# Math 110A
# Final Exam

Thursday, July 29, 2010

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

| Problem | Max | Score |
| --- | --- | --- |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 20 | |
| 6 | 20 | |
| Total | 80 | |

1. (10 pts) Solve the following congruences. If there are multiple solutions, be sure to find all of them. If there is no solution, say so, and be sure to justify your answer.

   (a) (5 pts) $8x \equiv 11 \pmod{30}$

   (b) (5 pts) $8x \equiv 11 \pmod{31}$

2. (10 pts) Let $a = 3X^2 + X + 2 \in \mathbb{Z}_7[X]$. Compute the inverse of $[a]$ in $\mathbb{Z}_7[X]/(p)$ where $p = X^3 + 4$.

3. (10 pts) Give an example of a field of order 125. Be sure to explain why your example is a field. *(Hint:* $125 = 5^3$*)*

4. (10 pts) Let $R$ be a commutative ring with identity. Prove that $R$ is an integral domain if and only if the ideal $(0)$ is prime.

5. (a) (6 pts) Let $R$ be a ring and let $I$ and $J$ be ideals in $R$. Let

$$I + J = \{i + j \mid i \in I \text{ and } j \in J\}.$$

Show that $I + J$ is an ideal in $R$.

(b) (7 pts) Let $a, b \in \mathbb{Z}$ with $a$ and $b$ not both 0, and let $d$ be the gcd of $a$ and $b$. Show that

$$(a) + (b) = (d).$$

(Note: Here $(n)$ refers to the principal ideal in $\mathbb{Z}$ generated by the integer $n$, and $(a)+(b)$ refers to the sum of the ideals $(a)$ and $(b)$ as defined on the previous page.)

(c) (7 pts) Let $F$ be a field, and let $p$ and $q$ be relatively prime polynomials in $F[X]$. Show that

$$(p) \cap (q) = (pq).$$

(Note: Here $(f)$ refers to the principal ideal in $F[X]$ generated by the polynomial $f$.)

6. (20 pts) Recall that $M_2(\mathbb{Z})$ denotes the ring of $2 \times 2$ matrices with integer coefficients. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z} \right\}.$$

(a) (5 pts) Show that $R$ is a commutative subring of $M_2(\mathbb{Z})$.

(b) (5 pts) Define a function $f : R \to \mathbb{Z}$ by $f(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)) = a$. Show that $f$ is a surjective homomorphism.

(c) (3 pts) What is the kernel of $f$?

(d) (3 pts) Prove that $R/\mathrm{Ker}(f) \cong \mathbb{Z}$.

(e) (4 pts) Is $\mathrm{Ker}(f)$ a prime ideal? Is it a maximal ideal? (Justify your answers.)