another logical statement. We can also use the connectives "and", "or", "if and only if", and "implies" to build compound statements. For each choice of truth values for the component statements, the compound statement has a specified truth value; this constitutes the defininition of the connective.

2.16. Definition. *Logical connectives.* In the following table, we define the operations named in the first column by the truth values specified in the last column.

Name	Symbol	Meaning	Condition for truth
Negation	$\neg P$	not P	P false
Conjunction	$P \wedge Q$	P and Q	both true
Disjunction	$P \vee Q$	P or Q	at least one true
Biconditional	$P \Leftrightarrow Q$	P if & only if Q	same truth value
Conditional	$P \Rightarrow Q$	P implies Q	Q true whenever P true

2.17. Remark. *Disjunctions.* The meaning of "or" in mathematics differs from its common usage in English. In response to "Are you going home or not?", the answer "Yes" causes annoyance despite being logically correct; in common English the word "or" means "one or the other but not both". In mathematics, this usage is **exclusive-or**; we reserve **or** for disjunction.

Disjunction is more common in mathematics than exclusive-or because *and* and *or* act as quantifiers. A conjunction is true if *all* of its component statements are true; thus *and* is a universal quantifier. A disjunction is true if *at least one* of its component statements is true; thus *or* is an existential quantifier. ■

In the conditional statement $P \Rightarrow Q$, we call P the **hypothesis** and Q the **conclusion**. The statement $Q \Rightarrow P$ is the **converse** of $P \Rightarrow Q$.

2.18. Remark. *Conditionals.* Conditional statements are the only type in Definition 2.16 whose meaning changes when P and Q are interchanged. There is no general relationship between the truth values of $P \Rightarrow Q$ and $Q \Rightarrow P$. Consider three statements about a real number x: P is "$x > 0$", Q is "$x^2 > 0$", and R is "$x + 1 > 1$". Here $P \Rightarrow Q$ is true but $Q \Rightarrow P$ is false. On the other hand, both $P \Rightarrow R$ and $R \Rightarrow P$ are true.

Note that here x is a variable. We have dropped x from the notation for the statements because the context is clear. Technically, when we write $P \Rightarrow Q$ here, we mean $(\forall x \in \mathbb{R})(P(x) \Rightarrow Q(x))$.

A conditional statement is false when and only when the hypothesis is true and the conclusion is false. When the hypothesis is false, the conditional statement will be true regardless of what the conclusion says and whether it is true. For example, if S is "This book was published in the year 73", then $S \Rightarrow P$ is true, no matter what P is.

It may be helpful to read the conditional as "if-then" instead of "implies". Below we list several ways to say $P \Rightarrow Q$ in English. ■

If P (is true), then Q (is true).	P is true only if Q is true.
Q is true whenever P is true.	P is a sufficient condition for Q.
Q is true if P is true.	Q is a necessary condition for P.

When a logical statement is built from elementary statements using connectives, we treat the elementary statements as variables in the universe {True, False}. Given their values, Definition 2.16 yields the truth value of the full expression. A listing of these computations for each choice of truth values of the elementary statements is a **truth table**.

2.19. Example. We give one example of a truth table to emphasize again the meaning of conditional statements. We want to know whether the expression R given by $(P \Rightarrow Q) \Leftrightarrow ((\neg P) \vee Q)$ is always true, no matter what P and Q represent. Such an expression is called a **tautology**. Each of P and Q may be true or false; we consider all cases. ■

P	Q	$P \Rightarrow Q$	$\neg P$	$(\neg P) \vee Q$	R
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Two logical expressions X, Y are **logically equivalent** if they have the same truth value for each assignment of truth values to the variables. Equivalences allow us to rephrase statements in more convenient ways.

2.20. Remark. *Elementary logical equivalences.* We may substitute P for $\neg(\neg P)$ whenever we wish, and vice versa. Similarly, $P \vee Q$ is equivalent to $Q \vee P$, and $P \wedge Q$ is equivalent to $Q \wedge P$. Whenever P and Q are statements, we may substitute the expression in the right column below for the corresponding expression in the left column (or vice versa); they always have the same truth value. We could verify these equivalences by manipulating symbols in truth tables, but it is more productive to understand them using the English meanings of the connectives.

a)	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
b)	$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
c)	$\neg(P \Rightarrow Q)$	$P \wedge (\neg Q)$
d)	$P \Leftrightarrow Q$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
e)	$P \vee Q$	$(\neg P) \Rightarrow Q$
f)	$P \Rightarrow Q$	$(\neg Q) \Rightarrow (\neg P)$

Equivalences (a) and (b) present our understanding of "and" and "or" as universal and existential quantifiers, respectively, over their component statements (see Remark 2.17). These two equivalences are called **de Morgan's laws** in honor of the logician Augustus de Morgan (1806–1871).

Equivalences (c) and (d) restate the definitions of the conditional and biconditional. A conditional statement is false precisely when the hypothesis is true and the conclusion is false. The biconditional is true precisely when the conditional and its converse are both true.

Each side of (e) is false precisely when P fails and Q fails. Each side of (f) fails precisely when P is true and Q is false. ∎

2.21. Remark. *Logical connectives and membership in sets.* Let $P(x)$ and $Q(x)$ be statements about an element x from a universe U. Often we write a conditional statement $(\forall x \in U)(P(x) \Rightarrow Q(x))$ as $P(x) \Rightarrow Q(x)$ or simply $P \Rightarrow Q$ with an implicit universal quantifier.

The hypothesis $P(x)$ can be interpreted as a universal quantifier in another way. With $A = \{x \in U : P(x) \text{ is true}\}$, the statement $P(x) \Rightarrow Q(x)$ can be written as $(\forall x \in A)Q(x)$.

