the proof is wrong. Note that the "proof" never used the hypothesis that $x, y \geq 0$, and when $x = y = -1$ the claimed inequality fails.

One must always distinguish a statement from its converse. Deriving a true statement Q from the desired statement P does not prove P! Let P be the assertion "$x + 1 = x + 2$". When we multiply both sides of P by 0, we obtain the true statement "$0 = 0$"; call this Q. Although Q is true for all x and we have proved $P \Rightarrow Q$, the statement P is true for no x.

EXERCISES

2.1. Find the flaw in Example 2.35.

2.2. Show that the following statement is false: "If a and b are integers, then there are integers m, n such that $a = m + n$ and $b = m - n$." What can be added to the hypothesis of the statement to make it true?

2.3. Consider the following sentence: "If a is a real number, then $ax = 0$ implies $x = 0$". Write this sentence using quantifiers, letting $P(a, x)$ be the assertion "$ax = 0$" and $Q(x)$ be the assertion "$x = 0$". Show that the implication is false, and find a small change in the quantifiers to make it true.

2.4. Let A and B be sets of real numbers, let f be a function from $\mathbb{R}$ to $\mathbb{R}$, and let P be the set of positive real numbers. Without using words of negation, for each statement below write a sentence that expresses its negation.

a) For all $x \in A$, there is a $b \in B$ such that $b > x$.

b) There is an $x \in A$ such that, for all $b \in B$, $b > x$.

c) For all $x, y \in \mathbb{R}$, $f(x) = f(y) \Rightarrow x = y$.

d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$.

e) For all $x, y \in \mathbb{R}$ and all $\epsilon \in P$, there is a $\delta \in P$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

f) For all $\epsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

2.5. (–) Prove the following statements.

a) For all real numbers y, b, m with $m \neq 0$, there is a unique real number x such that $y = mx + b$.

b) For all real numbers y, m, there exist $b, x \in \mathbb{R}$ such that $y = mx + b$.

2.6. (–) *Usage of language.*

a) The following sentence appeared on a restaurant menu: "Please note that every alternative may not be available at this time". Describe the unintended meaning. Rewrite the sentence to state the intended meaning clearly.

b) Give an example of an English sentence that has different meanings depending on inflection, pronunciation, or context.

2.7. (–) Describe how the notion of an *alibi* in a criminal trial fits into our discussion of conditional statements.

2.8. From outside mathematics, give an example of statements A, B, C such that A and B together imply C, but such that neither A nor B alone implies C.

2.9. (–) The negation of the statement "No slow learners attend this school" is:†

a) All slow learners attend this school.

b) All slow learners do not attend this school.

c) Some slow learners attend this school.

d) Some slow learners do not attend this school.

e) No slow learners attend this school.

2.10. Express each of the following statements as a conditional statement in "if-then" form or as a universally quantified statement. Also write the negation (without phrases like "it is false that").

a) Every odd number is prime.

b) The sum of the angles of a triangle is 180 degrees.

c) Passing the test requires solving all the problems.

d) Being first in line guarantees getting a good seat.

e) Lockers must be turned in by the last day of class.

f) Haste makes waste.

g) I get mad whenever you do that.

h) I won't say that unless I mean it.

2.11. (!) Suppose I have a penny, a dime, and a dollar, and I say, "If you make a true statement, I will give you one of the coins. If you make a false statement, I will give you nothing." What should you say to obtain the best coin?

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2.12. A telephone bill y (in cents) is determined by $y = mx + b$, where x is the number of calls during the month, and b is a constant monthly charge. Suppose that the bill is \$5.48 when 8 calls are made and is \$5.72 when 12 calls are made. Determine what the bill will be when 20 calls are made.

2.13. "In one year, my wife will be one-third as old as my house. In nine years, I will be half as old as my house. I am ten years older than my wife. How old are my house, my wife, and I?" Answer the question, stating the needed equations.

2.14. A **circle** is the set of ordered pairs $(x, y) \in \mathbb{R}^2$ such that x and y satisfy an equation of the form $x^2 + y^2 + ax + by = c$, where $c > -(a^2 + b^2)/4$. The circle is specified by the parameters a, b, c.

a) Using this definition, give examples of two circles such that

i) the circles do not intersect.

ii) the circles intersect in exactly one common element.

iii) the circles intersect in two common elements.

b) Explain why the parameter c is restricted as given.

