Matrix Airy functions for compact Lie groups

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Abstract

The classical Airy function and its many applications are well known. A few years ago Kontsevich defined a matrix generalization of the Airy integral and found that its theory had many applications to intersection theory on the moduli space of curves.

In my talk which will be as elementary as possible I shall show that the Kontsevich integral has a very wide generalization and discuss the properties of this generalization. The work is joint with my former student Rahul Fernandez.

The original Airy function

First discovery due to *Sir George Biddell Airy* in 1838 in his investigations of the intensity of light in the neighborhood of a caustic:

$$\int_0^\infty \cos\left(\frac{\pi}{2}\left(\frac{\omega^3}{3} - m\omega\right)\right) d\omega$$

It is more convenient to work with

$$A(x) = \int_{-\infty}^{\infty} e^{i((1/3)y^3 - xy)} dy$$

The integral is not convergent and so has to be interpreted suitably, for instance as an improper Riemann integral.

The differential equation

$A(x)$ is the (projectively)unique solution of the differential equation

$$\frac{d^2u}{dx^2} + xu = 0$$

which is of polynomial growth.

It extends to an entire function on **C**.

This is one of the simplest differential equations over **C** with an irregular singularity at infinity.

Kontsevich's generalization

In his work on intersection theory on the moduli space of curves Kontsevich introduced a generalization of the Airy integral:

$$A(X) = \int_{\mathcal{H}(N)} e^{i\mathrm{Tr}((1/3)Y^3 - XY)} dY$$

where $\mathcal{H}(N)$ is the space of hermitian $N \times N$ matrices and dY is Lebesgue measure.

With suitable interpretation it satisfies the elliptic equation

$$\Delta A(X) + \mathrm{Tr}(X) A(X) = 0$$

and so is a smooth function.

The Airy distribution

Let V be a real finite dimensional vector space with a symmetric nondegenerate biliner form (x, y), p a real polynomial on V. The Airy integral associated to p is:

$$A(x) = \int_V e^{i(p(y)-(x,y))} dy$$

We interpret this as a distribution, *the Airy distribution*:

$$A_p = \mathcal{F}(e^{ip}) \quad \mathcal{F} = \text{Fourier transform}$$

This makes sense because e^{ip} is bounded and so defines a tempered distribution.

The Airy property

The polynomial p has the *Airy property* if A_p is a smooth function of moderate growth, i.e., it and its derivatives are of polynomial growth and it extends to an entire function on $V_{\mathbf{C}} = \mathbf{C} \otimes V$.

If p is linear or depends only on a proper subset of the coordinates on V, A_p will be a distribution supported on a proper affine subspace of V and so will not be a function.

Generalization to compact Lie groups

The Kontsevich integral can be taken over $i\mathcal{H}(N)$ instead of $\mathcal{H}(N)$ and so generalizes to the integral

$$A(X) = \int_{\mathfrak{g}} e^{i(p(Y)-(X,Y))} dY$$

where $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, (X, Y) is the Cartan–Killing form, and p is a G-invariant real polynomial.

For $G = U(N)$, the unitary group in N variables, $\mathfrak{g} = i\mathcal{H}(N)$, and $p(Y) = \mathrm{Tr}((1/3)(Y^3))$ we obtain the Kontsevich integral.

Some general questions

The following are some of the questions that arise.

- Determine the class of p with the Airy property

- Find, if possible, explicit formulae for A_p in terms of the Airy functions on one variable

- Study the differential equations satisfied by the Airy functions

- The definitions make sense when we work not over $\mathbf{R}$ but over *any local field*. What are the corresponding questions and answers?

The case when the number of variables is 1

Theorem. *Any polynomial in one variable of degree* ≥ 2 *has the Airy property.*

Discussion. The Airy integral will become convergent when we take the path of integration into the complex domain. The paths have to be chosen carefully depending on whether the degree of p is odd or even.

The paths

Odd degree ≥ 3

The path is from $-\infty + i\eta$ to $\infty + i\eta$, $\eta > 0$.

$$\left| e^{Im(p(\xi+i\eta))} \right| \leq Ce^{-\eta D \xi^{2k}} \qquad (\deg(p) = 2k+1, D > 0)$$

Even degree ≥ 4

The above method fails because the leading term in $Im(p(\xi + i\eta))$ is like $-\eta\xi^{2k-1}$ and $\exp(-\eta\xi^{2k-1})$ is not integrable for $\xi < 0$. So the path starts out from $-\infty - i\eta$ and then changes over to $\infty + i\eta$ in a finite part of $\mathbf{C}$. The estimate is

$$\left| e^{Im(p(\xi+i\eta))} \right| \leq Ce^{-\eta D |\xi|^{2k-1}} \qquad (\deg(p) = 2k, D > 0)$$

The case of n variables

Theorem. *If m is even, $m = 2k$, let $p_m = \sum_{|\alpha|=k} c_\alpha y^{2\alpha}$, where the coefficients c_α are non-negative and the coefficients of y_j^{2k} are strictly positive for $1 \leq j \leq n$.*

If m is odd, $m = 2k+1$, let $p_m = \sum_{|\alpha|=2k+1} c_\alpha y^\alpha$, where all the coefficients c_α are non-negative, and the coefficients of y_j^{2k+1} are strictly positive for $1 \leq j \leq n$.