Another interpretation of $P(x) \Rightarrow Q(x)$ uses set inclusion. With $B = \{x \in U : Q(x) \text{ is true}\}$, the conditional statement has the same meaning as the statement $A \subseteq B$. The converse statement $Q(x) \Rightarrow P(x)$ is equivalent to $B \subseteq A$; thus the biconditional $P \Leftrightarrow Q$ is equivalent to $A = B$.

We can alternatively interpret operations with sets using logical connectives and membership statements. When P is the statement of membership in A and Q is the statement of membership in B, the statement $A = B$ has the same meaning as $P \Leftrightarrow Q$. Below we list the correspondence for other set operations. ∎

$$
\begin{array}{ccccc}
x \in A^c & \Leftrightarrow & \text{not } (x \in A) & \Leftrightarrow & \neg(x \in A) \\
x \in A \cup B & \Leftrightarrow & (x \in A) \text{ or } (x \in B) & \Leftrightarrow & (x \in A) \vee (x \in B) \\
x \in A \cap B & \Leftrightarrow & (x \in A) \text{ and } (x \in B) & \Leftrightarrow & (x \in A) \wedge (x \in B) \\
A \subseteq B & \Leftrightarrow & (\forall x \in A)(x \in B) & \Leftrightarrow & (x \in A) \Rightarrow (x \in B)
\end{array}
$$

The understanding of union and intersection in terms of quantifiers allows us to extend the definitions of union and intersection to apply to more than two sets. The **intersection** of a collection of sets consists of all elements that belong to all of the sets. The **union** of a collection of sets consists of all elements that belong to at least one of the sets.

2.22. Remark. The correspondence between $P \Leftrightarrow Q$ and $A = B$ in Remark 2.21 highlights an important phenomenon. Expressions that represent "being the same" can be interpreted as two instances of comparison. When x and y are numbers, the statement $x = y$ includes two pieces of information, $x \le y$ and $y \le x$. When A and B are sets, the equality $A = B$ includes two pieces of information, $A \subseteq B$ and $B \subseteq A$. For logical statements P and Q, similarly, $P \Leftrightarrow Q$ means both $P \Rightarrow Q$ and $Q \Rightarrow P$.

In some contexts, we prove equality by proving both comparisons. In other contexts, we can prove equality directly, by using manipulations that preserve the value, set, or meaning while transforming the first description into the second. ∎

2.23. Example. *de Morgan's laws for sets.* In the language of sets, de Morgan's laws (Remark 2.20a,b) become (1) $(A \cap B)^c = A^c \cup B^c$, and (2) $(A \cup B)^c = A^c \cap B^c$. We verify (1) by translation into a logical equivalence about membership, leaving (2) to Exercise 50. Given an element x, let P be the property $x \in A$, and let Q be the property $x \in B$. Remarks 2.20–2.21 imply that

$$x \in (A \cap B)^c \Leftrightarrow \neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q) \Leftrightarrow (x \notin A) \vee (x \notin B)$$

Alternatively, a Venn diagram makes the reasoning clear. ∎

Although relationships between sets correspond to logical statements about membership, the two expressions tell the same story in different languages. One must not mix them. For example, $A \cap B$ is a set, not a statement; it has no truth value. The notation "$(A \cap B)^c \Leftrightarrow A^c \cup B^c$" has no meaning, but $(A \cap B)^c = A^c \cup B^c$ is true whenever A and B are sets.

ELEMENTARY PROOF TECHNIQUES

The business of mathematics is deriving consequences from hypotheses—that is, proving conditional statements. Although we prove some biconditionals by chains of equivalences, as in Example 2.23, usually we prove a biconditional by proving a conditional and its converse, as suggested by Remark 2.20d. Also, we can prove the universally quantified statement "$(\forall x \in A)Q(x)$" by proving the conditional statement "If $x \in A$, then $Q(x)$"; the two have the same meaning. (For example, consider the two sentences when A is the set of even numbers and $Q(x)$ is "x^2 is even".)

2.24. Remark. *Elementary methods of proving $P \Rightarrow Q$.* The *direct method* of proving $P \Rightarrow Q$ is to assume that P is true and then to apply mathematical reasoning to deduce that Q is true. When P is "$x \in A$" and Q is "$Q(x)$", the direct method considers an *arbitrary* $x \in A$ and deduces $Q(x)$. This must not be confused with the invalid "proof by example". The proof must apply to every member of A as a possible instance of x, because "$(x \in A) \Rightarrow Q(x)$" is a universally quantified statement.

Remark 2.20f suggests another method. The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. The equivalence between a conditional and its contrapositive allows us to prove $P \Rightarrow Q$ by proving $\neg Q \Rightarrow \neg P$. This is the *contrapositive method.*

Remark 2.20c suggests another method. Negating both sides $(P \Rightarrow Q) \Leftrightarrow \neg[P \wedge (\neg Q)]$. Hence we can prove $P \Rightarrow Q$ by proving that P and $\neg Q$ cannot both be true. We do this by obtaining a contradiction after assuming both P and $\neg Q$. This is the *method of contradiction* or **indirect proof.** We summarize these methods below:

Direct Proof: Assume P, follow logical deductions, conclude Q.
Contrapositive: Assume $\neg Q$, follow deductions, conclude $\neg P$.
Method of Contradiction: Assume P and $\neg Q$, follow deductions, obtain a contradiction. ∎

We begin with easy examples of the direct method, including statements used in proving Theorem 2.3.

2.25. Example. *If integers x and y are both odd, then $x + y$ is even.* Suppose that x and y are odd. By the definition of "odd", there exist integers k, l such that $x = 2k + 1$ and $y = 2l + 1$. By the properties of addition and the distributive law, $x + y = 2k + 2l + 2 = 2(k + l + 1)$. This is twice an integer, so $x + y$ is even.

