DEFORMATION CONDITIONS FOR PSEUDOREPRESENTATIONS

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ABSTRACT. Given a property of representations satisfying a basic stability condition, Ramakrishna developed a variant of Mazur's Galois deformation theory for representations with that property. We introduce an axiomatic definition of pseudorepresentations with such a property. Among other things, we show that pseudorepresentations with a property enjoy a good deformation theory, generalizing Ramakrishna's theory to pseudorepresentations.

CONTENTS

1. INTRODUCTION

The deformation theory of Galois representations has been used extensively to realize and study the proposed Langlands correspondence between Galois representations and automorphic representations. Depending on the setting in which the correspondence is being studied, one wants to set up the deformation theory so that it parameterizes exactly the Galois representations with certain properties. A collection of such properties is known as a "deformation condition" or "deformation datum." Ramakrishna [Ram93] proved that for any deformation condition $\mathcal{C}$ satisfying a basic stability property, the deformation problem for $\mathcal{C}$ is representable relative to the unrestricted deformation problem.

In applications, and especially when working with residually reducible Galois representations, Galois pseudorepresentations are often more accessible than Galois representations. A Galois pseudorepresentation is the data of a characteristic polynomial for each element of the Galois group, satisfying appropriate compatibility conditions. However, it has not been clear how to apply deformation conditions – which apply to Galois modules – to pseudorepresentations.

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The point of this paper is to solve this problem by axiomatically translating deformation conditions for representations into deformation conditions for pseudorepresentations. In particular, we construct universal deformation rings for pseudorepresentations satisfying such a condition. Also, it follows from our techniques that a deformation condition cuts out a closed locus in any family of Galois representations, generalizing the case of local families considered in [Ram93]. We expect that the various constructions we make, especially the universal Cayley-Hamilton algebra defined by a condition, will find many applications.

We were stimulated to develop this theory for application in the companion paper [WWE17]. Also, we expect that there exist Galois cohomological data controlling the deformation theory of these pseudodeformation rings, along the lines of the unrestricted case to be explained in [WE].

1.1. **Background.** To give a more thorough explanation of our results, we define our scope. Fix a prime p and a profinite group G. We assume that G satisfies the finiteness condition Φ_p (see §1.5). We are interested in studying the categories $\mathcal{R}\mathrm{ep}_G$ of representations of G and PsR_G of pseudorepresentations of G, and related moduli spaces Rep_G and PsR_G.

Let $\mathcal{C}$ be a condition on finite cardinality $\mathbb{Z}_p[G]$-modules such that $\mathcal{C}$ is closed under isomorphisms, subquotients, and finite direct sums (we will call such a condition *stable*). Ramakrishna [Ram93] showed that the formal deformation functor of representations with $\mathcal{C}$ is representable.

The work [WE15], building on works of Chenevier [Che14] and Bellaïche-Chenevier [BC09], furnishes the category $\mathcal{CH}_G$ of *Cayley-Hamilton representations of* G, which we use as the bridge between G-pseudorepresentations and G-modules. This allows us to formulate condition $\mathcal{C}$ for pseudorepresentations. The crucial properties of $\mathcal{CH}_G$ can be summarized as follows:

- $\mathcal{CH}_G$ broadens the category of representations of G, in the sense that there is an inclusion $\mathcal{R}\mathrm{ep}_G \hookrightarrow \mathcal{CH}_G$ and a functor $\psi : \mathcal{CH}_G \to \mathrm{PsR}_G$ extending the usual functor $\mathcal{R}\mathrm{ep}_G \to \mathrm{PsR}_G$ associating pseudorepresentations to representations.
- ψ is essentially surjective, which is not true for $\psi|_{\mathcal{R}\mathrm{ep}_G}$.
- $\mathcal{CH}_G$ has a universal object, the "universal Cayley-Hamilton representation" of G, which we denote by E_G.
- E_G has useful finiteness properties.

We review this theory in §2.2.

1.2. **Results.** In §2, relying on the results of [WE15], we develop the Cayley-Hamilton representations of G with condition $\mathcal{C}$ (Definition 2.5.2), cutting out a full subcategory $\mathcal{CH}_G^{\mathcal{C}} \subset \mathcal{CH}_G$. We develop the theory as follows:

(1) The category $\mathcal{CH}_G^{\mathcal{C}}$ has a universal object $E_G^{\mathcal{C}}$, which is a quotient of the universal object E_G in $\mathcal{CH}_G$ (Theorem 2.5.3).
(2) We define the category of pseudorepresentations satisfying condition $\mathcal{C}$, denoted by $\mathrm{PsR}_G^{\mathcal{C}}$, as the essential image $\mathrm{PsR}_G^{\mathcal{C}} \subset \mathrm{PsR}_G$ of ψ restricted to $\mathcal{CH}_G^{\mathcal{C}} \subset \mathcal{CH}_G$ (Definition 2.5.4).
(3) We define the notion of a faithful Cayley-Hamilton G-module M (it is a G-module with extra structures giving rise to a Cayley-Hamilton representation E_M of G). We prove that M satisfies $\mathcal{C}$ if and only if E_M satisfies $\mathcal{C}$ (Theorem 2.6.3).

(4) Property $\mathcal{C}$ is a Zariski closed condition on both Rep_G and PsR_G (Corollary 2.7.1).

The governing idea behind the arguments is that for an associative algebra $\mathcal{A}$, the left $\mathcal{A}$-module $\mathcal{A}$ has a useful universal property. While $\mathbb{Z}_p[G]$ seems too "large" to test $\mathcal{C}$, E_G is "small" enough to apply $\mathcal{C}$ yet still "large" enough to be universal.

In the remainder of the paper, we restrict our attention to the subcategories of $\mathcal{CH}_G$, $\mathcal{R}\mathrm{ep}_G$, and PsR_G that are *residually multiplicity-free* (see Definition 3.2.1). For the sake of this introduction, we abuse notation and use the same letters for the residually multiplicity-free versions. With this restriction, it is known that E_G admits the extra structure of a *generalized matrix algebra*, a theory developed by Bellaïche-Chenevier (see [BC09, §1.5] and [WWE15a, §4.1]). Using this extra structure, we prove the following results in §3 and §4, respectively:

(5) $E_G^{\mathcal{C}}$ is uniquely characterized in terms of $\mathcal{R}\mathrm{ep}_G^{\mathcal{C}}$ (Theorem 3.3.1).
(6) We relate the structure of $E_G^{\mathcal{C}}$ to Ext-groups with condition $\mathcal{C}$ (Theorem 4.3.5).

Remark 1.2.1. In the body of the paper, we mostly discuss variants of these results for the versions of the categories that have a fixed residual pseudorepresentation $\bar{D}$. To translate into the form stated in this introduction, one can apply the results of §2.2, particularly Theorem 2.2.5.

1.3. **Context.** We are motivated by the case that G is a quotient of an absolute Galois group of a global field and $\mathcal{C} = \{\mathcal{C}_v\}$ is a condition on the restrictions of representations of G to its decomposition groups G_v, especially those conditions $\mathcal{C}_v$ that arise from p-adic Hodge theory when $v \mid p$. We discuss the examples we have in mind in §5.

The unavailability of a notion of pseudorepresentations satisfying said conditions has been an obstacle to generalizing the use of the deformation theory of Galois representations to the residually reducible case. In particular, in the paper [WWE17], we require the particular condition $\mathcal{C}_v$ that a $\mathbb{Z}_p[G_v]$-module arises as the generic fiber of a finite flat group scheme over the ring of integers of a p-adic local field.

In our previous work [WWE15b, WWE15a], we considered the ordinary condition on 2-dimensional global Galois representations (see also [CS16] and [WE15, §7]). However, the ordinary condition is of a rather different flavor, as it does not apply readily to a finite cardinality $\mathbb{Z}_p[G]$-module without extra structure.

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1.5. **Notation and conventions.** Rings R are commutative and Noetherian, unless otherwise noted. Algebras are associative but not necessarily commutative, and are usually finitely generated. For an algebra E, the term "E-module" is used to mean "left E-module", unless otherwise stated, and we let Mod_E denote the category of left E-modules.

When A an local ring, to say that B is commutative local A-algebra means that B is a local ring equipped with a morphism of local rings $A \to B$.

We study integral p-adic representations and pseudorepresentations of a profinite group G, which is assumed to satisfy the Φ_p finiteness condition of Mazur, i.e. the maximal pro-p quotient of every finite index subgroup of G is topologically finitely generated.

We will work with categories of topological rings R discussed in §2.2, giving finitely generated R-modules a natural topology. Algebras E over R are also understood to have their natural topology, either being finitely generated as R-modules or of the form $R[G]$. Actions of G and homomorphisms $G \to E^\times$ are understood to be continuous. Pseudorepresentations $D : E \to R$ (resp. $D : G \to R$) are also understood to be continuous, which means that the characteristic polynomial coefficient functions $E \to R$ (resp. $G \to R$) defined in Definition 2.1.1(8) are continuous.

2. Deformation conditions for Cayley-Hamilton representations

In this section, we develop the theory of Cayley-Hamilton representations with a deformation condition $\mathcal{C}$.

We begin by reviewing from [Che14] and [WE15, §2] the non-topological theory Cayley-Hamilton algebras, their compatible representations, and moduli spaces of these. We refer the reader to *loc. cit.* and [WWE15b] for further background and examples.

2.1. **Pseudorepresentations, compatible representations, and Cayley-Hamilton algebras.**

The following definitions are due to Chenevier [Che14].

Definitions 2.1.1. Let $f : R \to R'$ be a ring homomorphism, and let E (resp. E') be an R-algebra (resp. R'-algebra), and let $g : E \otimes_R R' \to E'$ be a morphism of R'-algebras. Let $\mathcal{G}$ be a group.

(1) A *pseudorepresentation*, denoted $D : E \to R$ or (E, D), is a multiplicative polynomial law[1] of degree d from E to R, for some $d \geq 1$. We call d the *dimension* of D, and R the *scalar ring* of (E, D). The data of D includes a function $E \otimes_R A \to A$ for each commutative R-algebra A, which we denote by D_A.

(2) A *pseudorepresentation of $\mathcal{G}$ with coefficients in R*, denoted $D : \mathcal{G} \to R$, is a pseudorepresentation of $R[\mathcal{G}]$.

(3) Let $D : E \to R$ be a pseudorepresentation. The *base-change of D by f*, denoted $f \circ D : E \otimes_R R' \to R'$, is the pseudorepresentation of $E \otimes_R R'$ defined by $(f \circ D)_A(x) = D_A(x)$, where A is a commutative R'-algebra, and $x \in E \otimes_R A = (E \otimes_R R') \otimes_{R'} A$.

When f is understood, we write $D \otimes_R R'$ instead of $f \circ D$.

(4) Let $D' : E' \to R'$ be a pseudorepresentation. The *composition of D' and g*, denoted $D' \circ g : E \otimes_R R' \to R'$, is the pseudorepresentation of $E \otimes_R R'$ defined by $(D' \circ g)_A(x) = D'_A((g \otimes_{R'} A)(x))$, where A is a commutative R'-algebra, $x \in E \otimes_{R'} A$.

(5) A *morphism of pseudorepresentations* $\rho : (E, D) \to (E', D')$ is the data of a pair (f, g), such that $f \circ D = D' \circ g$. We define $\psi(\rho) = f \circ D = D' \circ g$.

[1]A polynomial law is a natural transformation $D : \underline{E \otimes_R (-)} \to \underline{(-)}$ of set-valued functors on commutative R-algebras, where $\underline{(-)}$ is the forgetful functor from R-algebras to sets. It is multiplicative if, for each commutative R-algebra A, the function $D_A : E \otimes_R A \to A$ satisfies $D_A(1) = 1$, $D_A(xy) = D_A(x)D_A(y)$ for all $x, y \in E \otimes_R A$. It has degree d if, for each commutative R-algebra A, we have $D_A(bx) = b^d D_A(x)$ for all $x \in E \otimes_R A$ and all $b \in A$.

A *morphism of pseudorepresentations of* $\mathcal{G}$ is a morphism of pseudorepresentations $(R[\mathcal{G}], D) \to (R'[\mathcal{G}], D \otimes_R R')$ where the homomorphism $R'[\mathcal{G}] \to R'[\mathcal{G}]$ is the identity.

(6) If $D : E \to R$ is a pseudorepresentation and $\rho : \mathcal{G} \to E^\times$ is a homomorphism, there is an induced homomorphism $\tilde{\rho} : R[\mathcal{G}] \to E$, which defines a morphism of pseudorepresentations $(R[\mathcal{G}], D \circ \tilde{\rho}) \to (E, D)$. We abuse notation and denote this morphism by ρ, and write $\psi(\rho) : \mathcal{G} \to R$ for the resulting pseudorepresentation of $\mathcal{G}$.

(7) If $D : E \to R$ is a pseudorepresentation, and $x \in E$, we define the *characteristic polynomial* $\chi_D(x, t) \in R[t]$ by $\chi_D(x, t) = D_{R[t]}(t - x)$. It is monic of degree equal to the dimension d of D. We define the *trace* $\mathrm{Tr}_D(x)$ to be the additive inverse of the coefficient of t^{d-1} in $\chi_D(x, t)$.

Remark 2.1.2. This notion of pseudorepresentation is called a "determinant" by Chenevier. The notion of "pseudorepresentation" of a group was first considered by Wiles [Wil88, Lemma 2.2.3] in the case $d = 2$, and by Taylor [Tay91, §1] in general. Taylor's definition was also considered by Rouquier [Rou96], who considered R-algebras (not just groups), and called the resulting objects "pseudo-caractères" (or "pseudocharacters", in English). By [Che14, Lem. 1.12], $\mathrm{Tr}_D : E \to R$ is a "pseudocharacter" in the sense of Taylor and Rouquier. By [Che14, Prop. 1.29], the map $D \mapsto \mathrm{Tr}_D$ is a bijection if $(2d)! \in R^\times$.

Example 2.1.3. There is a d-dimensional pseudorepresentation $\det : M_d(R) \to R$ defined by letting $\det_A : M_d(A) \to A$ be the determinant for all commutative R-algebras A. More generally, if E is an Azumaya R-algebra of degree d, e.g. V is a projective R-module of rank d and $E = \mathrm{End}_R(V)$, then there is a d-dimensional pseudorepresentation $\det : E \to R$ given by the reduced norm on E.