2.15. *The quadratic formula, revisited.* We derive the quadratic formula by solving a system of linear equations for the two unknown solutions. The equation $ax^2 + bx + c = 0$ with $a \neq 0$ has real solutions r, s if and only if $ax^2 + bx + c = a(x - r)(x - s)$ (see Exercise 1.20). The calculation below shows that the factorization exists if and only if $b^2 - 4ac \geq 0$ and expresses r, s in terms of a, b, c.

a) By equating coefficients of corresponding powers of x, obtain the equations

†From the 1955 High School Mathematics Exam (C. T. Salkind, *Annual High School Mathematics Examinations 1950–1960*, Math. Assoc. Amer. 1961, p. 37.)

$r+s=-b/a$ and $rs=c/a$. Use these to prove that $(r-s)^2=(b^2-4ac)/a^2$.

b) From (a), obtain $r+s=-b/a$ and $r-s=\sqrt{b^2-4ac}/a$. Solve this system for r,s in terms of a,b,c.

c) What happens if the second equation in (b) is $r-s=-\sqrt{b^2-4ac}/a$?

2.16. (!) Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.

a) Prove that f can be expressed in a unique way as the sum of two functions g and h such that $g(-x)=g(x)$ for all $x\in\mathbb{R}$ and $h(-x)=-h(x)$ for all $x\in\mathbb{R}$. (Hint: Find a system of linear equations for the unknowns $g(x)$ and $h(x)$ in terms of the known values $f(x)$ and $f(-x)$.)

b) When f is a polynomial, express g and h in terms of the coefficients of f.

2.17. Given $f\colon\mathbb{R}\to\mathbb{R}$, let $g(x)=\frac{x}{2}+\frac{x}{f(x)-1}$ for all x such that $f(x)\neq 1$. Suppose $g(x)=g(-x)$ for all such x. Prove that $f(x)f(-x)=1$ for all such x.

2.18. (!) Given a polynomial p, let A be the sum of the coefficients of the even powers, and let B be the sum of the coefficients of the odd powers. Prove that $A^2-B^2=p(1)p(-1)$.

2.19. Abraham Lincoln said, "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time." Write this sentence in logical notation, negate the symbolic sentence, and state the negation in English. Which statement seems to be true?

2.20. Using quantifiers, explain what it would mean for the first player to have a "winning strategy" in Tic-Tac-Toe. (Don't consider whether the statement is true.)

2.21. Consider the sentence "For every integer $n>0$ there is some real number $x>0$ such that $x<1/n$". Without using words of negation, write a complete sentence that means the same as "It is false that for every integer $n>0$ there is some real number $x>0$ such that $x<1/n$". Which sentence is true?

2.22. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Without using words of negation, write the meaning of "f is not an increasing function".

2.23. Consider $f\colon\mathbb{R}\to\mathbb{R}$. Let S be the set of functions defined by putting $g\in S$ if there exist positive constants $c,a\in\mathbb{R}$ such that $|g(x)|\le c\,|f(x)|$ for all $x>a$. Without words of negation, state the meaning of "$g\notin S$". (Comment: The set S (written as $O(f)$) is used to compare the "order of growth" of functions.)

2.24. In simpler language, describe the meaning of the following two statements and their negations. Which one implies the other, and why?

a) There is a number M such that, for every x in the set S, $|x|\le M$.

b) For every x in the set S, there is a number M such that $|x|\le M$.

2.25. For $a\in\mathbb{R}$ and $f\colon\mathbb{R}\to\mathbb{R}$, show that (a) and (b) have different meanings.

a) $(\forall\epsilon>0)(\exists\delta>0)[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$

b) $(\exists\delta>0)(\forall\epsilon>0)[(|x-a|<\delta)\Rightarrow(|f(x)-f(a)|<\epsilon)]$

2.26. For $f\colon\mathbb{R}\to\mathbb{R}$, which of the statements below implies the other? Does there exist a function for which both statements are true?

a) For every $\epsilon>0$ and every real number a, there is a $\delta>0$ such that $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$.

b) For every $\epsilon>0$, there is a $\delta>0$ such that $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$ and a is a real number.