The converse is false. When x, y are integers, it is possible that $x + y$ is even but x, y are not both odd. Compare this with the next example. ∎

2.26. Example. *An integer is even if and only if it is the sum of two odd integers.* First we clarify what must be proved. Formally, the statement is $(\forall x \in \mathbb{Z})[(\exists k \in \mathbb{Z})(x = 2k) \Leftrightarrow (\exists y, z \in O)(x = y + z)]$, where O is the set of odd numbers. If $x = 2k$ is even, then $x = (2k - 1) + 1$, which expresses x as the sum of two odd integers. Conversely, let y and z be odd. By the definition of "odd", there exist integers k, l such that $y = 2k + 1$ and $z = 2l + 1$. Then $y + z = 2k + 1 + 2l + 1 = 2(k + l + 1)$, which is even. ∎

2.27. Example. *If x and y are odd, then xy is odd.* If x and y are odd, then there are integers k, l such that $x = 2k + 1$ and $y = 2l + 1$. Now $xy = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$. Since this is one more than twice an integer, xy also is odd. ∎

A special case of Example 2.27 is "x odd $\Rightarrow x^2$ odd". Here the conclusion is "There is an integer m such that $x^2 = 2m + 1$". We can prove an existential conclusion by providing an *example*: in this case a value m (constructed in terms of x) such that the statement is true. The direct method often succeeds when the conclusion is existentially quantified.

2.28. Example. *An integer is even if and only if its square is even.* If n is even, then we can write $n = 2k$, where k is an integer. Now $n^2 = 4k^2 = 2(2k^2)$, proving that "n even" implies "n^2 even" by the direct method. For the converse, we want to prove "n^2 even implies n even", but this we have already done! Since integers are even or odd, the desired implication is the contrapositive of "n odd implies n^2 odd". ∎

2.29. Remark. *Converse versus contrapositive.* Proving the biconditional statement $P \Leftrightarrow Q$ requires proving one statement from each column below. Each statement is the converse of the other in its row. Each statement is the contrapositive of the other in its column. Every conditional is equivalent to its contrapositive, so proving the two statements in one column would be proving the same fact twice.

$$\begin{array}{cc} P \Rightarrow Q & Q \Rightarrow P \\ \neg Q \Rightarrow \neg P & \neg P \Rightarrow \neg Q \end{array}$$

For example, consider "the product of two nonzero real numbers is positive if and only if they have the same sign". The axioms for real numbers imply that if x and y have the same sign, then xy is positive. We might then argue, "Now suppose that xy is negative. This implies that x and y have opposite signs." This accomplishes nothing; we have proved the contrapositive of the first conditional, not its converse. Instead, we must prove "If xy is positive, then x and y have the same sign" or "If x and y have opposite signs, then xy is negative".

We can interpret the first line of the display above as the direct method and the second line as the contrapositive method. To include the method of contradiction, we could add the line below:

$$\neg(P \wedge \neg Q) \qquad \neg(Q \wedge \neg P).$$ ∎

The next example uses the contrapositive and illustrates that care must be taken to avoid unjustified assumptions.

2.30. Example. Consider the statement *"If $f(x) = mx + b$ and $x \neq y$, then $f(x) \neq f(y)$."* The direct method considers $x < y$ and $x > y$ separately and obtains $f(x) < f(y)$ or $f(x) > f(y)$. This unsatisfying analysis by cases results from "not equals" being a messier condition than "equals".

We can use the contrapositive to retain the language of equalities and reduce analysis by cases. When $f(x) = f(y)$, we obtain $mx + b = my + b$ and then $mx = my$. If $m \neq 0$, then we obtain $x = y$.

If $m = 0$, then we cannot divide by m, and actually the statement is false. The difficulty is that m is a variable in the statement we want to prove, and we cannot determine its truth without quantifying m. The statement is true if and only if $m \neq 0$. ∎

A universally quantified statement like "$(\forall x \in U)[P(x) \Rightarrow Q(x)]$" can be disproved by finding an element x in U such that $P(x)$ is true and $Q(x)$ is false. Such an element x is a **counterexample**. In Example 2.30, $m = 0$ is a counterexample to a claim that the implication holds for all m.

We continue with another example of proof by contrapositive.

2.31. Example. *If a is less than or equal to every real number greater than b, then $a \leq b$.* The direct method goes nowhere, but when we say "suppose not", the light begins to dawn. If $a > b$, then $a > \frac{a+b}{2} > b$. Thus a is *not* less than or equal to every number greater than b. We have proved the contrapositive of the desired statement. ∎

When the hypothesis of $P \Rightarrow Q$ is universally quantified, its negation is existentially quantified. This can make the contrapositive easy; given $\neg Q$, we need only construct a counterexample to P. This is the scenario in Example 2.31; having assumed $a > b$, we need only construct a counterexample to "a is less than every real x that is greater than b".

The method of contradiction proves $P \Rightarrow Q$ by proving that P and $\neg Q$ cannot both hold, thereby proving that $P \Rightarrow Q$ cannot be false.

2.32. Example. *Among the numbers $y_1, \ldots, y_n$, some number is as large as the average.* Let $Y = y_1 + \cdots + y_n$. The **average** z is Y/n.

An indirect proof of the claim begins, "suppose that the conclusion is false". Thus $y_i < z$ for all y_i in the list. If we sum these inequalities, we obtain $Y < nz$, but this contradicts the definition of z, which yields $Y = nz$. Hence the assumption that each element is too small must be false.

A direct proof constructs the desired number. Let y^* be the largest number in the set. We prove that this candidate *is* as large as the average. Since $y_i \le y^*$ for all i, we sum the inequalties to obtain $Y \le ny^*$ and then divide by n to obtain $z \le y^*$. ∎

In Example 2.32, we did not derive the negation of the hypothesis; we obtained a different contradiction. This is the method of contradiction. Like the contrapositive method, it begins by assuming $\neg Q$ when proving $P \Rightarrow Q$. We need not decide in advance whether to deduce $\neg P$ or to use both P and $\neg Q$ to obtain some other contradiction.