Definition 2.1.4. Given a d-dimensional pseudorepresentation $D : E \to R$, by [WE15, §2] there exists an affine $\operatorname{Spec} R$-scheme representing the set-valued functor $\mathrm{Rep}^\square_{E,D}$ on commutative R-algebras given by

$$A \mapsto \{R\text{-algebra homomorphism } \rho_A : E \to M_d(A) \text{ such that } D \otimes_R A = \det \circ \rho_A\}.$$

We call elements of $\mathrm{Rep}^\square_{E,D}(A)$ *framed compatible representations of* (E, D).

We also consider the $\operatorname{Spec} R$-groupoid

$$\begin{aligned}\mathrm{Rep}_{E,D} : A \mapsto \{(&V_A \text{ a projective rank } d \text{ } A\text{-module}, \rho_A : E \to \mathrm{End}_A(V) \\ &\text{an } R\text{-algebra homomorphism}) \text{ such that } D \otimes_R A = \det \circ \rho_A\},\end{aligned}$$

where morphisms are the isomorphisms of such data. We call objects of $\mathrm{Rep}_{E,D}(A)$ *compatible representations of* (E, D).

The natural adjoint action of GL_d on $\mathrm{Rep}^\square_{E,D}$ provides a smooth presentation for $\mathrm{Rep}_{E,D}$ as a $\operatorname{Spec} R$-algebraic stack.

Definition 2.1.5 ([Che14, §1.17])**.** We call a pseudorepresentation $D : E \to R$ *Cayley-Hamilton* when E is finitely generated as an R-algebra and every element $x \in E$ satisfies the characteristic polynomial $\chi_D(x, t) \in R[t]$ associated to it by D. That is, D is Cayley-Hamilton when $\chi_D(x, x) = 0$ for all $x \in E$. When $D : E \to R$ is Cayley-Hamilton, we call the pair (E, D) a *Cayley-Hamilton R-algebra*.

If (E', D') is a Cayley-Hamilton algebra, and we refer to a morphism of pseudorepresentations $\rho : (E, D) \to (E', D')$ as a *Cayley-Hamilton representation* of

(E, D). In the case that $E = R[\mathcal{G}]$ for a group $\mathcal{G}$ and the map $E \to E'$ comes from a homomorphism $\rho : \mathcal{G} \to E'^{\times}$, we call this a *Cayley-Hamilton representation of* $\mathcal{G}$, and write it as (E', ρ, D'). If (E, D) is also Cayley-Hamilton, we also refer to a Cayley-Hamilton representation of (E, D) as a *morphism of Cayley-Hamilton algebras*.

A *morphism of Cayley-Hamilton representations of* $\mathcal{G}$, written $(E, \rho, D) \to (E', \rho', D')$, is a morphism of Cayley-Hamilton algebras $(E, D) \to (E', D')$ such that $\rho' = (E \to E') \circ \rho$.

We let $\mathcal{CH}_{\mathcal{G}}$ denote the category of Cayley-Hamilton representations of $\mathcal{G}$.

Proposition 2.1.6. *If (E, D) is a Cayley-Hamilton R-algebra, then E is finitely generated as an R-module.*

Proof. This follows from [WE15, Prop. 2.13] because E is finitely generated as an R-algebra (by definition of Cayley-Hamilton algebra) and R is Noetherian by convention. □

Example 2.1.7. A compatible representation $(E, D) \to (\mathrm{End}_A(V_A), \det)$ is an example of a Cayley-Hamilton representation of (E, D). Indeed, det is a Cayley-Hamilton pseudorepresentation by the Cayley-Hamilton theorem. This shows how the definition of Cayley-Hamilton representation broadens the notion of a compatible representation of (E, D).

2.2. **Representations of a profinite group.** For the remainder of the section, we fix a profinite group G that satisfies the Φ_p condition. Recall the conventions from §1.5 regarding continuity.

Let $\mathrm{Adm}_{\mathbb{Z}_p}$ denote the category of admissible topological (not necessarily Noetherian) rings where p is topologically nilpotent (see [Gro60, Def. 0.7.1.2, pg. 60] for the definition of admissible). We let $\mathrm{Tfg}_{\mathbb{Z}_p} \subset \mathrm{Adm}_{\mathbb{Z}_p}$ be the full subcategory of topologically finitely generated objects (i.e. those $A \in \mathrm{Adm}_{\mathbb{Z}_p}$ for which there exists a (non-topological) homomorphism $\mathbb{Z}_p[x_1, \dots, x_n] \to A$ with dense image), which are Noetherian rings. For $R \in \mathrm{Tfg}_{\mathbb{Z}_p}$ we define Tfg_R as the slice category of $\mathrm{Tfg}_{\mathbb{Z}_p}$ over R. We often use the following fact.

Lemma 2.2.1. *Let $R \in \mathrm{Tfg}_{\mathbb{Z}_p}$ and let $A \in \mathrm{Tfg}_R$. For any $x \in A$ non-zero, there exists an commutative local R-algebra of finite cardinality and an R-algebra homomorphism $f : A \to B$ such that $f(x) \neq 0$. In particular, a surjection $g : A \to A'$ in Tfg_R is determined by the natural transformation $\mathrm{Hom}_{R\text{-alg}}(A', -) \to \mathrm{Hom}_{R\text{-alg}}(A, -)$ of functors on commutative local R-algebras of finite cardinality.*

Proof. We leave this as an exercise. The main point is that, for any ideal of definition I for A, the ring A/I^n is a finitely generated $\mathbb{Z}$-algebra, and consequently a Noetherian Jacobson ring. □

Definition 2.2.2. For $A \in \mathrm{Tfg}_{\mathbb{Z}_p}$, a *representation of G with coefficients in A* is a finitely generated projective A-module V_A of constant rank with an A-linear G-action ρ_A. We write Rep_G for the category of representations, fibered in groupoids over $\mathrm{Tfg}_{\mathbb{Z}_p}$ via the forgetful functor $(V_A, \rho_A) \mapsto A$. We write $\mathrm{Rep}_G^{\square}$ for the category defined just as Rep_G, but with the additional data of an A-basis for V_A.

Write PsR_G for the category of pseudorepresentations of G, fibered in groupoids over $\mathrm{Tfg}_{\mathbb{Z}_p}$.

We write ψ for the functor $\psi : \mathrm{Rep}_G \to \mathrm{PsR}_G$ over $\mathrm{Tfg}_{\mathbb{Z}_p}$ that sends a representation ρ_A to its induced pseudorepresentation $\psi(\rho_A)$.

We decompose each of these categories by dimension $d \geq 1$, writing $\mathrm{Rep}_G^d \subset \mathrm{Rep}_G$, etc.

To understand PsR_G^d, we introduce deformations. Given a finite field $\mathbb{F}/\mathbb{F}_p$, we let $\hat{\mathcal{C}}_{W(\mathbb{F})} \subset \mathrm{Tfg}_{\mathbb{Z}_p}$ be the category of complete Noetherian commutative local $W(\mathbb{F})$-algebras $(A, \mathfrak{m}_A)$ with residue field $\mathbb{F}$.

Definition 2.2.3. Let $\bar{D} : G \to \mathbb{F}$ be a pseudorepresentation. Its deformation functor $\mathrm{PsDef}_{\bar{D}} : \hat{\mathcal{C}}_{W(\mathbb{F})} \to \mathrm{Sets}$ is

$$A \mapsto \{D : A[G] \to A \text{ such that } D \otimes_A \mathbb{F} \simeq \bar{D}\} \tag{2.2.4}$$

and elements of $\mathrm{PsDef}_{\bar{D}}(A)$ are called *deformations of* $\bar{D}$ or *pseudodeformations*.

Remarkably, when one varies $\bar{D}$ over a certain set of finite field-valued pseudorepresentations known as *residual pseudorepresentations* (see [WE15, Def. 3.4] for the definition), $\mathrm{PsDef}_{\bar{D}}$ captures all of PsR_G^d, in the following sense.

Theorem 2.2.5 (Chenevier)**.** *Assume G satisfies Φ_p.*

(1) Given a finite field $\mathbb{F}$ and a pseudorepresentation $\bar{D} : G \to \mathbb{F}$, the functor $\mathrm{PsDef}_{\bar{D}}$ is represented by an object $(R_{\bar{D}}, \mathfrak{m}_{\bar{D}}) \in \hat{\mathcal{C}}_{W(\mathbb{F})}$.

(2) There is an isomorphism

$$\mathrm{PsR}_G^d \cong \coprod_{\bar{D}} \mathrm{Spf}\, R_{\bar{D}}, \tag{2.2.6}$$

where $\bar{D}$ varies over d-dimensional residual pseudorepresentations.

Proof. See [Che14, Prop. 3.3, Prop. 3.7 and Cor. 3.14]. □

Let $\mathrm{Rep}_{\bar{D}}$ (resp. $\mathrm{Rep}_{\bar{D}}^{\square}$) denote the fiber in Rep_G (resp. $\mathrm{Rep}_G^{\square}$) of ψ over $\mathrm{Spf}\, R_{\bar{D}}$, where $\bar{D}$ is a residual pseudorepresentation, so that we have

$$\mathrm{Rep}_G^d \cong \coprod_{\bar{D}} \mathrm{Rep}_{\bar{D}}, \qquad \mathrm{Rep}_G^{\square,d} \cong \coprod_{\bar{D}} \mathrm{Rep}_{\bar{D}}^{\square} \tag{2.2.7}$$

where $\bar{D}$ varies over d-dimensional residual pseudorepresentations.

Now, and for the rest of this section, we fix a finite field $\mathbb{F}/\mathbb{F}_p$ and a pseudorepresentation $\bar{D} : G \to \mathbb{F}$. In light of the decompositions (2.2.6) and (2.2.7), we lose no scope in our study of p-adic families by fixing this choice.

Definition 2.2.8. Let $A \in \mathrm{Tfg}_{\mathbb{Z}_p}$. We say that a pseudorepresentation $D : G \to A$ *has residual pseudorepresentation* $\bar{D}$ when $\mathrm{Spf}\, A \to \mathrm{PsR}_G^d$ is concentrated over $\mathrm{Spf}\, R_{\bar{D}}$. We write $D_{\bar{D}}^u : G \to R_{\bar{D}}$ for the universal pseudodeformation of $\bar{D}$.

A Cayley-Hamilton representation $(E, \rho : G \to E^\times, D : E \to A)$ of G over $A \in \mathrm{Tfg}_{\mathbb{Z}_p}$ *has residual pseudorepresentation* $\bar{D}$ if $\psi(\rho) : G \to A$ has residual pseudorepresentation $\bar{D}$. In particular, a representation $(V_A, \rho_A) \in \mathrm{Rep}_G^d(A)$ has residual pseudorepresentation $\bar{D}$ when $\psi(\rho_A)$ does.

We let $\mathcal{CH}_{G,\bar{D}}$ denote the full subcategory of $\mathcal{CH}_G$ whose objects have residual pseudorepresentation $\bar{D}$.

We observe that $\mathrm{Rep}_{\bar{D}}$ parameterizes representations of G with residual pseudorepresentation $\bar{D}$. From [WE15, Thm. 3.8] we know that $\mathrm{Rep}_{\bar{D}}$ and $\mathrm{Rep}^{\square}_{\bar{D}}$ admit natural algebraic models over $\mathrm{Spec}\, R_{\bar{D}}$. By this we mean that there exists a finite type affine scheme (resp. algebraic stack) over $\mathrm{Spec}\, R_{\bar{D}}$ whose $\mathfrak{m}_{\bar{D}}$-adic completion is $\mathrm{Rep}^{\square}_{\bar{D}}$ (resp. $\mathrm{Rep}_{\bar{D}}$). This implies that $\mathrm{Rep}^{\square}_{\bar{D}}$ is an affine formal scheme and $\mathrm{Rep}_{\bar{D}}$ is a formal algebraic stack, both formally of finite type over $\mathrm{Spf}\, R_{\bar{D}}$.

This algebraic model arises from a canonical universal Cayley-Hamilton representation of G with residual pseudorepresentation $\bar{D}$, which we now define.

Theorem 2.2.9. *The category $\mathcal{CH}_{G,\bar{D}}$ has a universal object $(E_{\bar{D}}, \rho^u : G \to E^{\times}_{\bar{D}}, D^u_{\bar{D}} : E_{\bar{D}} \to R_{\bar{D}})$. In particular, $E_{\bar{D}}$ is a finitely generated $R_{\bar{D}}$-module. The map $\rho^u : R_{\bar{D}}[G] \to E_{\bar{D}}$ is surjective and $D^u_{\bar{D}} : E_{\bar{D}} \to R_{\bar{D}}$ is a factorization of the universal pseudodeformation $G \to R_{\bar{D}}$ through $E_{\bar{D}}$.*

Proof. See [WE15, Prop. 3.6]. □

We will continue to abuse notation by writing $D^u_{\bar{D}}$ for the factorization of the universal pseudorepresentation $D^u_{\bar{D}} : G \to R_{\bar{D}}$ through $E_{\bar{D}}$.

Theorem 2.2.10 ([WE15, Thm. 3.7])**.** *There is an isomorphism of topologically finite type formal algebraic stacks (resp. formal schemes) on $\mathrm{Tfg}_{\mathbb{Z}_p}$,*

$$\mathrm{Rep}_{\bar{D}} \xrightarrow{\sim} \mathrm{Rep}_{E_{\bar{D}}, D^u_{\bar{D}}}, \qquad \mathrm{Rep}^{\square}_{\bar{D}} \xrightarrow{\sim} \mathrm{Rep}^{\square}_{E_{\bar{D}}, D^u_{\bar{D}}}.$$

2.3. **Stability.** Let $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]} \subset \mathrm{Mod}_{\mathbb{Z}_p[G]}$ be the full subcategory whose objects have finite cardinality.

Definition 2.3.1. A *condition* on $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ is a full subcategory $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$. We will say an object of $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ *satisfies condition* $\mathcal{C}$ or *has* $\mathcal{C}$ if the object is in $\mathcal{C}$.

