The Quantum Dice

V. S. Varadarajan

University of California, Los Angeles, CA, USA

UCLA, February 26, 2008

Abstract

Albert Einstein once said that he does not believe in a God who plays dice. He was referring to the fundamental principle that quantum theory permits only statistical assertions concerning the values of physical quantities. This issue, namely, whether the description of Nature in statistical terms is complete or even consistent, was the subject of a very lively debate-dialogue between Einstein and **Niels Bohr** in the years 1920-1930. Nowadays there is no debate as the principles of quantum theory have been verified countless times in experiments.

In this lecture (which requires no previous background) I shall discuss both classical and quantum probabilities. Although the meaning of probability remains the same in both contexts, the method of calculating them is different. I shall explain the strange but beautiful manner in which quantum probabilities are calculated, explaining the mathematical foundations in an elementary manner.

The meaning of probability

Jakob Bernoulli

- If we want to determine the probability p of an event, find out the number f of times it has occurred in a very large number N of trials. Then

$$p \approx \frac{f}{N}$$

Pierre Simon Laplace

- If the possible number of outcomes is n and the number of outcomes that correspond to the event in question is m, then the probability p of the event is

$$p = \frac{m}{n}.$$

Jakob Bernoulli (1654-1705)

- One of the pioneers of Calculus

- Wrote the first great treatise **Ars Conjectandi** (**Art of Conjecture**) on Probability. He introduced the concept of successive trials which are independent (**Bernoulli trials**), constructed a mathematical model for it, and proved that *in this model*, the probability that the frequency ratio f/N is close to p goes to 1 as $N \to \infty$ (**Law of large numbers)**.

- The Law of large numbers is the basis for **all** applications of probability.

Pierre Simon Lapace (1749-1827)

- **Celestial Mechanics**

  Wrote his monumental *Mécanique Céleste*.

- **Probability**

  His book *Théorie analytique des probabilités* was the first systematic treatise on diverse problems in probability theory.

- **Black holes**

  Pointed out that a star can be so massive that even light cannot escape its gravitational pull.

- **Central limit theorem**

  He had the insight to realize that sums of independent variables were nearly Gaussian but could not prove it.

He dominated so many areas that he was called the French Newton. He transformed both the Theory of Probability and Mathematical and Observational Astronomy into mighty disciplines.

Classical models for Probability

We identify the possible outcomes ω of some phenomenon that we are studying, and introduce the set Ω of **all** possible ω. For any event we associate the subset A of Ω of all outcomes ω that correspond to the event in question. Then $P(A)$ satisfies the following.

1. $0 \leq P(A) \leq 1$.
2. $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if the events A_1, A_2 are mutually exclusive

Remark. If Ω is finite, then P is defined for all subsets. For infinite Ω, P has to satisfy a stronger version of 2 (*countable additivity*).

$$P(A_1 \cup A_2 \cup \ldots \cup A_n \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$$

for events $A_1, A_2, \ldots, A_n, \ldots$ which are pairwise mutually exclusive. In uncountable Ω one has to restrict the definition of P to certain classes of subsets. P is a *probability measure*. This model for probability was created by **A. N. Kolmogorov**, one of 20^{th} century's greatest mathematicians.

Andrei Nikolaevich Kolmogorov (1903–1987)

Kolmogorov made profound contributions to an extraordinary number of areas in both pure and applied mathematics.

- Probability Theory
- Fourier Series
- Statistics
- Logic
- Computational complexity
- Celestial Mechanics
- Superposition of functions
- Turbulence
- Topology
- Teaching of Mathematics

The double slit experiment

Electrons from a source, all with the same energy, pass through two slits on a screen and fall on a plate. We place a counter at a variable point X on the plate and record the arrival of the electrons. If the source is a *weak* one, we will be able to count the electrons one by one as they arrive. Let the holes be numbered 1 and 2. Let $A_i = A_i(X)$ be the event that corresponds to the situation when only hole i is open, $i = 1, 2$. Let $A(X)$ correspond to the situation when both holes are open. If we use the classical model for probability, then $A(X) = A_1(X) \cup A_2(X)$ while the two events A_i are mutually exclusive. Write

$$P(X) = P(A(X)), \quad P_i(X) = P(A_i(X)).$$

Then we should expect that

$$P(X) = P_1(X) + P_2(X).$$

However experiments conclusively show that

$$P(X) \neq P_1(X) + P_2(X).$$

If we put a light source at the slits to see if the electrons pass through the slits, then

$$P(X) = P_1(X) + P_2(X).$$

How are we to understand this bizarre state of affairs?

Discussion

The *qualitative* explanation is based on the fact that all particles possess *both* wave and particle properties. This is the *wave-particle duality* and is a special case of the *complementarity principle*. It is the *experimental arrangement* that decides which kind of property is to be expected. When we do not check whether the electrons pass through the holes, the experimental arrangement presumably allows interference between the electrons and so results in a complicated pattern.

The problem is to construct probability models when we have to deal with the phenomenon of complementarity. There are two *equivalent* approaches due respectively to

- von Neumann
- Feynman.