2.33. Example. *There is no largest real number.* If there is a largest real number z, then for all $x \in \mathbb{R}$, we have $z \ge x$. When x is the real number $z+1$, this yields $z \ge z+1$. Subtracting z from both sides yields $0 \ge 1$. This is a contradiction, and thus there is no largest real number. ∎

The method of contradiction works well when the conclusion is a statement of non-existence or impossibility, because negating the conclusion provides an *example* to use, like p/q in the proof of Theorem 2.3 or z in Example 2.33. In one sense the method of contradiction ("indirect proof") has more power than the contrapositive, since we start with more information (P *and* $\neg Q$), but in another sense it is less satisfying, because we start with a situation that (we hope) cannot be true.

2.34. Remark. *The consequences of false statements.* Recall that a conditional statement is false only if the hypothesis is true and the conclusion is false. When the hypothesis cannot be true, we say the conditional follows *vacuously*. Similarly, every statement universally quantified over an empty set is true; when there are no dogs in the class, the statement "Every dog in the class has three heads" is true. In contrast, every statement existentially quantified over an empty set is false; when there are no dogs in the class, the statement "Some dog in the class has four legs" is false!

Returning to the conditional, we have argued that $P \Rightarrow Q$ is true whenever P is false. This explains why a proof containing a single error in reasoning cannot be considered "nearly correct"; we can derive any conclusion from a single false statement (see Exercise 44a). Bertrand Russell (1872–1970) once stated this in a public lecture and was challenged to start with the assumption that $1 = 2$ and prove that he was God. He replied, "Consider the set {Russell, God}. If $1 = 2$, then the two elements of the set are one element, and therefore Russell = God." ∎

Students sometimes wonder about the meanings of the words "theorem", "lemma", "corollary", etc. The usage of these words is part of mathematical convention, like the notation $f\colon A \to B$ for functions and the designations $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ for the number systems. (By the way, $\mathbb{Q}$ stands for "quotient" and $\mathbb{Z}$ stands for "Zahlen", the German word for numbers.)

In Greek, *lemma* means "premise" and *theorema* means "thesis to be proved". Thus a theorem is a major result whose proof may require considerable effort. A lemma is a lesser statement, usually proved in order to help prove other statements. A proposition is something "proposed" to be proved; typically this is a less important statement or requires less effort than a theorem. The word *corollary* comes from Latin, as a modification of a word meaning "gift"; a corollary follows easily from a theorem or proposition, without much additional work.

Theorems, Propositions, Corollaries, and Lemmas may all be used to prove other results. In this book, these embody the central mathematical development, while Examples, Solutions, Applications, and Remarks are particular uses of or commentary on the mathematics. These two streams are interwoven but can be distinguished by the titles of the items. The first stream comprises the mathematical results that students might want to remember for later application, while the second illuminates the first and provides additional examples of problem-solving.

HOW TO APPROACH PROBLEMS

In this chapter we have discussed the language of mathematics and elementary techniques of proof. We review some of these issues and discuss several additional ones that arise when solving problems.

Methods of proof.

The first step is making sure that one understands exactly what the problem is asking. Definitions may provide a road map for what needs to be verified. Sometimes, the desired statement follows from a theorem already proved, and then one needs to verify that its hypotheses hold.

Most problems request proofs of conditional statements. These state that given circumstances produce certain results. Such sentences are often written using "if" and "then", but implication can be expressed with universal quantifiers and in many other ways (see Remark 2.8, Remark 2.18, and Exercise 10). Examples cannot provide proofs of such statements. Implications need to be proved in Exercises 34–42.

The elementary techniques for proving implications are direct proof, proof by contradiction, and proof of the contrapositive. The latter two methods are called "indirect" proofs. When seeking a direct proof, one can work from both ends. List statements that follow from the hypothesis. List statements that suffice to imply the conclusion. When some statement appears in both lists, the problem is solved.

When unsuccesful with the direct method, consider what would happen if the conclusion were false. If this leads to impossibility of some consequence of the hypothesis (or of other known facts), then again the problem is solved, using the method of contradiction. If the negation of the hypothesis is obtained, then the contrapositive has been proved.

Students often wonder when to use indirect proof. The form of the conclusion can provide a clue; when its negation provides something useful to work with, indirect proof may be appropriate. This can happen with obvious-sounding statements like Example 2.31. Often indirect proof is appropriate for statements of nonexistence, as in Theorem 2.3, Example 2.33, and Exercise 40. The negation of the conclusion provides an example, an object with specified properties. (In contrast, one can often prove that something *does* exist by constructing an example and proving that it has the desired properties; this is the direct method.)

Be aware of hypotheses and quantifiers.

An implication is true when the truth of its hypotheses guarantees the truth of its conclusion. The sentence "if we add two even integers, then the result is even" is true and easily proved, but the sentence "if we add two integers, then the result is even" is false. The second sentence is obviously missing a hypothesis (that the integers are even) that is needed to make the conclusion true.

In more subtle statements, the same principles apply. Carefully distinguish the hypotheses and the desired conclusions. Remember that hypotheses can be expressed as universal quantifiers: "for all $x \in A$" means the same as "if $x \in A$". In writing a solution, check where the hypotheses are used. If a hypothesis is not used, then either it is unnecessary (and the proof yields a stronger statement) or an error has been made.

Solving a problem may require determining whether a statement with many quantified variables is true or false. One must be able to identify the universal and existential quantifiers, put them in proper order (see items 2.9–2.11), and negate a quantified statement (see items 2.12–2.15).

More about cases.

A universally quantified statement must be proved for all instances of the variables. This includes statements phrased in the singular, like "The square of an even number is even." Writing $(-4)^2 = 16 = 2 \cdot 8$ does not prove this, because here "an" means *each individual*. The sentence means "If x is an even number, then x^2 is an even number." Similarly, "Let x be a positive real number" and "For $x > 0$" are universal quantifications; the claim to be proved must be proved for every positive real number x.