A condition $\mathcal{C}$ is *stable* if it is preserved under isomorphisms, subquotients, and finite direct sums in $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$. In other words, $\mathcal{C}$ is stable if

(1) for every object A in $\mathcal{C}$ and all isomorphisms $f : A \to B$ in $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$, the object B is also in $\mathcal{C}$, and
(2) for every object A in $\mathcal{C}$ and all morphisms $f : A \to B$ and $g : C \to A$ in $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$, the kernel of f and cokernel of g are in $\mathcal{C}$, and
(3) for every finite collection of objects $A_1, \dots, A_n$ of $\mathcal{C}$, the direct sum $A_1 \oplus \cdots \oplus A_n$ in $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ is an object of $\mathcal{C}$.

Example 2.3.2. Let $H \subset G$ be a normal subgroup. Let $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ be the full subcategory of objects where the G action factors through the quotient $G \to G/H$. Then $\mathcal{C}$ is stable (as $\mathcal{C} \cong \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G/H]}$).

Example 2.3.3. Let $H_1, \dots, H_n \subset G$ be subgroups, and, for $i = 1, \dots, n$, let $\mathcal{C}_i \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[H_i]}$ be a condition. Then there is a condition $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ defined by the Cartesian square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \prod_{i=1}^n \mathcal{C}_i \\ \downarrow & & \downarrow \\ \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]} & \longrightarrow & \prod_{i=1}^n \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[H_i]}. \end{array}$$

In other words, an object $M \in \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ has $\mathcal{C}$ if and only if the restriction $M|_{H_i} \in \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[H_i]}$ has $\mathcal{C}_i$ for all $i = 1, \dots, r$. If all $\mathcal{C}_i$ are stable, then this $\mathcal{C}$ is stable.

For examples of conditions $\mathcal{C}$ that are of use in arithmetic, see §5. For the rest of this section, we fix a stable condition $\mathcal{C}$.

Theorem 2.3.4 (Ramakrishna)**.** *Let A be a complete commutative Noetherian local $\mathbb{Z}_p$-algebra and let V_A be a finitely generated free A-module with an A-linear G-action. Then there exists a maximal quotient $A \twoheadrightarrow A^{\mathcal{C}}$ such that for an commutative local A-algebra B of finite cardinality, the $\mathbb{Z}_p[G]$-module $V_A \otimes_A B$ satisfies $\mathcal{C}$ if and only if $A \to B$ factors through $A^{\mathcal{C}}$.*

Proof. This follows immediately from [Ram93, Thm. 1.1]. □

Lemma 2.3.5. *Let $V \in \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ and let A be a commutative $\mathbb{Z}_p$-algebra.*

1. *There is a unique quotient $V \twoheadrightarrow V^{\mathcal{C}}$ such that $V^{\mathcal{C}}$ has $\mathcal{C}$ and such that any other quotient $V \twoheadrightarrow W$ where W has $\mathcal{C}$ factors uniquely through $V \to V^{\mathcal{C}}$.*
2. *Assume that $V \in \mathrm{Mod}_{A[G]}$ as well. Then $V^{\mathcal{C}} \in \mathrm{Mod}_{A[G]}$.*

Proof. (1) The uniqueness is clear because this is a universal property, so we must show existence. Since V has finite cardinality, there are a finite number of quotients of V. Let $\{V_1, \dots, V_n\}$ be the (possibly empty) set all of quotients of V that have property $\mathcal{C}$. Define $V \twoheadrightarrow V^{\mathcal{C}}$ to be the coimage of V under $V \to \oplus_{i=1}^n V_i$. Then $V^{\mathcal{C}}$ is isomorphic to a submodule of $\oplus_{i=1}^n V_i$. Since $\mathcal{C}$ is closed under isomorphisms, subobjects, and finite direct sums, we see that $V^{\mathcal{C}}$ satisfies $\mathcal{C}$. By definition, any of the maps $V \twoheadrightarrow V_i$ factor through $V \twoheadrightarrow V^{\mathcal{C}}$, and this factoring is unique since $V \twoheadrightarrow V^{\mathcal{C}}$ is an epimorphism.

(2) It suffices to show that $\ker(V \to V^{\mathcal{C}})$ is preserved by the A-action. Let $x \in \ker(V \to V^{\mathcal{C}})$ and $a \in A$, and let $m_a : V \to V$ denote multiplication by a. For a quotient $f : V \twoheadrightarrow W$ where W has $\mathcal{C}$, we have to show that $f(ax) = 0$. Let $g : V \twoheadrightarrow W'$ be the coimage of $f \circ m_a$, and let $V \xrightarrow{g} W' \xrightarrow{g'} W$ be the resulting factoring of $f \circ m_a$. Since g' is injective, we see that W' is isomorphic to a submodule of W and so W' has $\mathcal{C}$. This implies that $g(x) = 0$, and so $0 = g'(g(x)) = (f \circ m_a)(x) = f(ax)$. □

2.4. **Constructions.** In what follows, we will treat finite cardinality left $E_{\bar{D}}$-modules as objects of $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ via the map $\mathbb{Z}_p[G] \to E_{\bar{D}}$. By Proposition 2.1.6, $E_{\bar{D}}$ is finite as a $R_{\bar{D}}$-module. In particular, for any $i \geq 1$, the $\mathbb{Z}_p[G]$-module $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$ has finite cardinality, and thus is an object of $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$.

Lemma 2.4.1. *For any $i \geq 1$, there is a unique $E_{\bar{D}}$-module quotient $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \twoheadrightarrow E^{\mathcal{C}}_{\bar{D}}(i)$ such that $E^{\mathcal{C}}_{\bar{D}}(i)$ has $\mathcal{C}$ and such that any $E_{\bar{D}}$-module quotient $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \twoheadrightarrow W$ where W has $\mathcal{C}$ factors uniquely through $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \twoheadrightarrow E^{\mathcal{C}}_{\bar{D}}(i)$.*

Proof. By Theorem 2.2.9, $R_{\bar{D}}[G] \to E_{\bar{D}}$ is surjective, so the lattice of $E_{\bar{D}}$-quotients and $R_{\bar{D}}[G]$-quotients of $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$ are identical. By Lemma 2.3.5, we can take $E^{\mathcal{C}}_{\bar{D}}(i) = (E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}})^{\mathcal{C}}$. □

Lemma 2.4.2. *For any $i \geq 1$, there is a canonical isomorphism $E^{\mathcal{C}}_{\bar{D}}(i+1) \otimes_{R_{\bar{D}}} R_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} \xrightarrow{\sim} E^{\mathcal{C}}_{\bar{D}}(i)$.*

Proof. Let $E' = E^{\mathcal{C}}_{\bar{D}}(i+1) \otimes_{R_{\bar{D}}} R_{\bar{D}}/\mathfrak{m}^i_{\bar{D}}$. We show that E' satisfies the universal property of Lemma 2.4.1. The composite

$$E_{\bar{D}}/\mathfrak{m}^{i+i}_{\bar{D}} E_{\bar{D}} \twoheadrightarrow E^{\mathcal{C}}_{\bar{D}}(i+1) \twoheadrightarrow E'$$

factors through $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$, so E' is a quotient of $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$. Since $E^{\mathcal{C}}_{\bar{D}}(i+1)$ has $\mathcal{C}$ and E' is a quotient of it, we see that E' has $\mathcal{C}$.

Now suppose that $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \twoheadrightarrow W$ where W has $\mathcal{C}$. By the universal property of $E^{\mathcal{C}}_{\bar{D}}(i+1)$, the composite

$$E_{\bar{D}}/\mathfrak{m}^{i+1}_{\bar{D}} E_{\bar{D}} \twoheadrightarrow E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \twoheadrightarrow W$$

factors uniquely through a map $E^{\mathcal{C}}_{\bar{D}}(i+1) \twoheadrightarrow W$. Since W is a $R/\mathfrak{m}^i_{\bar{D}}$-module, this factors uniquely through $E' \twoheadrightarrow W$. □

Lemma 2.4.3. *For any $i \geq 1$, the module quotient $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \twoheadrightarrow E^{\mathcal{C}}_{\bar{D}}(i)$ has the following properties.*

(1) Let N be a left $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$-module that has finite cardinality. Then N satisfies condition $\mathcal{C}$ as a $\mathbb{Z}_p[G]$-module if and only if every map of left $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$-modules $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \to N$ factors through $E^{\mathcal{C}}_{\bar{D}}(i)$.

(2) There is a natural right action of $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$ on $E^{\mathcal{C}}_{\bar{D}}(i)$, making $E^{\mathcal{C}}_{\bar{D}}(i)$ a quotient $R_{\bar{D}}$-algebra of $E_{\bar{D}}$.

(3) An $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$-module N that has finite cardinality satisfies condition $\mathcal{C}$ if and only if its $E_{\bar{D}}$-action factors through $E^{\mathcal{C}}_{\bar{D}}(i)$.

Proof. (1) If N satisfies $\mathcal{C}$, then so does the image of any $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \to N$, so this arrow factors through $E^{\mathcal{C}}_{\bar{D}}(i)$ by Lemma 2.4.1. Conversely, if every such arrow factors through $E^{\mathcal{C}}_{\bar{D}}(i)$, then because of the finiteness assumption on N there exists some $m \in \mathbb{Z}_{\geq 1}$ and a surjective map $(E^{\mathcal{C}}_{\bar{D}}(i))^{\oplus m} \twoheadrightarrow N$. Consequently N satisfies $\mathcal{C}$.

(2) Choose $z \in E_{\bar{D}}$ and consider the composite morphism of left $E_{\bar{D}}$-modules (they are also in $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$)

$$E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \xrightarrow{(\ )\cdot z} E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \longrightarrow E^{\mathcal{C}}_{\bar{D}}(i)$$

where the leftmost arrow is right multiplication by z. The composite must factor through $E^{\mathcal{C}}_{\bar{D}}(i)$ by (1). The resulting map $E^{\mathcal{C}}_{\bar{D}}(i) \to E^{\mathcal{C}}_{\bar{D}}(i)$ gives the desired right action of z on $E^{\mathcal{C}}_{\bar{D}}(i)$ and shows that $E^{\mathcal{C}}_{\bar{D}}(i)$ is an $R_{\bar{D}}$-algebra.

(3) This follows directly from (1) and (2) in light of the following general fact: for an algebra E and a left E-module M, the E-action on M factors through a quotient algebra $E \twoheadrightarrow Q$ if and only if every morphism of left E-modules $E \to M$ factors through Q. This follows from the fact that any such $E \to M$ is of the form $x \mapsto x \cdot m$ for some $m \in M$. □

By Lemmas 2.4.2 and 2.4.3(2), we have an inverse system $\{E^{\mathcal{C}}_{\bar{D}}(i)\}$ of $R_{\bar{D}}$-algebra quotients of $E_{\bar{D}}$. Let $K^{\mathcal{C}}$ be the kernel of the $R_{\bar{D}}$-algebra quotient

$$E_{\bar{D}} \twoheadrightarrow \varprojlim_i E^{\mathcal{C}}_{\bar{D}}(i). \tag{2.4.4}$$

We construct the maximal Cayley-Hamilton quotient of $E_{\bar{D}}$ in which $K^{\mathcal{C}} = 0$.

Lemma 2.4.5. *Let $(E, D : E \to R)$ be a Cayley-Hamilton R-algebra. Let $I \subset E$ be a two-sided ideal. Let $J \subset R$ be the ideal generated by the non-constant coefficients of the polynomials in the set*

$$\{D_{R[t]}(1 - xt) \mid x \in I\} \subset R[t].$$

Let $E' := E/(I, J)$ and $R' := R/J$, and let $f : R \to R'$ and $g : E \to E'$ be the quotient maps.

(1) *There is a unique Cayley-Hamilton pseudorepresentation $D' : E' \to R'$ such that the pair (f, g) gives a morphism $(E, D) \to (E', D')$ of Cayley-Hamilton algebras.*
(2) *For any Cayley-Hamilton representation $\rho : (E, D) \to (E_A, D_A : E_A \to A)$, the map $E \to E_A$ sends I to 0 if and only if ρ factors through $(E, D) \to (E', D')$.*

Proof. (1) The uniqueness follows from (2). To show existence, we start with the pseudorepresentation $\tilde{D} := (f \circ D) : E \otimes_R R' \to R'$. To construct D', we use Chenevier's notion of *kernel of a pseudorepresentation* [Che14, §1.17]. It is defined by the following universal property: there is an R'-algebra quotient $h : E \otimes_R R' \to (E \otimes_R R')/\ker(\tilde{D})$ and a pseudorepresentation $\tilde{D}' : (E \otimes_R R')/\ker(\tilde{D}) \to R'$ such that $\tilde{D} = \tilde{D}' \circ h$, and $\ker(\tilde{D})$ is the maximal ideal with this property.

By [CS16, Lem. 2.1.2], there is an equality

$$\ker(\tilde{D}) = \{x \in E \otimes_R R' \mid \tilde{D}_{R'[t]}(1 - xyt) = 1 \ \forall y \in E \otimes_R R'\}.$$

By the definition of J, we have $\tilde{D}_{R'[t]}(1 - (x \otimes 1)t) = 1$ for all $x \in I$, and hence $\tilde{D}_{R'[t]}(1 - (x \otimes 1)yt) = 1$ for all $x \in I$ and $y \in E \otimes_R R'$, since I is a two-sided ideal. This implies that $I \otimes_R R' \subset \ker(\tilde{D})$. By the property of $\ker(\tilde{D})$, we have $D' : E' \to R'$ such that $f \circ D = D' \circ g$.

(2) If ρ factors through $(E, D) \to (E', D')$, then the map $E \to E_A$ factors through $E \to E' \to E_A$, so $E \to E_A$ sends I to 0. Conversely, let $\rho : (E, D) \to (E_A, D_A)$ be given by $f_A : R \to A$ and $g_A : E \to E_A$, and assume that $g_A(I) = 0$. To show that ρ factors through $(E, D) \to (E', D')$, it suffices to show that $f_A(J) = 0$. For any $x \in E$, the naturality of D implies that the image of $D_{R[t]}(1 - xt)$ in $A[t]$ under f_A is given by $D_{A[t]}(1 - xt) = (f_A \circ D)_{A[t]}(1 - xt)$. Hence it is enough to show that, for $x \in I$, we have $(f_A \circ D)_{A[t]}(1 - xt) = 1$. However, since ρ is a morphism of Cayley-Hamilton algebras, we have

$$(f_A \circ D)_{A[t]}(1 - xt) = (D_A \circ g_A)_{A[t]}(1 - xt) = (D_A)_{A[t]}(g_A(1 - xt)) = (D_A)_{A[t]}(1) = 1,$$

where we use the fact that $g_A(x) = 0$. □

Example 2.4.6. Let (E, D) be a Cayley-Hamilton R-algebra and let $I = JE$ with $J \subset R$ being an ideal. Then the Cayley-Hamilton quotient (E', D') of (E, D) by I has $E' = E/JE$ and scalar ring $R' = R/J$. In other words, $(E', D') = (E/JE, D \otimes_R R/J : E/JE \to R/J)$.