Then $p = p_m + q$, where q is arbitrary and $\deg(q) < m$, has the Airy property. In particular, $p = c_1 y_1^m + \ldots + c_n y_n^m + q$ has the Airy property if all c_i are > 0 and $\deg(q) < m$.

The paths (or cycles) are generalizations of the ones when we have only one variable. The integral is independent of the path and is identified with the Airy distribution by letting $\eta \to 0$. Since $\eta \to 0$ we cannot use the above estimates but rather some Payley-Wiener estimates.

The derivatives of A_p of order r are $O(|x|^{(r+n)/(m-1)})$ as $|x| \to \infty$

Invariant Airy functions on Lie algebras of compact type

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra and π the product of roots of a positive system. Let $\partial(\pi)$ be the corresponding differential operator on $\mathfrak{h}$.

Theorem. *Let p be a real invariant polynomial and $p_{\mathfrak{h}}$ its restriction to $\mathfrak{h}$. If $p_{\mathfrak{h}}$ has the Airy property on $\mathfrak{h}$, then p has the Airy property on $\mathfrak{g}$. If $A_{p_{\mathfrak{h}}}, A_p$ are the respective Airy functions, then*

$$A_p\Big|_{\mathfrak{h}} = \frac{1}{\pi}\partial(\pi)A_{p_{\mathfrak{h}}}$$

The proofs use the theory of invariant integrals and differential operators on semi simple Lie algebras, developed by Harish-Chandra in the 1950's.

Example

For $G = U(N)$ we have $\mathfrak{g} = \mathcal{H}(N)$, and $\mathfrak{h}$ is the diagonal subalgebra. If y_j is the linear function on $\mathfrak{h}$ taking $\mathrm{diag}(ia_1, \ldots, ia_N)$ to a_j, then

$$\pi = \prod_{k<\ell}(y_k - y_\ell), \qquad \partial(\pi) = \prod_{k<\ell}(\partial_k - \partial_\ell)$$

where

$$\partial_k = \frac{\partial}{\partial y_k}.$$

A formula of Kontsevich

For the case when $G = U(N)$ we work over $\mathcal{H}(N)$ rather than $i\mathcal{H}(N)$. Let

$$p(Y) = \mathrm{Tr}(Y^r) \qquad (Y \in \mathcal{H}(N))$$

Let A_m be the one-dimensional Airy function for the polynomial y^m. Let $y_1, \ldots, y_n$ be the linear coordinates on $\mathfrak{h}$. Then the theorem above leads to the following formula for A_p which was obtained by Kontsevich ($A_m^{(j-1)}$ is the $(j-1)^{\mathrm{th}}$ derivative of A_m):

$$\prod_{k>\ell}(y_k - y_\ell) A_p(\mathrm{diag}(y_1, \ldots, y_n)) = \det\left(A_m^{(j-1)}(y_i)\right).$$

The Kontsevich formula has some additional constants which appear because he works with standard Lebesgue measure. The constants disappear when we use the *self-dual* Lebesgue measure.

Airy functions on local fields

The definitions of Airy functions and Airy property make sense when $\mathbf{R}$ is replaced by a local field K. Let V be a finite dimensional vector space over K with a symmetric nondegenerate bilinear form. Then for a polynomial function $p : V \longrightarrow K$ the Airy distribution A_p is the Fourier transform of $\psi(p)$ where ψ is a non-trivial additive character of K. It appears that there is a corresponding theory in this case. For example we have the following result.

Theorem. *Let K be of characteristic 0. Then for any polynomial $p : K \longrightarrow K$ of degree ≥ 2 the Airy distribution is a locally constant function which is at most $O(|x|)$ as $|x| \rightarrow \infty$.*

There is a corresponding result when K has positive characteristic.

Compact p-adic Lie groups

Let D be a division algebra over K and G the compact p-adic ie group of units (elements of norm 1) of D. The computation of the Airy functions for D is a very interesting problem. It is also interesting to ask if these Airy functions have some geometric interpretation.

The work on p-adic Airy functions is ongoing and is joint with Rahul Fernandez and David Weisbart.