Analysis by cases can arise when an argument is valid for some instances but not for all. Consider showing that $x(x+1)/2$ is an integer whenever x is an integer. When x is even, we write $x = 2k$ and compute $2k(2k+1)/2 = k(2k+1)$, where k is an integer. For odd x, we need a different computation. We can avoid cases by observing that one of $\{x, x+1\}$ is even and is divisible by 2. Combining cases via a unified argument leads to a concise solution that captures the essence of the proof.

When several cases are treated in the same way, it may be possible to reduce to a single case by using symmetry. We did this in proving Theorem 2.2. Having disposed of the case where all four coefficients are zero (which uses a different argument), we may assume that some coefficient is nonzero. We would use the same arguments no matter which it is. By writing the equations in the opposite order and/or switching the names of the variables, we can arrange that the coefficient d is nonzero. We say that symmetry allows us to reduce to the case where d is nonzero.

Similarly, when proving a statement about distinct real numbers x, y, it may be helpful to assume by symmetry that $x > y$. The same argument with the roles of x and y switched would apply when $y > x$, and we use the symmetry in the problem to avoid writing out the argument twice.

On the other hand, sometimes a problem becomes simpler when we introduce an additional hypothesis. This leads to two cases: when the assumption is true and when it is false. Consider Exercise 33. The first child knows that her hat is black or red. She considers these two cases to seek a contradiction that will eliminate one. Perhaps further assumptions will be needed, leading to subcases. Exercise 32 is similar; we consider various assumptions. Assuming that Person A tells the truth yields an immediate contradiction; knowing that A lies leads to further conclusions.

This method is known informally as "process of elimination". If a particular assumption seems to lead nowhere, try another! Remember that eventually all possibilities must be considered. For example, when the roles of variables x and y are not interchangeable in a problem, we cannot use symmetry to reduce to $x \le y$, but considering the cases $x < y$, $x = y$, and $x > y$ separately might lead to different arguments that work.

Finally, beware of overlooking cases that result from introducing unwanted hypotheses. In particular, be aware of the conditions under which symbolic manipulations are valid. Since we cannot divide by zero, the equation $y = mx$ can be solved for x only when $m \neq 0$. For all real y and all nonzero real m, there is a unique x with $y = mx$. The case $m = 0$ has not been considered and must be treated in some other way.

Taking square roots also requires care. For example, Exercise 1.19 has no solution for some choices of the perimeter p and area a, because the algebraic solution involves a square root. Square roots exists only for nonnegative numbers; this constrains the values p and a.

Equations and algebraic manipulations.

Consider the equation $x^2-10x+5 = -20$. Manipulating the equation yields $(x-5)^2 = 0$, which implies $x = 5$. This can be interpreted as the conditional statement "If the equation holds, then $x = 5$". Checking the answer shows the converse assertion "If $x = 5$, then the equation holds". Together, the two steps yield the statement "The set of solutions to the equation $x^2 - 10x + 5 = -20$ is $\{5\}$".

Consider also the equation $x^2 = 5x$. Dividing both sides by x yields $x = 5$. Checking 5 in the equation yields "if $x = 5$, then the equation holds". The statement "If the equation holds, then $x = 5$" is false. The correct assertion is "If the equation holds, then $x = 0$ or $x = 5$". The problem is that the division was valid only under the hypothesis that $x \neq 0$. The solution in the remaining case was lost.

Algebraic manipulations can also introduce extraneous solutions. Consider the equation $x = 4$. If we next write $x^2 = 4x$, then we obtain $x^2 - 4x = 0$, with solutions $x = 4$ and $x = 0$. Multiplying by x introduced the extraneous solution $x = 0$; it changed the solution set. Substituting the results of invalid manipulations into the original equation may or may not expose the error.

Multiplying both sides of an equation in x by an expression $f(x)$ introduces all the zeroes of f as solutions; some may be extraneous. Dividing by $f(x)$ is invalid when $f(x)$ can be zero; in this case solutions may be lost. When manipulating an equation to seek equivalent statements, one must check that the set of solutions never changes or analyze separately the cases where it may change.

2.35. Example. The following argument alleges to prove that $2 = 1$; it must be wrong! What is the flaw?

Let x, y be real numbers, and suppose that $x = y$. This yields $x^2 = xy$, which implies $x^2 - y^2 = xy - y^2$ by subtracting y^2 from both sides. Factoring yields $(x + y)(x - y) = y(x - y)$, and thus $x + y = y$. In the special case $x = y = 1$, we obtain $2 = 1$. ∎

Sets and membership.

Various exercises in this chapter involve identities involving unions, intersections, and differences of sets. These can be understood using Venn diagrams. Equality of expressions involving sets can be proved by showing that an element belongs to the set given by one expression if and only if it belongs to the set given by the other.

Reasoning about sets and subsets is parallel to reasoning about conditional statements. The set-theoretic statement $S \subseteq T$ can be interpreted as "If $x \in S$, then $x \in T$". Thus the logical statement $P \Leftrightarrow Q$ is parallel to the set-theoretic equality $S = T$ (see Remarks 2.21–2.22).

Identities involving operations on sets (Exercises 50–53) and equivalences involving logical connectives and statements (Exercises 43–46) are universally quantified, with variables representing sets or statements. Thus the proof must be valid for all instances.

In several of the exercises, two sets of real numbers are specified by numerical constraints, and the problem is to show that the two sets are the same. One can prove that each set is contained in the other, or one can manipulate the constraints in ways that do not change the set of solutions. In either approach, words should be used to explain the arguments.

Communicating mathematics.

Solutions to problems should be written using sentences that explain the argument. Notation introduced to represent concepts in the discussion should be clearly defined, and a symbol should not be used with different meanings in a single discussion.