Definition 2.4.7. With the notation of the lemma, we call (E', D') the *Cayley-Hamilton quotient of (E, D) by I.*

Let $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}} : E^{\mathcal{C}}_{\bar{D}} \to R^{\mathcal{C}}_{\bar{D}})$ be the Cayley-Hamilton quotient of $(E_{\bar{D}}, D^{u}_{\bar{D}})$ by $K^{\mathcal{C}}$.

2.5. **Extending condition $\mathcal{C}$ to pseudorepresentations and Cayley-Hamilton representations.** We extend $\mathcal{C}$ to $A[G]$-modules that may not have finite cardinality in the following way.

Definition 2.5.1. Let $\mathcal{C}$ be a stable condition on objects of $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$. Let $(A, \mathfrak{m}_A)$ be a complete commutative Noetherian local $\mathbb{Z}_p$-algebra. For an $A[G]$-module M that is finitely generated as an A-module, we say that M *satisfies condition* $\mathcal{C}$ when $M/\mathfrak{m}_A^i M$ satisfies $\mathcal{C}$ for all $i \geq 1$.

Note that, for $(A, \mathfrak{m}_A)$ and M as in the definition, the canonical map $M \to \varprojlim_i M/\mathfrak{m}_A^i M$ is an isomorphism. Using this, one can check that this extension of $\mathcal{C}$ is stable in the same sense as Definition 2.3.1. We will use this extension of condition $\mathcal{C}$ without further comment.

Now we give the definition of condition $\mathcal{C}$ for Cayley-Hamilton representations.

Definition 2.5.2. Let $(A, \mathfrak{m}_A)$ be a complete commutative Noetherian local $\mathbb{Z}_p$-algebra and let (E, ρ, D) be a Cayley-Hamilton representation of G with scalar ring A and residual pseudorepresentation $\bar{D}$. We say that (E, ρ, D) *satisfies condition* $\mathcal{C}$ if E satisfies condition $\mathcal{C}$ as an $A[G]$-module. (Note that E is finitely generated as an A-module by Proposition 2.1.6.)

We let $\mathcal{CH}^{\mathcal{C}}_{G,\bar{D}}$ denote the full subcategory of $\mathcal{CH}_{G,\bar{D}}$ whose objects satisfy condition $\mathcal{C}$.

We can be confident that this notion behaves well by finding a universal object.

Theorem 2.5.3. *The Cayley-Hamilton representation* $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}} : E^{\mathcal{C}}_{\bar{D}} \to R^{\mathcal{C}}_{\bar{D}})$ *is the universal object in* $\mathcal{CH}^{\mathcal{C}}_{G,\bar{D}}$.

Proof. By the definition of $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}} : E^{\mathcal{C}}_{\bar{D}} \to R^{\mathcal{C}}_{\bar{D}})$, we see that the map $E_{\bar{D}} \to E^{\mathcal{C}}_{\bar{D}}$ sends $K^{\mathcal{C}}$ to 0. By Lemma 2.4.3, we see that $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}} : E^{\mathcal{C}}_{\bar{D}} \to R^{\mathcal{C}}_{\bar{D}})$ satisfies condition $\mathcal{C}$. We now show that it has the universal property.

Let A be a complete commutative Noetherian local $\mathbb{Z}_p$-algebra, and let (E, ρ, D) be a Cayley-Hamilton representation with scalar ring A and residual pseudorepresentation $\bar{D}$. We have to show that (E, ρ, D) satisfies condition $\mathcal{C}$ if and only if the map of Cayley-Hamilton algebras $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$ induced by the universal property of $(E_{\bar{D}}, \rho^u, D^u_{\bar{D}})$ factors through $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}})$.

For any $i \geq 1$, $E_{\bar{D}} \to E/\mathfrak{m}_A^i E$ factors through $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}}$. (Recall that a local homomorphism of scalar rings $R_{\bar{D}} \to A$ is implicit in $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$.)

By Lemma 2.4.3, (E, ρ, D) satisfies $\mathcal{C}$ if and only if $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \to E/\mathfrak{m}_A^i E$ factors through $E^{\mathcal{C}}_{\bar{D}}(i)$ for every $i \geq 1$. Equivalently, (E, ρ, D) satisfies $\mathcal{C}$ if and only if $E_{\bar{D}} \to E$ maps $K^{\mathcal{C}}$ to 0. By Lemma 2.4.5, $K^{\mathcal{C}}$ maps to 0 in $E_{\bar{D}} \to E$ if and only if $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$ factors through $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}})$. □

Following the pattern of [WWE15b, Defn. 5.9.1], we define condition $\mathcal{C}$ on pseudorepresentations.

Definition 2.5.4. Let A be an complete commutative Noetherian local $\mathbb{Z}_p$-algebra. Let $D : G \to A$ be a pseudorepresentation with residual pseudorepresentation $\bar{D}$. Then D *satisfies condition* $\mathcal{C}$ provided that there exists a Cayley-Hamilton representation (E, ρ, D') satisfying condition $\mathcal{C}$ such that $D = \psi(\rho) := D' \circ \rho$.

We define the $\mathcal{C}$-pseudodeformation functor $\mathrm{PsDef}^{\mathcal{C}}_{\bar{D}} : \hat{\mathcal{C}}_{W(\mathbb{F})} \to \mathrm{Sets}$ by

$$\mathrm{PsDef}^{\mathcal{C}}_{\bar{D}}(A) = \{\text{pseudodeformations } D : G \to A \text{ of } \bar{D} \text{ satisfying } \mathcal{C}\}.$$

Theorem 2.5.5. *The functor* $\mathrm{PsDef}^{\mathcal{C}}_{\bar{D}}$ *is represented by* $R^{\mathcal{C}}_{\bar{D}}$.

Proof. Let $A \in \hat{\mathcal{C}}_{W(\mathbb{F})}$, and let $D \in \mathrm{PsDef}_{\bar{D}}(A)$. By Theorem 2.2.5, there is unique $R_{\bar{D}} \to A$ such that $D \cong D^u_{\bar{D}} \otimes_{R_{\bar{D}}} A$. We have to show that $D \in \mathrm{PsDef}^{\mathcal{C}}_{\bar{D}}(A)$ if and only if $R_{\bar{D}} \to A$ factors through $R_{\bar{D}} \twoheadrightarrow R^{\mathcal{C}}_{\bar{D}}$.

If $R_{\bar{D}} \to A$ factors through $R_{\bar{D}} \twoheadrightarrow R^{\mathcal{C}}_{\bar{D}}$, then the Cayley-Hamilton representation

$$(E^{\mathcal{C}}_{\bar{D}} \otimes_{R^{\mathcal{C}}_{\bar{D}}} A, \rho^{\mathcal{C}} \otimes_{R_{\bar{D}}} A : G \to (E^{\mathcal{C}}_{\bar{D}} \otimes_{R^{\mathcal{C}}_{\bar{D}}} A)^\times, D^{\mathcal{C}}_{\bar{D}} \otimes_{R^{\mathcal{C}}_{\bar{D}}} A)$$

induces D via $D = (R^{\mathcal{C}}_{\bar{D}} \to A) \circ D^{\mathcal{C}}_{\bar{D}}$ and satisfies condition $\mathcal{C}$ by Theorem 2.5.3. Consequently D satisfies $\mathcal{C}$.

Now assume D satisfies condition $\mathcal{C}$, i.e. there exists a Cayley-Hamilton representation (E, ρ, D') satisfying $\mathcal{C}$ such that $D = D' \circ \rho$. By Theorem 2.5.3, there exists a morphism of Cayley-Hamilton algebras $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}}) \to (E, D)$ inducing ρ. In particular, the implicit morphism of scalar rings $R^{\mathcal{C}}_{\bar{D}} \to A$ factors $R_{\bar{D}} \to A$. □

2.6. **Modules with Cayley-Hamilton structure.** We introduce the notion of Cayley-Hamilton G-module.

Definition 2.6.1. Let $A \in \hat{\mathcal{C}}_{W(\mathbb{F})}$. A *Cayley-Hamilton G-module* over A is the data of a Cayley-Hamilton representation (E, ρ, D) of G with scalar ring A, and an E-module N. We consider N as a $A[G]$-module via the map $\rho : A[G] \to E$. We often refer to a to a Cayley-Hamilton G-module simply by the letter N, and call (E, D) the *Cayley-Hamilton algebra of N*. We say N is *faithful* if it is faithful as E-module.

Example 2.6.2. If N is an $A[G]$-module and there is a Cayley-Hamilton pseudorepresentation $D : \mathrm{End}_A(N) \to A$, then the canonical action of $\mathrm{End}_A(N)$ on N makes N a faithful Cayley-Hamilton G-module with Cayley-Hamilton algebra $(\mathrm{End}_A(N), \rho, D)$, where $\rho : G \to \mathrm{End}_A(N)$ is the action map.

Theorem 2.6.3. *Let N be a faithful Cayley-Hamilton G-module with Cayley-Hamilton A-algebra (E, D). Then N satisfies condition $\mathcal{C}$ as an $A[G]$-module if and only if (E, ρ, D) satisfies condition $\mathcal{C}$ as a Cayley-Hamilton representation.*

Proof. By Definition 2.5.1, it suffices to prove the theorem in the case that A is Artinian. Choose $i \geq 1$ such that $\mathfrak{m}^i_{\bar{D}} \cdot A = 0$. Let $\bar{D} : G \to \mathbb{F}$ be the residual pseudorepresentation of $D \circ \rho : G \to A$.

By Theorem 2.2.9, there is a morphism of Cayley-Hamilton algebras $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$. Then the action map $\mathbb{Z}_p[G] \to \mathrm{End}_A(N)$ factors as

$$\mathbb{Z}_p[G] \to E_{\bar{D}} \to E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \to E \to \mathrm{End}_A(N). \tag{2.6.4}$$

Then we claim that the following are equivalent:

(1) The $\mathbb{Z}_p[G]$-module N satisfies $\mathcal{C}$
(2) The map $E_{\bar{D}}/\mathfrak{m}^i_{\bar{D}} E_{\bar{D}} \to \mathrm{End}_A(N)$ factors through $E^{\mathcal{C}}_{\bar{D}}(i)$
(3) The map $E_{\bar{D}} \to \mathrm{End}_A(N)$ sends $K^{\mathcal{C}}$ to 0
(4) The map $E_{\bar{D}} \to E$ sends $K^{\mathcal{C}}$ to 0
(5) The morphism of Cayley-Hamilton algebras $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$ factors through $(E^{\mathcal{C}}_{\bar{D}}, D^{\mathcal{C}}_{\bar{D}})$

(6) The Cayley-Hamilton representation (E, D) satisfies condition $\mathcal{C}$.

The equivalences are proven as follows:

(1) $\iff$ (2): Lemma 2.4.3(3).
(2) $\iff$ (3): From the definition of $K^{\mathcal{C}}$ and (2.6.4).
(3) $\iff$ (4): Since N is faithful, the map $E \to \mathrm{End}_A(N)$ is injective.
(4) $\iff$ (5): Lemma 2.4.5.
(5) $\iff$ (6): Theorem 2.5.3. $\square$

2.7. **Formal moduli of representations with $\mathcal{C}$.** We generalize Ramakrishna's result (Theorem 2.3.4) to any family of integral p-adic representations of G.

Theorem 2.7.1. *There exists a unique closed formal substack (resp. closed formal subscheme)*

$$\mathrm{Rep}_G^{\mathcal{C},d} \subset \mathrm{Rep}_G^d, \qquad (\textit{resp. } \mathrm{Rep}_G^{\square,\mathcal{C},d} \subset \mathrm{Rep}_G^{\square,d})$$

characterized by the following property. For any commutative local $\mathbb{Z}_p$-algebra B of finite cardinality and free rank d B-module V_B with a B-linear G-action (resp. and fixed basis), the corresponding B-point of Rep_G^d lies in $\mathrm{Rep}_G^{\mathcal{C},d}$ (resp. of $\mathrm{Rep}_G^{\square,d}$ lies in $\mathrm{Rep}_G^{\square,\mathcal{C},d}$) if and only if V_B has $\mathcal{C}$.

Proof. It suffices to consider the case of $\mathrm{Rep}_G^{\square,d}$. Indeed, since condition $\mathcal{C}$ does not depend on the choice of basis, the closed subscheme $\mathrm{Rep}_G^{\square,\mathcal{C},d} \subset \mathrm{Rep}_G^{\square,d}$ descends to a closed locus in Rep_G^d.

By Theorem 2.2.5, we may consider a fixed residual pseudorepresentation $\bar{D}$ and produce $\mathrm{Rep}_{\bar{D}}^{\square,\mathcal{C}} \subset \mathrm{Rep}_{\bar{D}}^{\square}$. We define $\mathrm{Rep}_{\bar{D}}^{\square,\mathcal{C}}$ via the pullback diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\bar{D}}^{\square,\mathcal{C}} & \longrightarrow & \mathrm{Rep}_{\bar{D}}^{\square} \\ \downarrow & & \downarrow \wr \\ \mathrm{Rep}_{E_{\bar{D}}^{\mathcal{C}},D_{\bar{D}}^{\mathcal{C}}}^{\square} & \longrightarrow & \mathrm{Rep}_{E_{\bar{D}},D_{\bar{D}}}^{\square} \end{array}$$

where the right vertical arrow is the isomorphism in Theorem 2.2.10.

