Math 115A
Final Exam

Friday, September 10, 2010

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Problem	Max	Score
1	10	
2	20	
3	10	
4	10	
5	10	
6	10	
Total	70	

1. (10 pts) Let F be a field, and let $S : P_3(F) \to F^6$ and $T : F^6 \to P_3(F)$ be linear transformations. Answer the following, and be sure to justify each answer briefly.

   (a) (2 pts) What are the maximum and minimum possible values of $\text{rank}(S)$?

   (b) (2 pts) What are the maximum and minimum possible values of $\text{rank}(T)$?

   (c) (2 pts) What are the maximum and minimum possible values of $\text{nullity}(T)$?

   (d) (2 pts) What are the maximum and minimum possible values of $\text{nullity}(ST)$?

   (e) (2 pts) What are the maximum and minimum possible values of $\text{rank}(ST)$?

2. (20 pts) Define a linear operator $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T(aX^2 + bX + c) = (4a + b)X^2 + (a + 4b)X + (2a + 2b + 3c).$$

(a) (5 pts) Let $\gamma = \{1, X, X^2\}$ be the standard basis of $P_2(\mathbb{R})$. Compute $[T]_\gamma$.

(b) (5 pts) Find the eigenvalues of T. State the algebraic multiplicity of each eigenvalue.

(c) (5 pts) For each eigenvalue λ, find a basis of the eigenspace E_λ of T (not just of $[T]_\gamma$!), and state the geometric multiplicity of λ.

(d) (5 pts) Is T diagonalizable? If so, write down an ordered basis β of $P_2(\mathbb{R})$ such that $[T]_\beta$ is diagonal, and write down the matrix $[T]_\beta$.

3. (10 pts) Let $V = M_{2\times 2}(\mathbb{R})$, and let

$$\beta = \left\{ \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -3 & 1 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix} \right\},$$
$$\gamma = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Both β and γ are bases of V. (You do not need to prove this.) Compute the change of coordinates matrix Q that changes β-coordinates to γ-coordinates. *(Hint:* $\dim(V) = 4$*, so your answer should be a* 4×4 *matrix.)*

4. (10 pts) Let $F = \mathbb{R}$ or $F = \mathbb{C}$, and let V be an inner product space over F. Prove that for any $x, y \in V$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Bonus: (3 pts) The formula above is called the parallelogram law. Explain geometrically what it says about vectors in $\mathbb{R}^2$. (A picture might help.)

5. (10 pts) Let V be a vector space, and let U and W be subspaces.

(a) (5 pts) Assume $V = U \oplus W$. Prove that for every $v \in V$, there exist *unique* vectors $u \in U$ and $w \in W$ such that $v = u + w$.

(b) (5 pts) Conversely, assume that for every $v \in V$, there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$. Prove that $V = U \oplus W$.

6. (10 pts) Let V be any vector space, and let T be a linear operator on V such that $T^2 = T$.

   (a) (5 pts) Prove that $V = R(T) + N(T)$. *(Hint: For $x \in V$, write $x = T(x) + (x - T(x))$, and show that $x - T(x) \in N(T)$.)*

   (b) (5 pts) Prove that $V = R(T) \oplus N(T)$.