A convincing proof cannot depend on asking the reader to guess what the writer intended. A well-written argument may begin with an overview or with an indication of the method of proof. Such an indication is particularly helpful when using the contrapositive or the method of contradiction.

When the writer gives no explanation of the method of proof and merely lists some formulas, the reader can only assume that a direct proof is being given, with each line derived from the previous line. This gets students into trouble when they reduce a desired statement to a known statement. In attempting to prove the AGM Inequality for all nonnegative real numbers x, y, some students will write

$$\begin{aligned} \sqrt{xy} &\le (x+y)/2 \\ xy &\le (x+y)^2/4 \\ 4xy &\le x^2 + 2xy + y^2 \\ 0 &\le x^2 - 2xy + y^2 \\ 0 &\le (x-y)^2, \quad \text{which is true.} \end{aligned}$$

Here the student has derived a true statement from the desired statement; this does not prove the desired statement. Within the set of pairs of nonnegative real numbers, these manipulations of the inequality have not changed the set of solutions, so the steps are reversible to obtain the desired inequality. Without words to indicate that this is what is intended, the proof is wrong. Note that the "proof" never used the hypothesis that $x, y \geq 0$, and when $x = y = -1$ the claimed inequality fails.

One must always distinguish a statement from its converse. Deriving a true statement Q from the desired statement P does not prove P! Let P be the assertion "$x + 1 = x + 2$". When we multiply both sides of P by 0, we obtain the true statement "$0 = 0$"; call this Q. Although Q is true for all x and we have proved $P \Rightarrow Q$, the statement P is true for no x.

EXERCISES

2.1. Find the flaw in Example 2.35.

2.2. Show that the following statement is false: "If a and b are integers, then there are integers m, n such that $a = m + n$ and $b = m - n$." What can be added to the hypothesis of the statement to make it true?

2.3. Consider the following sentence: "If a is a real number, then $ax = 0$ implies $x = 0$". Write this sentence using quantifiers, letting $P(a, x)$ be the assertion "$ax = 0$" and $Q(x)$ be the assertion "$x = 0$". Show that the implication is false, and find a small change in the quantifiers to make it true.

2.4. Let A and B be sets of real numbers, let f be a function from $\mathbb{R}$ to $\mathbb{R}$, and let P be the set of positive real numbers. Without using words of negation, for each statement below write a sentence that expresses its negation.

a) For all $x \in A$, there is a $b \in B$ such that $b > x$.

b) There is an $x \in A$ such that, for all $b \in B$, $b > x$.

c) For all $x, y \in \mathbb{R}$, $f(x) = f(y) \Rightarrow x = y$.

d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$.

e) For all $x, y \in \mathbb{R}$ and all $\epsilon \in P$, there is a $\delta \in P$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

f) For all $\epsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

2.5. (–) Prove the following statements.

a) For all real numbers y, b, m with $m \neq 0$, there is a unique real number x such that $y = mx + b$.

b) For all real numbers y, m, there exist $b, x \in \mathbb{R}$ such that $y = mx + b$.

2.6. (–) *Usage of language.*

a) The following sentence appeared on a restaurant menu: "Please note that every alternative may not be available at this time". Describe the unintended meaning. Rewrite the sentence to state the intended meaning clearly.

b) Give an example of an English sentence that has different meanings depending on inflection, pronunciation, or context.

2.7. (–) Describe how the notion of an *alibi* in a criminal trial fits into our discussion of conditional statements.

2.8. From outside mathematics, give an example of statements A, B, C such that A and B together imply C, but such that neither A nor B alone implies C.

2.9. (–) The negation of the statement "No slow learners attend this school" is:†

a) All slow learners attend this school.

b) All slow learners do not attend this school.

c) Some slow learners attend this school.

d) Some slow learners do not attend this school.

e) No slow learners attend this school.

2.10. Express each of the following statements as a conditional statement in "if-then" form or as a universally quantified statement. Also write the negation (without phrases like "it is false that").

a) Every odd number is prime.

b) The sum of the angles of a triangle is 180 degrees.

c) Passing the test requires solving all the problems.

d) Being first in line guarantees getting a good seat.

e) Lockers must be turned in by the last day of class.

f) Haste makes waste.

g) I get mad whenever you do that.

h) I won't say that unless I mean it.

2.11. (!) Suppose I have a penny, a dime, and a dollar, and I say, "If you make a true statement, I will give you one of the coins. If you make a false statement, I will give you nothing." What should you say to obtain the best coin?

• • • • •

2.12. A telephone bill y (in cents) is determined by $y = mx + b$, where x is the number of calls during the month, and b is a constant monthly charge. Suppose that the bill is \$5.48 when 8 calls are made and is \$5.72 when 12 calls are made. Determine what the bill will be when 20 calls are made.

2.13. "In one year, my wife will be one-third as old as my house. In nine years, I will be half as old as my house. I am ten years older than my wife. How old are my house, my wife, and I?" Answer the question, stating the needed equations.

2.14. A **circle** is the set of ordered pairs $(x, y) \in \mathbb{R}^2$ such that x and y satisfy an equation of the form $x^2 + y^2 + ax + by = c$, where $c > -(a^2 + b^2)/4$. The circle is specified by the parameters a, b, c.

a) Using this definition, give examples of two circles such that

i) the circles do not intersect.

ii) the circles intersect in exactly one common element.

iii) the circles intersect in two common elements.

b) Explain why the parameter c is restricted as given.

2.15. *The quadratic formula, revisited.* We derive the quadratic formula by solving a system of linear equations for the two unknown solutions. The equation $ax^2 + bx + c = 0$ with $a \neq 0$ has real solutions r, s if and only if $ax^2 + bx + c = a(x - r)(x - s)$ (see Exercise 1.20). The calculation below shows that the factorization exists if and only if $b^2 - 4ac \geq 0$ and expresses r, s in terms of a, b, c.

a) By equating coefficients of corresponding powers of x, obtain the equations

†From the 1955 High School Mathematics Exam (C. T. Salkind, *Annual High School Mathematics Examinations 1950–1960*, Math. Assoc. Amer. 1961, p. 37.)