Let B be an local $\mathbb{Z}_p$-algebra of finite cardinality. By definition, a point $V_B \in \mathrm{Rep}_{\bar{D}}^{\square}(B)$ lies in $\mathrm{Rep}_{\bar{D}}^{\square,\mathcal{C}}(B)$ if and only if the map $(E_{\bar{D}}, D_{\bar{D}}) \to (\mathrm{End}_B(V_B), \det)$ factors through $(E_{\bar{D}}^{\mathcal{C}}, D_{\bar{D}}^{\mathcal{C}})$. By Theorem 2.5.3, this occurs if and only if the Cayley-Hamilton representation $(\mathrm{End}_B(V_B), \det)$ has $\mathcal{C}$, which, by Theorem 2.6.3, is if and only if V_B has $\mathcal{C}$. By Lemma 2.2.1, this characterizes $\mathrm{Rep}_{\bar{D}}^{\square,\mathcal{C}}$. $\square$

Corollary 2.7.2. *Let $A \in \mathrm{Tfg}_{\mathbb{Z}_p}$ and $(V_A, \rho_A) \in \mathrm{Rep}_G^d(A)$. There exists a unique quotient $A \twoheadrightarrow A^{\mathcal{C}}$ such that, for any local A-algebra B of finite cardinality, the object $V_A \otimes_A B$ of $\mathrm{Mod}_{\mathbb{Z}_p[G]}^{\mathrm{fin}}$ satisfies $\mathcal{C}$ if and only if the homomorphism $A \to B$ factors through $A^{\mathcal{C}}$.*

Proof. The quotient $A \to A^{\mathcal{C}}$ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{Spec}(A^{\mathcal{C}}) & \longrightarrow & \mathrm{Rep}_G^{\mathcal{C},d} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Rep}_G^d. \end{array}$$

The characterizing property follows from the characterizing property of $\mathrm{Rep}_G^{\mathcal{C},d}$. □

3. Deformation conditions for generalized matrix algebras

In this section, we assume that the pseudorepresentation $\bar{D}$ is multiplicity-free (see Definition 3.2.1). Under this assumption, the universal Cayley-Hamilton algebra with residual pseudorepresentation $\bar{D}$ admits the additional structure of a generalized matrix algebra (GMA). We show that this additional structure affords an alternate characterization of the the universal Cayley-Hamilton algebra with property $\mathcal{C}$. In the §4, we will use the GMA structure to study extensions.

3.1. Generalized matrix algebras and their adapted representations.

A generalized matrix algebra is a particular kind of Cayley-Hamilton algebra with extra data. We learned this notion from Bellaïche-Chenevier [BC09].

Definition 3.1.1 ([BC09, §1.3])**.** Let R be a ring. A *generalized matrix algebra over* R (or an *R-GMA*) is the data of:

(1) An R-algebra E that is finitely generated as an R-module,
(2) A set of orthogonal idempotents $e_1, \dots, e_r \in E$ such that $\sum_i e_i = 1$, and
(3) A set of isomorphisms of R-algebras $\phi_i : e_i E e_i \xrightarrow{\sim} M_{d_i}(R)$ for $i = 1, \dots, r$.

We call $\mathcal{E} = (\{e_i\}, \{\phi_i\})$ the *data of idempotents* or *GMA structure* of E, and write the R-GMA as $(E, \mathcal{E})$. We call the list of integers $(d_1, \dots, d_r)$ the *type* of $(E, \mathcal{E})$. These data are required to satisfy the condition that the function $\mathrm{Tr}_{\mathcal{E}} : E \to A$ defined by

$$\mathrm{Tr}_{\mathcal{E}}(x) := \sum_{i=1}^{r} \mathrm{tr}(\phi_i(e_i x e_i))$$

is a central function, i.e. $\mathrm{Tr}_{\mathcal{E}}(xy) = \mathrm{Tr}_{\mathcal{E}}(yx)$ for all $x, y \in E$.

Given an R-GMA $(E, \mathcal{E})$ and an R'-GMA $(E', \mathcal{E}')$, a *morphism of GMAs* $(E, \mathcal{E}) \to (E', \mathcal{E}')$ is the data of a ring homomorphism $f : R \to R'$ and morphism of R-algebras $g : E \to E'$ such that $\mathcal{E}$ and $\mathcal{E}'$ are of the same type $(d_1, \dots, d_r)$, we have $g(e_i) = e'_i$ and $f \circ \phi_i = \phi'_i \circ g$ for $i = 1, \dots, r$.

Example 3.1.2. The matrix algebra $M_d(R)$ comes with a natural R-GMA structure $\mathcal{E} = (1, \mathrm{id} : M_d(R) \xrightarrow{\sim} M_d(R))$ of type (d). More generally, given any ordered partition of $d = d_1 + \ldots + d_r$ of d, there is a natural R-GMA structure $\mathcal{E}_{\mathrm{block}}$ on $M_d(R)$ of type $(d_1, \dots, d_r)$. Namely, the natural R-algebra map with block-diagonal image

$$\nu_1 \times \cdots \times \nu_r : M_{d_1}(R) \times \cdots \times M_{d_r}(R) \hookrightarrow M_d(R)$$

induces $\mathcal{E}_{\mathrm{block}} = (e_i = \nu_i(1_i), \phi_i)$, where $1_i \in M_{d_i}(R)$ is the identity matrix, and ϕ_i is the inverse to ν_i on its image $e_i M_d(R) e_i$.

Lemma 3.1.3. *Given an R-GMA $(E, \mathcal{E})$, there is a canonical Cayley-Hamilton pseudorepresentation $D_{\mathcal{E}} : E \to R$, such that $\mathrm{Tr}_{D_{\mathcal{E}}} = \mathrm{Tr}_{\mathcal{E}}$. A morphism of R-GMAs $(E, \mathcal{E}) \to (E', \mathcal{E}')$ induces a morphism of Cayley-Hamilton algebras $(E, D_{\mathcal{E}}) \to (E', D_{\mathcal{E}'})$.*

Proof. See [WE15, Prop. 2.23]; its statement includes the first claim. The second claim follows from examining the formula for $D_{\mathcal{E}}$ given in *loc. cit.* and noting that a morphism of GMAs preserves the idempotents that are used to specify $D_{\mathcal{E}}$. □

This lemma gives a faithful embedding of the category of R-GMAs into the category of R-Cayley-Hamilton algebras. We will consider this embedding as an inclusion. We extend the definition of Cayley-Hamilton representation to GMAs as follows.

Definition 3.1.4. If $(E', \mathcal{E}')$ is a GMA, we refer to a morphism of pseudorepresentations $(E, D) \to (E', D_{\mathcal{E}'})$ as a *GMA representation* of (E, D). If $(E, \mathcal{E})$ is another GMA, we call a GMA representation $(E, D_{\mathcal{E}}) \to (E', D_{\mathcal{E}'})$ *adapted* if the same data give a morphism of GMAs $(E, \mathcal{E}) \to (E', \mathcal{E}')$.

Lemma 3.1.5. *Given an R-GMA $(E, \mathcal{E})$ of type $(d_1, \dots, d_r)$, we can associate to it the data of*

(1) R-modules $\mathcal{A}_{i,j}$ for $1 \le i, j \le r$,
(2) canonical isomorphisms $\mathcal{A}_{i,i} \xrightarrow{\sim} R$ for $1 \le i \le r$, and
(3) R-module homomorphisms $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_R \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ for $1 \le i, j, k \le r$,

such that $(\mathcal{A}_{i,j}, \varphi_{i,j,k})$ completely determine $(E, \mathcal{E})$ and there is an isomorphism of R-modules

$$E \xrightarrow{\sim} \begin{pmatrix} M_{d_1}(\mathcal{A}_{1,1}) & M_{d_1 \times d_2}(\mathcal{A}_{1,2}) & \cdots & M_{d_1 \times d_r}(\mathcal{A}_{1,r}) \\ M_{d_2 \times d_1}(\mathcal{A}_{2,1}) & M_{d_2}(\mathcal{A}_{2,2}) & \cdots & M_{d_2 \times d_r}(\mathcal{A}_{2,r}) \\ \vdots & \vdots & \vdots & \vdots \\ M_{d_r \times d_1}(\mathcal{A}_{r,1}) & M_{d_2}(\mathcal{A}_{r,2}) & \cdots & M_{d_r}(\mathcal{A}_{r,r}) \end{pmatrix}, \tag{3.1.6}$$

Moreover, the collection of maps $\varphi_{i,j,k}$ satisfies certain properties (UNIT), (COM), and (ASSO), and there is a bijection between R-GMAs of type $(d_1, \dots, d_r)$ and data $(\mathcal{A}_{i,j}, \varphi_{i,j,k})$ satisfying (UNIT), (COM), and (ASSO).

Proof. This is explained in [BC09, §1.3.1-§1.3.6]. The association is given as follows. Let $E_i := \phi_i^{-1}(\delta^{1,1})$, where $\delta^{1,1}$ denotes the elementary matrix with 1 as $(1,1)$th entry, and 0 otherwise. Define $\mathcal{A}_{i,j} := E_j E E_i$. The maps $\varphi_{i,j,k}$ are induced by the multiplication in E. In particular, note that ϕ_i induces a canonical isomorphism $\mathcal{A}_{i,i} \xrightarrow{\sim} R$. □

We will not spell out the bijection or the properties (UNIT), (COM), and (ASSO) in general. Instead, we explain the content of the lemma in the case of type $(1,1)$.

Example 3.1.7. There is a bijection between R-GMAs $(E, \mathcal{E})$ of type $(1,1)$ and triples (B, C, m) where B, C are finitely generated R-modules and $m : B \otimes_R C \to R$ is an R-module homomorphism, such that the squares

$$\begin{array}{ccc} B \otimes_R C \otimes_R B & \xrightarrow{1 \otimes (m \circ \iota)} & B \otimes_R R \\ \downarrow{\scriptstyle m \otimes 1} & & \downarrow \\ R \otimes_R B & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} C \otimes_R B \otimes_R C & \xrightarrow{1 \otimes m} & C \otimes_R R \\ \downarrow{\scriptstyle (m \circ \iota) \otimes 1} & & \downarrow \\ R \otimes_R C & \longrightarrow & C \end{array}$$

commute. Here $\iota : C \otimes_R B \xrightarrow{\sim} B \otimes_R C$ is the isomorphism given by $b \otimes c \mapsto c \otimes b$, and the unlabeled maps are the R-action maps.

The R-GMA associated to a triple (B, C, m) is

$$E = \begin{pmatrix} R & B \\ C & R \end{pmatrix}. \tag{3.1.8}$$

This means that $E = R \oplus B \oplus C \oplus R$ as an R-module, and the multiplication on E is given by 2×2-matrix multiplication, but where the action maps $R \otimes_R B \to B$, $R \otimes_R C \to C$ and the map m are used in place of the scalar multiplication. The idempotents e_1, e_2 are given in this notation by $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and the isomorphisms ϕ_i are given by the identifications $\left(\begin{smallmatrix} R & 0 \\ 0 & 0 \end{smallmatrix}\right) \xrightarrow{\sim} R$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & R \end{smallmatrix}\right) \xrightarrow{\sim} R$.

In the notation of the lemma, $\mathcal{A}_{1,2} = B$ and $\mathcal{A}_{2,1} = C$, and the maps $\varphi_{i,j,k}$ are the action maps, except $\varphi_{1,2,1} = m$ and $\varphi_{2,1,2} = m \circ \iota$. Indeed, the (UNIT) property is exactly that, except for $(i, j, k) \in \{(1, 2, 1), (2, 1, 2)\}$, the maps $\varphi_{i,j,k}$ must be the action maps. The (ASSO) property is expressed by the commutative squares above, and the (COM) property is that $\varphi_{2,1,2} = \varphi_{1,2,1} \circ \iota$.

Example 3.1.9. In the foregoing example, setting m to be the zero map is always a valid choice.

Definition 3.1.10 ([BC09, Defn. 1.3.6])**.** Let A be a commutative R-algebra and let $(E, \mathcal{E})$ be an R-GMA of type $(d_1, \dots, d_r)$. Let $d = \sum_{i=1}^r d_i$ and let $(M_d(A), \mathcal{E}_{\mathrm{block}})$ be the A-GMA of type $(d_1, \dots, d_r)$ constructed in Example 3.1.2.

An *adapted representation of* $(E, \mathcal{E})$, denoted $(E, \mathcal{E}) \to M_d(A)$ is a morphism of GMAs $(E, \mathcal{E}) \to (M_d(A), \mathcal{E}_{\mathrm{block}})$. A pseudorepresentation $D : E \otimes_R A \to A$ is *adapted* if $D = D_{\mathcal{E}} \otimes_R A$.

Now fix $(E, \mathcal{E}) = (E, (\{e_i\}, \{\phi_i\}))$, an R-GMA of type $(d_1, \dots, d_r)$, and let $(\mathcal{A}_{i,j}, \varphi_{i,j,k})$ be the data associated to it by Lemma 3.1.5. We define a the set-valued functor on commutative R-algebras by

$$\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}} : A \mapsto \{\text{Adapted representations } (E, \mathcal{E}) \to (M_d(A), \mathcal{E}_{\mathrm{block}})\}.$$

There is an explicit presentation for $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}}$ as an affine $\operatorname{Spec} R$-scheme as follows.

Theorem 3.1.11. *The functor* $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}}$ *is represented by* $\operatorname{Spec} S$*, where* S *is an* R*-algebra quotient*

$$\mathrm{Sym}^*_R\left(\bigoplus_{1 \le i \ne j \le r} \mathcal{A}_{i,j}\right) \twoheadrightarrow S$$

by the ideal generated by the set of all $\varphi(b \otimes c) - b \otimes c$*, with* $b \in \mathcal{A}_{i,j}, c \in \mathcal{A}_{j,k}$ *and* $\varphi = \varphi_{i,j,k}$ *for all* $1 \le i, j, k \le r$*. In particular,* S *is a finitely generated* R*-algebra.*

Moreover, for $1 \le i, j \le r$*, the natural* R*-module maps* $\mathcal{A}_{i,j} \to S$ *are split injections of* R*-modules, inclusive of the case* $R \xrightarrow{\sim} \mathcal{A}_{i,i} \hookrightarrow S$*, which is the* R*-algebra structure map of* S*. The universal adapted representation* $\rho_{\mathrm{Ad}} : (E, \mathcal{E}) \to M_d(S)$ *is given by the isomorphism of* (3.1.6) *along with these* R*-module injections. In particular, the* R*-algebra homomorphism* $E \to M_d(S)$ *is injective.*

Proof. See [BC09, Prop. 1.3.9] and its proof, as well as [BC09, Prop. 1.3.13] for the split injectivity. □

By [WE15, Prop. 2.23], any adapted representation of $(E, \mathcal{E})$ is compatible with $D_{\mathcal{E}}$, resulting in a monomorphism $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}} \hookrightarrow \mathrm{Rep}^{\square}_{E,D_{\mathcal{E}}}$, which can easily be observed to be a closed immersion.