2.27. (+) For $c\in\mathbb{R}$ and $f\colon\mathbb{R}\to\mathbb{R}$, interpret each statement below.

a) For all $x\in\mathbb{R}$ and all $\delta>0$, there exists $\epsilon>0$ such that $|x|<\delta$ implies $|f(x)-c|<\epsilon$.

b) For all $x\in\mathbb{R}$, there exists $\delta>0$ such that, for all $\epsilon>0$, we have $|x|<\delta$ implies $|f(x)-c|<\epsilon$.

2.28. (!) Consider the equation $x^4y+ay+x=0$.

a) Show that the following statement is false. "For all $a,x\in\mathbb{R}$, there is a unique y such that $x^4y+ay+x=0$."

b) Find the set of real numbers a such that the following statement is true. "For all $x\in\mathbb{R}$, there is a unique y such that $x^4y+ay+x=0$."

2.29. (!) *Extremal problems.*

a) Let f be a real-valued function on S. In order to prove that the minimum value in the image of f is β, two statements must be proved. Express each of these statements using quantifiers.

b) Let T be the set of ordered pairs of positive real numbers. Define $f\colon T\to\mathbb{R}$ by $f(x,y)=\max\{x,y,\frac{1}{x}+\frac{1}{y}\}$. Determine the minimum value in the image of f. (Hint: What must a pair achieving the minimum satisfy?)

2.30. (!) Consider tokens that have some letter written on one side and some integer written on the other, in unknown combinations. The tokens are laid out, some with letter side up, some with number side up. Explain which tokens must be turned over to determine whether these statements are true:

a) Whenever the letter side is a vowel, the number side is odd.

b) The letter side is a vowel if and only if the number side is odd.

2.31. Which of these statements are believable? (Hint: Consider Remark 2.34.)

a) "All of my 5-legged dogs can fly."

b) "I have no 5-legged dog that cannot fly."

c) "Some of my 5-legged dogs cannot fly."

d) "I have a 5-legged dog that cannot fly."

2.32. A fraternity has a rule for new members: each must always tell the truth or always lie. They know who does which. If I meet three of them on the street and they make the statements below, which ones (if any) should I believe?

A says: "All three of us are liars."

B says: "Exactly two of us are liars."

C says: "The other two are liars."

2.33. Three children are in line. From a collection of two red hats and three black hats, the teacher places a hat on each child's head. The third child sees the hats on two heads, the middle child sees the hat on one head, and the first child sees no hats. The children, who reason carefully, are told to speak out as soon as they can determine the color of the hat they are wearing. After 30 seconds, the front child correctly names the color of her hat. Which color is it, and why?

2.34. (!) For each statement below about natural numbers, decide whether it is true or false, and prove your claim using only properties of the natural numbers.

a) If $n\in\mathbb{N}$ and $n^2+(n+1)^2=(n+2)^2$, then $n=3$.

b) For all $n\in\mathbb{N}$, it is false that $(n-1)^3+n^3=(n+1)^3$.

2.35. Prove that if x and y are distinct real numbers, then $(x+1)^2=(y+1)^2$ if and only if $x+y=-2$. How does the conclusion change if we allow $x=y$?

2.36. Let x be a real number. Prove that if $|x-1|<1$, then $\left|x^2-4x+3\right|<3$.

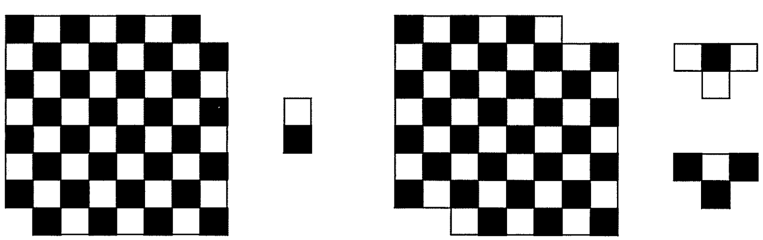
2.37. Given a real number x, let A be the statement "$\frac{1}{2}<x<\frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2=1$, and let D be the statement "$x=2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

2.38. Let x, y be integers. Determine the truth value of each statement below.
a) xy is odd if and only if x and y are odd.
b) xy is even if and only if x and y are even.