$r+s=-b/a$ and $rs=c/a$. Use these to prove that $(r-s)^2=(b^2-4ac)/a^2$.

b) From (a), obtain $r+s=-b/a$ and $r-s=\sqrt{b^2-4ac}/a$. Solve this system for r, s in terms of a, b, c.

c) What happens if the second equation in (b) is $r-s=-\sqrt{b^2-4ac}/a$?

2.16. (!) Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.

a) Prove that f can be expressed in a unique way as the sum of two functions g and h such that $g(-x)=g(x)$ for all $x\in\mathbb{R}$ and $h(-x)=-h(x)$ for all $x\in\mathbb{R}$. (Hint: Find a system of linear equations for the unknowns $g(x)$ and $h(x)$ in terms of the known values $f(x)$ and $f(-x)$.)

b) When f is a polynomial, express g and h in terms of the coefficients of f.

2.17. Given $f\colon \mathbb{R}\to\mathbb{R}$, let $g(x)=\frac{x}{2}+\frac{x}{f(x)-1}$ for all x such that $f(x)\neq 1$. Suppose $g(x)=g(-x)$ for all such x. Prove that $f(x)f(-x)=1$ for all such x.

2.18. (!) Given a polynomial p, let A be the sum of the coefficients of the even powers, and let B be the sum of the coefficients of the odd powers. Prove that $A^2-B^2=p(1)p(-1)$.

2.19. Abraham Lincoln said, "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time." Write this sentence in logical notation, negate the symbolic sentence, and state the negation in English. Which statement seems to be true?

2.20. Using quantifiers, explain what it would mean for the first player to have a "winning strategy" in Tic-Tac-Toe. (Don't consider whether the statement is true.)

2.21. Consider the sentence "For every integer $n>0$ there is some real number $x>0$ such that $x<1/n$". Without using words of negation, write a complete sentence that means the same as "It is false that for every integer $n>0$ there is some real number $x>0$ such that $x<1/n$". Which sentence is true?

2.22. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Without using words of negation, write the meaning of "f is not an increasing function".

2.23. Consider $f\colon \mathbb{R}\to\mathbb{R}$. Let S be the set of functions defined by putting $g\in S$ if there exist positive constants $c, a\in\mathbb{R}$ such that $|g(x)|\le c\,|f(x)|$ for all $x>a$. Without words of negation, state the meaning of "$g\notin S$". (Comment: The set S (written as $O(f)$) is used to compare the "order of growth" of functions.)

2.24. In simpler language, describe the meaning of the following two statements and their negations. Which one implies the other, and why?

a) There is a number M such that, for every x in the set S, $|x|\le M$.

b) For every x in the set S, there is a number M such that $|x|\le M$.

2.25. For $a\in\mathbb{R}$ and $f\colon \mathbb{R}\to\mathbb{R}$, show that (a) and (b) have different meanings.

a) $(\forall\epsilon>0)(\exists\delta>0)[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$

b) $(\exists\delta>0)(\forall\epsilon>0)[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$

2.26. For $f\colon \mathbb{R}\to\mathbb{R}$, which of the statements below implies the other? Does there exist a function for which both statements are true?

a) For every $\epsilon>0$ and every real number a, there is a $\delta>0$ such that $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$.

b) For every $\epsilon>0$, there is a $\delta>0$ such that $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$ and a is a real number.

2.27. (+) For $c\in\mathbb{R}$ and $f\colon \mathbb{R}\to\mathbb{R}$, interpret each statement below.

a) For all $x\in\mathbb{R}$ and all $\delta>0$, there exists $\epsilon>0$ such that $|x|<\delta$ implies $|f(x)-c|<\epsilon$.

b) For all $x\in\mathbb{R}$, there exists $\delta>0$ such that, for all $\epsilon>0$, we have $|x|<\delta$ implies $|f(x)-c|<\epsilon$.

2.28. (!) Consider the equation $x^4y+ay+x=0$.

a) Show that the following statement is false. "For all $a, x\in\mathbb{R}$, there is a unique y such that $x^4y+ay+x=0$."

b) Find the set of real numbers a such that the following statement is true. "For all $x\in\mathbb{R}$, there is a unique y such that $x^4y+ay+x=0$."

2.29. (!) *Extremal problems.*

a) Let f be a real-valued function on S. In order to prove that the minimum value in the image of f is β, two statements must be proved. Express each of these statements using quantifiers.

b) Let T be the set of ordered pairs of positive real numbers. Define $f\colon T\to\mathbb{R}$ by $f(x,y)=\max\{x, y, \frac{1}{x}+\frac{1}{y}\}$. Determine the minimum value in the image of f. (Hint: What must a pair achieving the minimum satisfy?)

2.30. (!) Consider tokens that have some letter written on one side and some integer written on the other, in unknown combinations. The tokens are laid out, some with letter side up, some with number side up. Explain which tokens must be turned over to determine whether these statements are true:

a) Whenever the letter side is a vowel, the number side is odd.

b) The letter side is a vowel if and only if the number side is odd.

2.31. Which of these statements are believable? (Hint: Consider Remark 2.34.)

a) "All of my 5-legged dogs can fly."

b) "I have no 5-legged dog that cannot fly."

c) "Some of my 5-legged dogs cannot fly."

d) "I have a 5-legged dog that cannot fly."

2.32. A fraternity has a rule for new members: each must always tell the truth or always lie. They know who does which. If I meet three of them on the street and they make the statements below, which ones (if any) should I believe?

A says: "All three of us are liars."

B says: "Exactly two of us are liars."

C says: "The other two are liars."