Let $\rho^{\square}$ denote the universal object of $\mathrm{Rep}^{\square}_{E,D_{\mathcal{E}}}$. Let $Z(\mathcal{E}) \subset \mathrm{GL}_d$ denote the stabilizer of $\{\rho^{\square}(e_1), \dots, \rho^{\square}(e_r)\}$ under the adjoint action of GL_d on $\mathrm{Rep}^{\square}_{E,D_{\mathcal{E}}}$, a torus of rank r.

Proposition 3.1.12 ([WE15, Thm. 2.27])**.** *For any R-GMA $(E, \mathcal{E})$, the map $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}} \hookrightarrow \mathrm{Rep}^{\square}_{E,D_{\mathcal{E}}}$ induces an isomorphism*

$$[\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}}/Z(\mathcal{E})] \xrightarrow{\sim} \mathrm{Rep}_{E,D_{\mathcal{E}}} \tag{3.1.13}$$

of Spec R*-algebraic stacks.*

3.2. **Residually multiplicity-free representations of profinite groups.** Let G be a profinite group satisfying the Φ_p finiteness condition, and let $\mathbb{F}$ be a finite field of characteristic p with algebraic closure $\overline{\mathbb{F}}$.

By [WE15, Cor. 2.9], for any pseudorepresentation $\bar{D}: G \to \mathbb{F}$, there is a unique semisimple representation $\rho^{ss}_{\bar{D}}: G \to \mathrm{GL}_d(\mathbb{F})$ such that $\bar{D} = \det \circ \rho^{ss}_{\bar{D}}$.

Definition 3.2.1. A residual pseudorepresentation $\bar{D}: G \to \mathbb{F}$ is *multiplicity-free* if $\rho^{ss}_{\bar{D}} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ has distinct simple factors. In this case, the factors are defined over $\mathbb{F}$. A representation $(V_A, A) \in \mathrm{Rep}^d_G(A)$ is *residually multiplicity-free* if $(V_A, A) \in \mathrm{Rep}_{\bar{D}}(A)$ with $\bar{D}$ multiplicity-free.

Theorem 3.2.2. *Let $\bar{D}: G \to \mathbb{F}$ be multiplicity-free, and let $(d_1, \dots, d_r)$ be the dimensions of the simple factors of $\rho^{ss}_{\bar{D}}$. Let A be a Noetherian Henselian local ring with residue field $\mathbb{F}$, and let (E, ρ, D) Cayley-Hamilton representation over A with residual pseudorepresentation $\bar{D}$. Then there is an A-GMA structure $\mathcal{E}$ of type $(d_1, \dots, d_r)$ on E such that $D = D_{\mathcal{E}}$.*

Moreover, given a morphism $(E, \rho, D) \to (E', \rho', D')$ of such objects, the structures $\mathcal{E}$ and $\mathcal{E}'$ may be chosen so that the map $(E, \mathcal{E}) \to (E', \mathcal{E}')$ is a morphism of GMAs.

Proof. The structure $\mathcal{E}$ is constructed in [Che14, Thm. 2.22(ii)], and it follows from this construction that $D = D_{\mathcal{E}}$ (see [WE15, Thm. 2.27]). Moreover, the construction only depends on the choice certain lifts of idempotents. If we first choose the structure $\mathcal{E} = (\{e_i\}, \{\phi_i\})$ on E, then the images of e_i in E' will give a choice of lifts of idempotents in E', and for the resulting GMA structure $\mathcal{E}'$, the map $(E, \mathcal{E}) \to (E', \mathcal{E}')$ is a morphism of GMAs. □

For the rest of this section, we fix a pseudorepresentation $\bar{D}: G \to \mathbb{F}$ that is multiplicity-free. By Theorem 2.2.5, the ring $R_{\bar{D}}$ is Noetherian and complete (and hence Henselian). By Theorem 3.2.2, we can and do fix a choice of A-GMA structure $\mathcal{E}_{\bar{D}}$ of type $(d_1, \dots, d_r)$ on $E_{\bar{D}}$ such that $D^u_{\bar{D}} = D_{\mathcal{E}_{\bar{D}}}$.

Corollary 3.2.3. *Assume that $\bar{D}$ is multiplicity-free, and let $\mathcal{E}_{\bar{D}}$ be choice of $R_{\bar{D}}$-GMA structure on $E_{\bar{D}}$ as in Theorem 3.2.2.*

(1) There are isomorphisms

$$[\mathrm{Rep}^{\square}_{(E_{\bar{D}},\mathcal{E}_{\bar{D}}),\mathrm{Ad}}/Z(\mathcal{E}_{\bar{D}})] \xrightarrow{\sim} \mathrm{Rep}_{E_{\bar{D}},D^u_{\bar{D}}} \xrightarrow{\sim} \mathrm{Rep}_{\bar{D}}$$

of stacks on $\mathrm{Tfg}_{\mathbb{Z}_p}$.

(2) Let B be a commutative Noetherian local $\mathbb{Z}_p$-algebra. Given an adapted representation $(E_{\bar{D}}, \mathcal{E}_{\bar{D}}) \to M_d(B)$, the map $E_{\bar{D}} \to M_d(B)$ is a compatible representation of $(E_{\bar{D}}, D^u_{\bar{D}})$.

(3) Let (E, ρ, D) be a Cayley-Hamilton representation of G with residual pseudorepresentation $\bar{D}$. Then there is an A-GMA structure $\mathcal{E}$ on E_A such that $D_{\mathcal{E}} = D$ and such that the map $(E_{\bar{D}}, D_{\mathcal{E}_{\bar{D}}}) \to (E, D_{\mathcal{E}})$ is adapted.

Remark 3.2.4. Let $(E, \mathcal{E})$ be an R-GMA where $R \in \mathrm{Tfg}_{\mathbb{Z}_p}$ is local. Let S be the R-algebra from Theorem 3.1.11 so that $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}} = \mathrm{Spec}(S)$. Restricting the functor $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}}$ to the subcategory Tfg_R of the category of R-algebras, we obtain an affine formal scheme $\mathrm{Spf}(\hat{S})$, where $\hat{S}$ is the $\mathfrak{m}_R S$-adic completion of S. We also denote $\mathrm{Spf}(\hat{S})$ by $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}}$, abusing notation. This is how we consider $[\mathrm{Rep}^{\square}_{(E_{\bar{D}},\mathcal{E}_{\bar{D}}),\mathrm{Ad}}/Z(\mathcal{E}_{\bar{D}})]$ as a formal stack on $\mathrm{Tfg}_{\mathbb{Z}_p}$.

Proof. Statement (1) follows from Proposition 3.1.12 and Theorem 2.2.10, while (2) follows from the statement of Theorem 3.2.2 that $D^u_{\bar{D}} = D_{\mathcal{E}_{\bar{D}}}$. Statement (3) follows from the second part of Theorem 3.2.2. □

Lemma 3.2.5. *Assume that $\bar{D}$ is multiplicity-free. Let $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$ be a morphism of Cayley-Hamilton algebras, where E is an R-algebra.*

1. *For any non-zero $x \in E$, there is a commutative local R-algebra B of finite cardinality and a compatible representation $\rho_B : E \to M_d(B)$ such that $\rho_B(x) \neq 0$.*
2. *Let $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E', D')$ be another such object. Assume that $E_{\bar{D}} \to E$ and $E_{\bar{D}} \to E'$ are surjective. If, for all B as in (1) and all compatible representations $\rho_B : E_{\bar{D}} \to M_d(B)$, the map ρ_B factors through E if and only if it factors through E', then there is a canonical isomorphism $(E, D) \xrightarrow{\sim} (E', D')$ of Cayley-Hamilton algebras.*

Proof. Since $\bar{D}$ is multiplicity-free, we may fix a GMA structure $\mathcal{E}_{\bar{D}}$ on $E_{\bar{D}}$ as in Theorem 3.2.2. By Corollary 3.2.3(3), this gives a GMA structure $\mathcal{E}$ on E such that $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$ induces a morphisms of GMAs $(E_{\bar{D}}, \mathcal{E}_{\bar{D}}) \to (E, \mathcal{E})$, and similarly for (E', D'). By Corollary 3.2.3(3) we may work with adapted representations of these GMAs in the place of compatible representations of the Cayley-Hamilton algebras. We will do this for the remainder of the proof.

(1) Let $x \in E$ be a non-zero element. By Theorem 3.1.11 and Remark 3.2.4, there is $\hat{S} \in \mathrm{Tfg}_R$ such that $\mathrm{Rep}^{\square}_{(E,\mathcal{E}),\mathrm{Ad}} = \mathrm{Spf}(\hat{S})$ and, moreover, $E \hookrightarrow M_d(S)$ splits as an R-module map. Therefore $\rho_{\mathrm{Ad}} : E \to M_d(\hat{S})$ remains injective, so $\rho_{\mathrm{Ad}}(x) \neq 0$. Let $y \in \hat{S}$ be a non-zero entry in the matrix $\rho_{\mathrm{Ad}}(x)$. By Lemma 2.2.1, there is a commutative local R-algebra B of finite cardinality and an R-algebra homomorphism $f : \hat{S} \to B$ such that $f(y) \neq 0$. Then the composite $f \circ \rho_{\mathrm{Ad}} : E \to M_d(B)$ is an adapted representation such that $(f \circ \rho_{\mathrm{Ad}})(x) \neq 0$.

(2) Since the maps $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D)$ and $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E', D')$ are morphisms of pseudorepresentations and the maps $g : E_{\bar{D}} \to E$ and $g' : E_{\bar{D}} \to E'$ are surjective, it suffices to show $\ker(g) = \ker(g')$. Assume for a contradiction that there exists $x \in \ker(g)$ with $x \notin \ker(g')$. Since $g'(x) \neq 0$, part (1) implies that there is a commutative local R-algebra B of finite cardinality and a compatible representation $\rho'_B : E' \to M_d(B)$ such that $\rho'_B(g'(x)) \neq 0$. The adapted representation $\tilde{\rho}_B = \rho'_B \circ g' : E_{\bar{D}} \to M_d(B)$ factors through E', so it must factor through E by assumption. This implies that $\tilde{\rho}_B = \rho_B \circ g$ for some adapted representation ρ_B of E. But then

$$0 = \rho_B(g(x)) = \tilde{\rho}_B = \rho'_B(g'(x)) \neq 0$$

a contradiction. □

3.3. **Condition $\mathcal{C}$ in the residually multiplicity-free case.** Let G be a profinite group satisfying the Φ_p finiteness condition. Fix a stable condition $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$

as in Definition 2.3.1 and a residual pseudorepresentation $\bar{D} : G \to \mathbb{F}$. By Theorem 2.5.3, there is a universal Cayley-Hamilton algebra with condition $\mathcal{C}$, denoted $(E_{\bar{D}}^{\mathcal{C}}, D_{\bar{D}}^{\mathcal{C}})$. In the case that $\bar{D}$ is multiplicity-free, the following theorem gives an alternate characterization of $(E_{\bar{D}}^{\mathcal{C}}, D_{\bar{D}}^{\mathcal{C}})$.

Theorem 3.3.1. *Assume that $\bar{D}$ is multiplicity-free.*

1. *Let B be a commutative local $R_{\bar{D}}$-algebra of finite cardinality. Let $\rho_B : (E_{\bar{D}}, D_{\bar{D}}^u) \to M_d(B)$ be a compatible representation, and let $V_B \cong B^d$ denote the corresponding object of $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$. Then V_B satisfies $\mathcal{C}$ if and only if ρ_B factors through $E_{\bar{D}}^{\mathcal{C}}$.*
2. *The property of (1) characterizes the quotient $E_{\bar{D}} \to E_{\bar{D}}^{\mathcal{C}}$.*

Proof. Part (1) follows from Theorem 2.6.3 and Theorem 2.5.3. Part (2) is immediate from Lemma 3.2.5(2). □

When $\bar{D}$ is not multiplicity-free we only know how to characterize $E_{\bar{D}}^{\mathcal{C}}$ by Theorem 2.5.3, cf. [BC09, §1.3.4].

4. Deformation conditions and extensions

In this section, we continue to assume that $\bar{D} : G \to \mathbb{F}$ is multiplicity-free. For simplicity, we also assume that $\bar{D}$ has dimension 2, but see Remark 4.3.6. We use the GMA structure discussed in the previous section to relate the structure of the universal Cayley-Hamilton algebra with residual pseudorepresentation $\bar{D}$ and property $\mathcal{C}$ to extension groups in the category $\mathcal{C}$.

4.1. **Conventions.** Throughout this section, we fix G, a profinite group satisfying the Φ_p finiteness condition, and two distinct characters $\bar{\chi}_1, \bar{\chi}_2 : G \to \mathbb{F}^\times$. We let $\bar{D} = \psi(\bar{\chi}_1 \oplus \bar{\chi}_2)$.

By Theorem 3.2.2, we can and do fix a $R_{\bar{D}}$-GMA structure $\mathcal{E}_{\bar{D}} = (\{e_1, e_2\}, \{\phi_1, \phi_2\})$ on $E_{\bar{D}}$. We write $(E_{\bar{D}}, \mathcal{E}_{\bar{D}})$ and $\rho^u : G \to E_{\bar{D}}^\times$ as

$$E_{\bar{D}} \cong \begin{pmatrix} R_{\bar{D}} & B^u \\ C^u & R_{\bar{D}} \end{pmatrix}, \qquad \rho^u(\sigma) = \begin{pmatrix} \rho^u_{1,1}(\sigma) & \rho^u_{1,2}(\sigma) \\ \rho^u_{2,1}(\sigma) & \rho^u_{2,2}(\sigma) \end{pmatrix} \tag{4.1.1}$$

and write $m : B^u \otimes_{R_{\bar{D}}} C^u \to R_{\bar{D}}$ for the induced map, as in Example 3.1.7. For $b \in B^u$ and $c \in C^u$, we define $b \cdot c = m(b \otimes c) \in R_{\bar{D}}$. We can and do assume that the idempotents are ordered so that the image of $\rho_{i,i}(\sigma)$ under $R_{\bar{D}} \to \mathbb{F}$ is $\bar{\chi}_i(\sigma)$.