2.39. (!) A particle starts at the point $(0,0) \in \mathbb{R}^2$ on day 0. On each day, it moves one unit in a horizontal or vertical direction. For $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$, prove that it is possible for the particle to reach (a, b) on day k if and only if (1) $|a|+|b| \le k$, and (2) $a+b$ has the same parity as k.

2.40. (!) *Checkerboard problems.* (Hint: Use the method of contradiction.)
a) Two opposite corner squares are deleted from an eight by eight checkerboard. Prove that the remaining squares cannot be covered exactly by dominoes (rectangles consisting of two adjacent squares).
b) Two squares from each of two opposite corners are deleted as shown on the right below. Prove that the remaining squares cannot be covered exactly by copies of the "T-shape" and its rotations.



2.41. A clerk returns n hats to n people who have checked them, but not necessarily in the right order. For which k is it possible that exactly k people get a wrong hat? Phrase your conclusion as a biconditional statement.

2.42. A closet contains n different pairs of shoes. Determine the minimum t such that every choice of t shoes from the closet includes at least one matching pair of shoes. For $n>1$, what is the minimum t to guarantee that two matching pairs of shoes are obtained?

2.43. Using the equivalences discussed in Remark 2.20, write a chain of symbolic equivalences to prove that $P \Leftrightarrow Q$ is logically equivalent to $Q \Leftrightarrow P$.

2.44. Let P and Q be statements. Prove that the following are true.
a) $(Q \wedge \neg Q) \Rightarrow P$.
b) $P \wedge Q \Rightarrow P$.
c) $P \Rightarrow P \vee Q$.

2.45. Prove that the statements $P \Rightarrow Q$ and $Q \Rightarrow R$ imply $P \Rightarrow R$, and that the statements $P \Leftrightarrow Q$ and $Q \Leftrightarrow R$ imply $P \Leftrightarrow R$. (Comment: This is the justification for using a chain of equivalences to prove an equivalence.)

2.46. Prove that the logical expression S is equivalent to the logical expression $\neg S \Rightarrow (R \wedge \neg R)$, and explain the relationship between this equivalence and the method of proof by contradiction.

2.47. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "x^2-1 is divisible by 8". Determine whether the following statements are true:
a) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$.
b) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

2.48. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "x is twice an integer". Determine whether the following statements are true:
a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$.
b) $(\forall x \in \mathbb{Z})(P(x)) \Rightarrow (\forall x \in \mathbb{Z})(Q(x))$.

2.49. Let $S=\{x \in \mathbb{R}: x^2>x+6\}$. Let $T=\{x \in \mathbb{R}: x>3\}$. Determine whether the following statements are true, and interpret these results in words:
a) $T \subseteq S$.
b) $S \subseteq T$.

2.50. Prove the following identities involving complementation of sets.
a) $(A \cup B)^c = A^c \cap B^c$. (This is de Morgan's second law.)
b) $A \cap [(A \cap B)^c] = A - B$.
c) $A \cap [(A \cap B^c)^c] = A \cap B$.
d) $(A \cup B) \cap A^c = B - A$.

2.51. *Distributive laws for set operations.* Using statements about membership, prove the statements below, where A, B, C are any sets. Use Venn diagrams to illustrate the results and guide the proofs.
a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2.52. Let A, B, C be sets. Prove that $A \cap (B-C) = (A \cap B) - (A \cap C)$.

2.53. (!) Let A, B, C be sets. Prove that $(A \cup B) - C$ must be a subset of $[A-(B \cup C)] \cup [B-(A \cap C)]$, but that equality need not hold.

2.54. (+) Consider three circles in the plane as shown below. Each bounded region contains a token that is white on one side and black on the other. At each step, we can either (a) flip all four tokens inside one circle, or (b) flip the tokens showing white inside one circle to make all four tokens in that circle show black. From the starting configuration with all tokens showing black, can we reach the indicated configuration with all showing black except the token in the central region? (Hint: Consider parity conditions and work backward from the desired configuration.)