2.33. Three children are in line. From a collection of two red hats and three black hats, the teacher places a hat on each child's head. The third child sees the hats on two heads, the middle child sees the hat on one head, and the first child sees no hats. The children, who reason carefully, are told to speak out as soon as they can determine the color of the hat they are wearing. After 30 seconds, the front child correctly names the color of her hat. Which color is it, and why?

2.34. (!) For each statement below about natural numbers, decide whether it is true or false, and prove your claim using only properties of the natural numbers.

a) If $n\in\mathbb{N}$ and $n^2+(n+1)^2=(n+2)^2$, then $n=3$.

b) For all $n\in\mathbb{N}$, it is false that $(n-1)^3+n^3=(n+1)^3$.

2.35. Prove that if x and y are distinct real numbers, then $(x+1)^2=(y+1)^2$ if and only if $x+y=-2$. How does the conclusion change if we allow $x=y$?

2.36. Let x be a real number. Prove that if $|x-1|<1$, then $|x^2-4x+3|<3$.

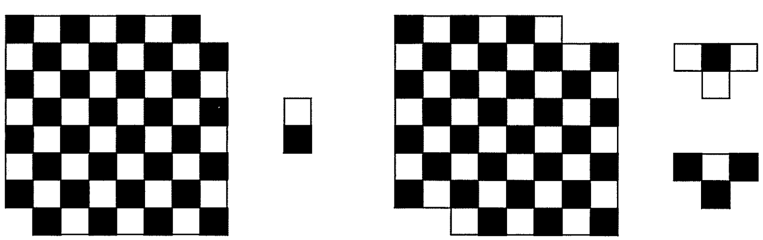
2.37. Given a real number x, let A be the statement "$\frac{1}{2}<x<\frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2=1$, and let D be the statement "$x=2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

2.38. Let x, y be integers. Determine the truth value of each statement below.
a) xy is odd if and only if x and y are odd.
b) xy is even if and only if x and y are even.

2.39. (!) A particle starts at the point $(0,0) \in \mathbb{R}^2$ on day 0. On each day, it moves one unit in a horizontal or vertical direction. For $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$, prove that it is possible for the particle to reach (a,b) on day k if and only if (1) $|a|+|b| \le k$, and (2) $a+b$ has the same parity as k.

2.40. (!) *Checkerboard problems.* (Hint: Use the method of contradiction.)
a) Two opposite corner squares are deleted from an eight by eight checkerboard. Prove that the remaining squares cannot be covered exactly by dominoes (rectangles consisting of two adjacent squares).
b) Two squares from each of two opposite corners are deleted as shown on the right below. Prove that the remaining squares cannot be covered exactly by copies of the "T-shape" and its rotations.



2.41. A clerk returns n hats to n people who have checked them, but not necessarily in the right order. For which k is it possible that exactly k people get a wrong hat? Phrase your conclusion as a biconditional statement.

2.42. A closet contains n different pairs of shoes. Determine the minimum t such that every choice of t shoes from the closet includes at least one matching pair of shoes. For $n>1$, what is the minimum t to guarantee that two matching pairs of shoes are obtained?

2.43. Using the equivalences discussed in Remark 2.20, write a chain of symbolic equivalences to prove that $P \Leftrightarrow Q$ is logically equivalent to $Q \Leftrightarrow P$.

2.44. Let P and Q be statements. Prove that the following are true.
a) $(Q \wedge \neg Q) \Rightarrow P$.
b) $P \wedge Q \Rightarrow P$.
c) $P \Rightarrow P \vee Q$.

2.45. Prove that the statements $P \Rightarrow Q$ and $Q \Rightarrow R$ imply $P \Rightarrow R$, and that the statements $P \Leftrightarrow Q$ and $Q \Leftrightarrow R$ imply $P \Leftrightarrow R$. (Comment: This is the justification for using a chain of equivalences to prove an equivalence.)

2.46. Prove that the logical expression S is equivalent to the logical expression $\neg S \Rightarrow (R \wedge \neg R)$, and explain the relationship between this equivalence and the method of proof by contradiction.

2.47. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "x^2-1 is divisible by 8". Determine whether the following statements are true:
a) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$.
b) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

2.48. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "x is twice an integer". Determine whether the following statements are true:
a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$.
b) $(\forall x \in \mathbb{Z})(P(x)) \Rightarrow (\forall x \in \mathbb{Z})(Q(x))$.

2.49. Let $S=\{x \in \mathbb{R}: x^2>x+6\}$. Let $T=\{x \in \mathbb{R}: x>3\}$. Determine whether the following statements are true, and interpret these results in words:
a) $T \subseteq S$.
b) $S \subseteq T$.

2.50. Prove the following identities involving complementation of sets.
a) $(A \cup B)^c = A^c \cap B^c$. (This is de Morgan's second law.)
b) $A \cap [(A \cap B)^c] = A - B$.
c) $A \cap [(A \cap B^c)^c] = A \cap B$.
d) $(A \cup B) \cap A^c = B - A$.

2.51. *Distributive laws for set operations.* Using statements about membership, prove the statements below, where A, B, C are any sets. Use Venn diagrams to illustrate the results and guide the proofs.
a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2.52. Let A, B, C be sets. Prove that $A \cap (B-C) = (A \cap B) - (A \cap C)$.

2.53. (!) Let A, B, C be sets. Prove that $(A \cup B) - C$ must be a subset of $[A-(B \cup C)] \cup [B-(A \cap C)]$, but that equality need not hold.

2.54. (+) Consider three circles in the plane as shown below. Each bounded region contains a token that is white on one side and black on the other. At each step, we can either (a) flip all four tokens inside one circle, or (b) flip the tokens showing white inside one circle to make all four tokens in that circle show black. From the starting configuration with all tokens showing black, can we reach the indicated configuration with all showing black except the token in the central region? (Hint: Consider parity conditions and work backward from the desired configuration.)