By Corollary 3.2.3(3), a Cayley-Hamilton representation (E, ρ, D) of G with residual pseudorepresentation $\bar{D}$ inherits a GMA structure from the data above. We will use matrix notation for E and ρ according to this structure, as in (4.1.1) (for example, writing the coordinates of ρ as $\rho_{i,j}$).

4.2. **Review of reducibility, extensions, and GMAs.** In this subsection, we review known results of [BC09, §1.5] relating the structure of GMAs to Ext-groups.

Definition 4.2.1. Let $A \in \hat{\mathcal{C}}_{W(\mathbb{F})}$. We call a pseudodeformation $D : G \to A$ of $\bar{D}$ *reducible* if $D = \psi(\chi_1 \oplus \chi_2)$ for characters $\chi_i : G \to A^\times$ deforming $\bar{\chi}_i$. Otherwise, we call D *irreducible.*

A GMA representation $(E, \rho : G \to E^\times, D_{\mathcal{E}})$ of G with residual pseudorepresentation $\bar{D}$ is called *reducible* (resp. *irreducible*) provided that pseudodeformation $D_{\mathcal{E}} \circ \rho : G \to A$ is reducible (resp. irreducible).

Proposition 4.2.2. *Let A be a commutative Noetherian local $\mathbb{Z}_p$-algebra. Let $D : G \to A$ pseudorepresentation with residual pseudorepresentation $\bar{D}$.*

(1) *D is reducible if and only if $D = \psi(\rho)$ for some GMA representation ρ with scalar ring A such that $\rho_{1,2}(G) \cdot \rho_{2,1}(G)$ is zero.*

(2) *Let $J = B^u \cdot C^u \subset R_{\bar{D}}$ be the image ideal of $B^u \otimes_{R_{\bar{D}}} C^u$ in $R_{\bar{D}}$ under $m : B^u \otimes_{R_{\bar{D}}} C^u \to R_{\bar{D}}$. Then D is reducible if and only if the corresponding local homomorphism $R_{\bar{D}} \to A$ kills J.*

Proof. See [BC09, §1.5.1]. □

In light of Proposition 4.2.2, we establish this terminology.

Definition 4.2.3. The ideal $J \subset R_{\bar{D}}$ of Proposition 4.2.2(2) is called the *reducibility ideal* of $R_{\bar{D}}$. The image of J under the map $R_{\bar{D}} \to A$ corresponding to a pseudodeformation $D : G \to A$ of $\bar{D}$ is called the *reducibility ideal of D*.

We define $E^{\mathrm{red}}_{\bar{D}}$ to be the Cayley-Hamilton quotient of $E_{\bar{D}}$ by $JE_{\bar{D}}$, which as in Example 2.4.6 is the usual algebra quotient $E^{\mathrm{red}}_{\bar{D}} = E_{\bar{D}}/JE_{\bar{D}}$. Its scalar ring is $R^{\mathrm{red}}_{\bar{D}} = R_{\bar{D}}/J$ and is called the *universal reducible pseudodeformation ring* for $\bar{D}$. We let $(E^{\mathrm{red}}_{\bar{D}}, \rho^{\mathrm{red}}, \mathcal{E}^{\mathrm{red}}_{\bar{D}})$ denote the corresponding GMA representation of G, and write the decomposition arising from (4.1.1) as

$$E^{\mathrm{red}}_{\bar{D}} = \begin{pmatrix} R^{\mathrm{red}}_{\bar{D}} & B^{\mathrm{red}} \\ C^{\mathrm{red}} & R^{\mathrm{red}}_{\bar{D}} \end{pmatrix}. \tag{4.2.4}$$

Let for $i = 1, 2$, let $R_{\bar{\chi}_i}$ denote the universal deformation ring of $\bar{\chi}_i$ and let $\chi_i^u : G \to R^{\times}_{\bar{\chi}_i}$ denote the universal deformation.

Proposition 4.2.5. (1) *If $\rho : G \to (E, D)$ is a GMA representation of G with scalar ring A and residual pseudorepresentation $\bar{D}$, then the resulting GMA map $(E_{\bar{D}}, \mathcal{E}_{\bar{D}}) \to (E, \mathcal{E})$ factors through $(E^{\mathrm{red}}_{\bar{D}}, \mathcal{E}^{\mathrm{red}}_{\bar{D}})$ if and only if ρ is reducible.*

(2) *There is a canonical isomorphism $R^{\mathrm{red}}_{\bar{D}} \cong R_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R_{\bar{\chi}_2}$. Letting $\chi_i = \chi_i^u \otimes_{R_{\bar{\chi}_i}} R^{\mathrm{red}}_{\bar{D}}$, the universal reducible pseudodeformation of $\bar{D}$ is $\psi(\chi_1 \oplus \chi_2)$.*

(3) *In terms of the decomposition (4.2.4), we have $\rho^{\mathrm{red}}_{i,i} = \chi_i$.*

Proof. (1) By Lemma 2.4.5, the GMA map $(E_{\bar{D}}, \mathcal{E}_{\bar{D}}) \to (E, \mathcal{E})$ factors through $(E^{\mathrm{red}}_{\bar{D}}, \mathcal{E}^{\mathrm{red}}_{\bar{D}})$ if and only if $E_{\bar{D}} \to E$ sends $JE_{\bar{D}}$ to zero. Because $R \to E$ is injective whenever there is a pseudorepresentation $D : R \to E$ [WWE15b, Lem. 5.2.5], $JE_{\bar{D}}$ maps to zero in E if and only if $R_{\bar{D}} \to A$ factors through $R^{\mathrm{red}}_{\bar{D}}$, which, by Proposition 4.2.2(2), is equivalent to $D \circ \rho$ being reducible.

(2) By Yoneda's lemma, it suffices to construct a canonical functorial isomorphism

$$\mathrm{Hom}(R^{\mathrm{red}}_{\bar{D}}, A) \cong \mathrm{Hom}(R_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R_{\bar{\chi}_2}, A)$$

for $A \in \hat{\mathcal{C}}_{W(\mathbb{F})}$. Given $R_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R_{\bar{\chi}_2} \to A$, we define a reducible deformation D of $\bar{D}$ by $D = \psi(\chi_1^u \otimes A \oplus \chi_2^u \otimes A)$, which determines an element of $\mathrm{Hom}(R^{\mathrm{red}}_{\bar{D}}, A)$ by Proposition 4.2.2(2) and the universal property of $R_{\bar{D}}$.

Conversely, given $R^{\mathrm{red}}_{\bar{D}} \to A$, consider the GMA representation $\rho^{\mathrm{red}} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A$. Since $B^{\mathrm{red}} \cdot C^{\mathrm{red}} = 0$, we have that $\rho^{\mathrm{red}}_{i,i} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A : G \to A^{\times}$ (for $i = 1, 2$) is a character that, by the conventions of §4.1, is a deformation of $\bar{\chi}_i$. This pair of characters determines an element of $\mathrm{Hom}(R_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R_{\bar{\chi}_2}, A)$.

We have defined maps between $\mathrm{Hom}(R^{\mathrm{red}}_{\bar{D}}, A)$ and $\mathrm{Hom}(R_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R_{\bar{\chi}_2}, A)$. The reader can check these are mutually inverse and functorial in A.

(3) Follows from the construction of the isomorphism in (2). $\square$

The following key result relates $R^{\mathrm{red}}_{\bar{D}}[G]$-module extensions to the structure of $E^{\mathrm{red}}_{\bar{D}}$. For ease of notation, we write χ_i for the base change from $R_{\bar{\chi}_i}$ to $R^{\mathrm{red}}_{\bar{D}}$ of the universal deformation χ_i^u of $\bar{\chi}_i$.

Proposition 4.2.6. *Let $A \in \hat{\mathcal{C}}_{W(\mathbb{F})}$ and let M be a finitely generated A-module. For $i = 1, 2$, let $\chi_{i,A} : G \to A^\times$ be characters deforming $\bar{\chi}_i$. By Proposition 4.2.5, this induces a unique local homomorphism $R^{\mathrm{red}}_{\bar{D}} \to A$. There is a natural isomorphism*

$$\mathrm{Hom}_A(B^{\mathrm{red}} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A, M) \xrightarrow{\sim} \mathrm{Ext}^1_{A[G]}(\chi_{2,A}, \chi_{1,A} \otimes_A M).$$

as well as a similar isomorphism in the C-coordinate.

Proof. The details may be found in [WWE15a, Lem. 4.1.5, proof of (4.1.7)], cf. also [BC09, Thm. 1.5.6]. We reproduce the construction of the map here with notation that will be convenient in §4.3.

Let $E_M = \left(\begin{smallmatrix} A & M \\ 0 & A \end{smallmatrix}\right)$, with GMA structure as in Example 3.1.9. Given a homomorphism $f : B^{\mathrm{red}} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A \to M$, we have morphism of GMAs $E^{\mathrm{red}}_{\bar{D}} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A \to E_M$ as the composition

$$E^{\mathrm{red}}_{\bar{D}} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{id} \end{smallmatrix}\right)} \begin{pmatrix} A & B^{\mathrm{red}} \otimes_{R^{\mathrm{red}}_{\bar{D}}} A \\ 0 & A \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & f \\ 0 & \mathrm{id} \end{smallmatrix}\right)} \begin{pmatrix} A & M \\ 0 & A \end{pmatrix}. \tag{4.2.7}$$

Using the fact that $B^{\mathrm{red}} \cdot C^{\mathrm{red}} = 0$, we see that $e_1 E_M e_2 = \left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$ is a left $E_{\bar{D}}$-submodule of $E_M e_2 = \left(\begin{smallmatrix} 0 & M \\ 0 & A \end{smallmatrix}\right)$. Noting that $e_1 E_M e_2 \cong \chi_{1,A} \otimes_A M$ and $E_M e_2 / e_1 E_M e_2 \cong \chi_{2,A}$ as $A[G]$-modules, we obtain an exact sequence

$$0 \longrightarrow \chi_{1,A} \otimes_A M \to E_M e_2 \longrightarrow \chi_{2,A} \longrightarrow 0, \tag{4.2.8}$$

which determines the corresponding element of $\mathrm{Ext}^1_{A[G]}(\chi_{2,A}, \chi_{1,A} \otimes_A M)$. Conversely, any such extension can be realized in the form $E_M e_2$. $\square$

4.3. **GMA structures corresponding to extensions with an abstract property.** Let $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ be a stable property, as in Definition 2.3.1. The following lemma is a well-known consequence of stability, and we leave the proof to the reader. To state the lemma clearly, we introduce some notation for extension classes. If E is an algebra and V_1, V_2 are E-modules, given an extension class $c \in \mathrm{Ext}^1_E(V_1, V_2)$, and an exact sequence

$$0 \to V_2 \to V \to V_1 \to 0 \tag{4.3.1}$$

representing c, we call V an *extension module* for c.

Let $V_1, V_2 \in \mathcal{C}$, and assume that $V_1, V_2 \in \mathrm{Mod}_{R[G]}$ as well for commutative $\mathbb{Z}_p$-algebra R. Define $\mathrm{Ext}^1_{R[G],\mathcal{C}}(V_1, V_2)$ as the subset of $\mathrm{Ext}^1_{R[G]}(V_1, V_2)$ consisting of extension classes c such that, for every extension module V for c, we have $V \in \mathcal{C}$.

Lemma 4.3.2. *With V_1, V_2 as above, we have the following.*

1. *For a class $c \in \mathrm{Ext}^1_{R[G]}(V_1, V_2)$, if some extension module V for c has $V \in \mathcal{C}$, then $c \in \mathrm{Ext}^1_{R[G],\mathcal{C}}(V_1, V_2)$.*
2. *The subset $\mathrm{Ext}^1_{R[G],\mathcal{C}}(V_1, V_2) \subset \mathrm{Ext}^1_{R[G]}(V_1, V_2)$ is a sub-R-module.*

Example 4.3.3. Let $H_1, \dots, H_n \subset G$ be subgroups, and, for $i = 1, \dots, n$, let $\mathcal{C}_i \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[H_i]}$ be a stable condition. Assume that the condition $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ arises from the $\mathcal{C}_i$ as in Example 2.3.3. Then $\mathrm{Ext}^1_{R[G],\mathcal{C}}(V_1, V_2)$ is the kernel of the map

$$\mathrm{Ext}^1_{R[G]}(V_1, V_2) \to \bigoplus_{i=1}^{n} \frac{\mathrm{Ext}^1_{R[H_i]}(V_1, V_2)}{\mathrm{Ext}^1_{R[H_i],\mathcal{C}_i}(V_1, V_2)}$$

given by the restrictions $\mathrm{Ext}^1_{R[G]}(V_1, V_2) \to \mathrm{Ext}^1_{R[H_i]}(V_1, V_2)$ followed by the quotients. (This is sometimes referred to as a Selmer group.)

Let $E^{\mathcal{C}}_{\bar{D}}$ be as in Theorem 3.3.1, and let $E^{\mathcal{C},\mathrm{red}}_{\bar{D}} = E^{\mathcal{C}}_{\bar{D}}/JE^{\mathcal{C}}_{\bar{D}}$ where $J \subset R_{\bar{D}}$ is the reducibility ideal. Following the notation of (4.1.1), we write them as

$$E^{\mathcal{C}}_{\bar{D}} = \begin{pmatrix} R^{\mathcal{C}}_{\bar{D}} & B^{\mathcal{C}} \\ C^{\mathcal{C}} & R^{\mathcal{C}}_{\bar{D}} \end{pmatrix}, \qquad E^{\mathcal{C},\mathrm{red}}_{\bar{D}} = \begin{pmatrix} R^{\mathcal{C},\mathrm{red}}_{\bar{D}} & B^{\mathcal{C},\mathrm{red}} \\ C^{\mathcal{C},\mathrm{red}} & R^{\mathcal{C},\mathrm{red}}_{\bar{D}} \end{pmatrix}.$$

We denote the Cayley-Hamilton representations by $\rho^{\mathcal{C}} : G \to (E^{\mathcal{C}}_{\bar{D}})^\times$ and $\rho^{\mathcal{C},\mathrm{red}} : G \to (E^{\mathcal{C},\mathrm{red}}_{\bar{D}})^\times$.

By Ramakrishna's result [Ram93, Prop. 1.2], for $i = 1, 2$, there is a quotient $R_{\bar{\chi}_i} \twoheadrightarrow R^{\mathcal{C}}_{\bar{\chi}_i}$ that represents the functor of deformations of $\bar{\chi}_i$ having property $\mathcal{C}$ and $\chi_i^{u,\mathcal{C}} = \chi_i^u \otimes_{R_{\bar{\chi}_i}} R^{\mathcal{C}}_{\bar{\chi}_i}$ is the universal deformation with property $\mathcal{C}$.

Proposition 4.3.4. *There is a canonical commutative diagram*

$$\begin{array}{ccc} R^{\mathcal{C},\mathrm{red}}_{\bar{D}} & \xrightarrow{\sim} & R^{\mathcal{C}}_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R^{\mathcal{C}}_{\bar{\chi}_2} \\ \uparrow & & \uparrow \\ R^{\mathrm{red}}_{\bar{D}} & \xrightarrow{\sim} & R_{\bar{\chi}_1} \otimes_{W(\mathbb{F})} R_{\bar{\chi}_2} \end{array}$$

where the vertical maps are the quotients and the lower horizontal map is the isomorphism of Proposition 4.2.5(2).

Proof. For simplicity of notation, let $R = R^{\mathcal{C}}_{\bar{\chi}_1} \hat{\otimes}_{W(\mathbb{F})} R^{\mathcal{C}}_{\bar{\chi}_2}$ and, for $i = 1, 2$, let $\chi_i^{\mathcal{C}} = \chi_i^{u,\mathcal{C}} \otimes_{R^{\mathcal{C}}_{\bar{\chi}_i}} R$.

By Proposition 4.2.5(2), the composite map $R^{\mathrm{red}}_{\bar{D}} \to R$ corresponds to the reducible pseudodeformation $D = \psi(\chi_1^{\mathcal{C}} \oplus \chi_2^{\mathcal{C}})$. The G-module $N = \chi_1^{\mathcal{C}} \oplus \chi_2^{\mathcal{C}}$ is a faithful Cayley-Hamilton G-module with Cayley-Hamilton algebra $(E = \left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right), D_{\mathcal{E}})$. By Theorem 2.6.3, the map $(E_{\bar{D}}, D^u_{\bar{D}}) \to (E, D_{\mathcal{E}})$ factors through $E^{\mathcal{C}}_{\bar{D}}$. Since D is reducible, Proposition 4.2.2 implies that it further factors through $E^{\mathcal{C},\mathrm{red}}_{\bar{D}}$. This implies that $R^{\mathrm{red}}_{\bar{D}} \to R$ factors through $R^{\mathcal{C},\mathrm{red}}_{\bar{D}}$.

On the other hand, the composite map $R_{\bar{\chi}_1} \otimes_{W(\mathbb{F})} R_{\bar{\chi}_2} \to R^{\mathcal{C},\mathrm{red}}_{\bar{D}}$ corresponds to the pair of characters $e_1\rho^{\mathrm{red},\mathcal{C}}e_1, e_2\rho^{\mathrm{red},\mathcal{C}}e2 : G \to R^{\mathcal{C},\mathrm{red}}_{\bar{D}}$. Since these characters are quotient $R^{\mathrm{red}}_{\bar{D}}[G]$-modules of the $R^{\mathrm{red}}_{\bar{D}}[G]$-module $E^{\mathcal{C},\mathrm{red}}_{\bar{D}}$ that has $\mathcal{C}$, we see that they both have $\mathcal{C}$ as well. This implies that the map $R_{\bar{\chi}_1} \otimes_{W(\mathbb{F})} R_{\bar{\chi}_2} \to R^{\mathcal{C},\mathrm{red}}_{\bar{D}}$ factors through R, completing the proof. □

We will write $\chi_i^{\mathcal{C}}$ for the base change from $R^{\mathcal{C}}_{\bar{\chi}_i}$ to $R^{\mathcal{C},\mathrm{red}}_{\bar{D}}$ of the universal deformation of $\bar{\chi}_i$ satisfying $\mathcal{C}$.

Theorem 4.3.5. *Let $A \in \hat{\mathcal{C}}_{W(\mathbb{F})}$ and let M be a finitely generated A-module. For $i = 1, 2$, let $\chi_{i,A} : G \to A^\times$ be characters deforming $\bar\chi_i$ and having property $\mathcal{C}$. By Proposition 4.3.4, this induces a unique local homomorphism $R^{\mathrm{red},\mathcal{C}}_{\bar D} \to A$. There is a natural isomorphism*

$$\mathrm{Hom}_A(B^{\mathcal{C},\mathrm{red}} \otimes_{R^{\mathrm{red},\mathcal{C}}_{\bar D}} A, M) \xrightarrow{\sim} \mathrm{Ext}^1_{A[G],\mathcal{C}}(\chi_{2,A}, \chi_{1,A} \otimes_A M).$$

as well as a similar isomorphism in the C-coordinate.

Remark 4.3.6. We expect that a more general result is true after a simple reformulation for general type GMAs. This result would be analogous to how [BC09, Thm. 1.5.6] is a more general version of Proposition 4.2.6. However, Theorem 4.3.5 is essentially already valid for (d_1, d_2) by Morita equivalence, cf. [BC09, §1.3.2].

Proof. We set $B = B^{\mathrm{red}} \otimes_{R^{\mathrm{red}}_{\bar D}} A$ and $B^{\mathcal{C}} = B^{\mathcal{C},\mathrm{red}} \otimes_{R^{\mathcal{C},\mathrm{red}}_{\bar D}} A$ to simplify notation. We consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(B^{\mathcal{C}}, M) & \overset{\sim}{\dashrightarrow} & \mathrm{Ext}^1_{A[G],\mathcal{C}}(\chi_{2,A}, \chi_{1,A} \otimes_A M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_A(B, M) & \xrightarrow[\sim]{\Psi} & \mathrm{Ext}^1_{A[G]}(\chi_{2,A}, \chi_{1,A} \otimes_A M), \end{array}$$

where Ψ isomorphism of Proposition 4.2.6, and where the dotted arrow is the isomorphism we wish to construct. Since this diagram is canonically isomorphic to the one obtained by replacing A by the image of $A \to \mathrm{End}_A(M)$, we can and do assume that M is a faithful A-module. Given a class $c \in \mathrm{Ext}^1_{A[G]}(\chi_{2,A}, \chi_{1,A} \otimes_A M)$, we have to show that the map $f_c = \Psi^{-1}(c) : B \to M$ factors through $B^{\mathcal{C}}$ if and only if $c \in \mathrm{Ext}^1_{A[G],\mathcal{C}}(\chi_{2,A}, \chi_{1,A} \otimes_A M)$.

Following the proof of Proposition 4.2.6, we see that the class c has an extension module $E_M e_2$ where $E_M = \left(\begin{smallmatrix} A & M \\ 0 & A \end{smallmatrix}\right)$ is a GMA and f_c induces a morphism of GMAs $E^{\mathrm{red}}_{\bar D} \otimes_{R^{\mathrm{red}}_{\bar D}} A \to E_M$ by

$$E^{\mathrm{red}}_{\bar D} \otimes_{R^{\mathrm{red}}_{\bar D}} A \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{id} \end{smallmatrix}\right)} \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & f_c \\ 0 & \mathrm{id} \end{smallmatrix}\right)} \begin{pmatrix} A & M \\ 0 & A \end{pmatrix}. \tag{4.3.7}$$

Since M is a faithful A-module, we see that $E_M e_2$ is a faithful Cayley-Hamilton G-module with Cayley-Hamilton algebra $(E_M, D_{\mathcal{E}_M})$. By Theorem 2.6.3, $E_M e_2$ has $\mathcal{C}$ if and only if $(E_M, D_{\mathcal{E}_M})$ has $\mathcal{C}$ as a Cayley-Hamilton representation of G. By Lemma 4.3.2, Lemma 2.4.5, and Theorem 2.5.3, we are reduced to showing that f_c factors through $B^{\mathcal{C}}$ if and only if the map $E_{\bar D} \twoheadrightarrow E^{\mathrm{red}}_{\bar D} \otimes_{R^{\mathrm{red}}_{\bar D}} A \to E_M$ given by (4.3.7) factors through $E^{\mathcal{C}}_{\bar D}$.

If f_c factors through $B^{\mathcal{C}}$, then we see that the map $E^{\mathrm{red}}_{\bar D} \otimes_{R^{\mathrm{red}}_{\bar D}} A \to E_M$ of (4.3.7) agrees with the map

$$E^{\mathrm{red}}_{\bar D} \otimes_{R^{\mathrm{red}}_{\bar D}} A \twoheadrightarrow E^{\mathcal{C},\mathrm{red}}_{\bar D} \otimes_{R^{\mathcal{C},\mathrm{red}}_{\bar D}} A \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{id} \end{smallmatrix}\right)} \begin{pmatrix} A & B^{\mathcal{C}} \\ 0 & A \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & f_c \\ 0 & \mathrm{id} \end{smallmatrix}\right)} \begin{pmatrix} A & M \\ 0 & A \end{pmatrix}.$$

Hence the map $E_{\bar D} \to E_M$ factors though $E^{\mathcal{C},\mathrm{red}}_{\bar D}$, which is a quotient of $E^{\mathcal{C}}_{\bar D}$.

Conversely, suppose that $(E_{\bar D}, D^u_{\bar D}) \to (E_M, D_{\mathcal{E}_M})$ factors through $(E^{\mathcal{C}}_{\bar D}, D^{\mathcal{C}}_{\bar D})$. Since $(E_M, D_{\mathcal{E}_M})$ is reducible, this implies that the map $E_{\bar D} \to E_M$ factors through a map $g : E^{\mathcal{C},\mathrm{red}}_{\bar{D}} \otimes_{R^{\mathcal{C},\mathrm{red}}_{\bar{D}}} A \to E_M$. By (4.3.7), we see that f_c factors through $e_1 g e_2 : B^{\mathcal{C}} \to M$. □

5. Examples

The follow examples of conditions $\mathcal{C}$ could be useful in arithmetic applications.

5.1. **Unramified local condition.**

Let ℓ be a prime number and let K be a finite field extension of $\mathbb{Q}_\ell$. Let $G = \mathrm{Gal}(\overline{K}/K)$, which satisfies the Φ_p finiteness condition because it is topologically finitely generated. Let $H = I_\ell \subset G$ be the inertia subgroup. The condition $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G/H]} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$, as in Example 2.3.2, is called *unramified*.

5.2. **Local conditions at $\ell = p$.**

Retain the same notation as the previous subsection, but now assume $\ell = p$. Let $O_K \subset K$ denote the ring of integers. In this case, there are many conditions on representations of G in $\mathbb{Q}_p$-vector spaces, coming from p-adic Hodge theory. Some of these conditions descend to $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ as follows.

Definitions 5.2.1. Let V be an object of $\mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ and let $a \leq b$ be integers.

(1) We call V *finite-flat* if there is a finite flat group scheme $\mathcal{G}$ over O_K such that $V \cong \mathcal{G}(\overline{K})$ as $\mathbb{Z}_p[G]$-modules.
(2) We call an object $V \in \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ *torsion crystalline (resp. semi-stable) with Hodge-Tate weights in* $[a, b]$ if there is a crystalline (resp. semi-stable) representation $\rho : G \to \mathrm{GL}(W)$ with Hodge-Tate weights in $[a, b]$ and a G-stable $\mathbb{Z}_p$-lattice $T \subset W$ such that V is isomorphic to a subquotient of T.

Let $\mathcal{C}_{\mathrm{flat}} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ denote the full subcategory of finite-flat objects. Ramakrishna has proven that $\mathcal{C}_{\mathrm{flat}}$ is stable [Ram93, §2].

Let $\mathcal{C}_{\mathrm{crys},[a,b]}, \mathcal{C}_{\mathrm{st},[a,b]} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ denote the full subcategories that are torsion crystalline (resp. semi-stable) with Hodge-Tate weights in $[a, b]$. Since the category of crystalline (resp. semi-stable) representations with Hodge-Tate weights in $[a, b]$ is closed under finite direct sums, we see that $\mathcal{C}_{\mathrm{crys},[a,b]}$ and $\mathcal{C}_{\mathrm{st},[a,b]}$ are closed under finite direct sums. They are also closed under isomorphisms and subquotients by definition, so we see that they are stable.

Remark 5.2.2. It is known that, for a $\mathbb{Z}_p[G]$-module M that is finitely generated and free as a $\mathbb{Z}_p$-module, $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline (resp. semi-stable) with Hodge-Tate weights in $[a, b]$ if and only if $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$ is torsion crystalline (resp. semi-stable) with Hodge-Tate weights in $[a, b]$ for all $n \geq 1$. This was conjectured by Fontaine, and proven by Liu using results of Kisin, following partial results of Ramakrishna, Berger, and Breuil. It is also known that there is an equivalence of categories $\mathcal{C}_{\mathrm{flat}} \cong \mathcal{C}_{\mathrm{crys},[0,1]}$ for $p > 2$. See [Liu07] and the references given there.

For the conditions $\mathcal{C}$ above, this paper's constructions of $\mathrm{Rep}^{\mathcal{C}}_{\bar{D}}$ and $R^{\mathcal{C}}_{\bar{D}}$ supersede those of [WE15, §7.1]; in particular, in *loc. cit.* $R^{\mathcal{C}}_{\bar{D}}$ was unconditionally constructed only in the residually multiplicity-free case. The two constructions agree on the generic fiber (over $\mathrm{Spec}\,\mathbb{Z}_p$), so the geometric properties of the generic fibers proved in Prop. 6.4.4 and Cor. 7.1.5 of *loc. cit.* apply to our constructions.

5.3. **Global conditions.** Let F be a number field with algebraic closure $\overline{F}$ and let S be a finite set of places of F. Let $G = \mathrm{Gal}(F_S/F)$, where $F_S \subset \overline{F}$ is the maximal extension of F unramified outside S. Then G satisfies Φ_p by class field theory.

For each $v \in S$, choose a decomposition group $G_v \subset G$ (so G_v is of the type considered in the previous subsections) and a stable condition $\mathcal{C}_v \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G_v]}$. Then there is a corresponding stable condition $\mathcal{C} \subset \mathrm{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$, as in Example 2.3.3. The Selmer groups of Example 4.3.3 correspond to such a $\mathcal{C}$.

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