John von Neumann (1903–1957)

One of the greatest mathematicians of all time, he worked in almost all parts of pure and applied mathematics. Here is a *partial* list of his achievements and the areas to which he contributed at a fundamental level.

- Logic
- Quantum mechanics

  His book, *Mathematical Foundations of Quantum Mechanics*, written when he was 29, is one of the great landmarks of 20$^{\text{th}}$ century Science.
- Infinite dimensional linear algebra
- Solution of Hilbert's 5$^{\text{th}}$ problem for compact groups
- Computer Science and the theory of automata

  His pioneering work on the theory of the computer entitles him to be regarded as the father of the computer age.
- Atomic and Hydrogen bombs

  Played an important role in the Manhattan project and later, with **Stanislaw Ulam**, in the creation of the hydrogen bomb.
- Numerical analysis, including weather prediction

Logic of quantum mechanics:

The von Neumann model

The double slit experiment suggests that the logic of quantum theory is *not* classical, i.e., not a *boolean algebra*, because of the complementarity principle. As long as the statements are made with respect to a *fixed experimental arrangement*, the logic is classical. But when statements are being made about *different experimental arrangements*, there is *interference* and one has to abandon boolean algebras.

In the von Neumann model the *experimental statements* (=*quantum events*) are arranged as in a *complex euclidean projective geometry*, namely as the set of *linear subspaces of a complex euclidean space*: inclusion corresponds to implication, and negation to taking orthogonal complements.

Complex scalars are essential for many reasons (charge conjugation for instance).

Probabilities in the von Neumann model

For each linear subspace S of a complex euclidean space (Hilbert space) $\mathcal{H}$ we have a probability $P(S)$ of the event that S represents. The classical additivity still holds but *only for orthogonal subspaces*:

$$P(A_1 \oplus A_2) = P(A_1) + P(A_2) \qquad (A_1 \perp A_2).$$

It is possible that for suitable A_1, A_2, we have $A = A_1 \oplus A_2$ where A_1 and A_2 are *not* orthogonal; in this case in general

$$P(A_1 \oplus A_2) \neq P(A_1) + P(A_2).$$

We shall say that the events A_1 and A_2 *interfere* with each other. Such a P is called a *quantum probability measure*. The statistical basis of quantum theory is the statement:

- P *is* the state of the system.

- *All* of quantum theory is subsumed under this model.

Description of states: Gleason-Mackey theorem

For any unit vector ϕ of $\mathcal{H}$ defines a quantum probability measure P_ϕ as follows:

$$P_\phi(S) = ||\phi_S||^2 \qquad (\phi_S = \text{ the projection of } \phi \text{ on } S).$$

The relation

$$||\phi_S||^2 = ||\phi_{S_1}||^2 + ||\phi_{S_2}||^2 + \ldots + ||\phi_{S_n}||^2$$

where

$$S = S_1 \oplus S_2 \oplus \ldots \oplus S_n, \qquad S_i \perp S_j (i \neq j)$$

shows that P_ϕ is a quantum probability measure. Moreover

$$P_\phi = P_\psi \Longleftrightarrow \phi = c\psi (|c| = 1)$$

Theorem (Gleason–Mackey) *If* $\dim(\mathcal{H}) \geq 3$, *then every quantum probability measure is a convex linear combination of the* P_ϕ.

Discussion

The Gleason-Mackey theorem is the basis of the principle in QM that states are represented by unit vectors and that ϕ and $c\phi(|c| = 1)$ define the same state. The *Schrödinger equation* describes how ϕ varies with time.

When $\dim(\mathcal{H}) = 3$ it is equivalent to saying that any function f defined on the unit sphere satisfying

(1) $0 \leq f(\phi) \leq 1$
(2) $f(\phi) = f(c\phi), |c| = 1$
(3) $f(\phi_1) + f(\phi_2) + f(\phi_3) = 1$ ((ϕ_i) an ON basis)

is of the form

$$f(\phi) = (A\phi, \phi)$$

where A is a hermitian matrix with eigenvalues ≥ 0 and trace 1. Try proving this!

Richard P. Feynman (1918–1998)

The most imaginative and iconolastic physicist of his generation, and an American original. He is most famous for

- Path integral (sum over histories) approach to QM
- *The Feynman Lectures on Physics*
- Theory of superfluidity

  The behaviour of liquid Helium at near zero (Absolute) temperature
- Quantum Electrodynamics

  Shared the Nobel prize with Julian Schwinger and Sin-Itiro-Tomonaga in 1965.
- Feynman diagrams
- Pioneered quantum computing and nano technology
- Role in the *Challenger Commission*

Calculation of probabilties in the double slit experiment

The general principle is that one calculates the *complex amplitude* ϕ_i of the path from the source S to the point X on the plate when only the slit i is open. Then

$$P(X) = |\phi|^2, \qquad P_i(X) = |\phi_i|^2$$

where

$$\phi = \phi_1 + \phi_2.$$

In general

$$|\phi_1 + \phi_2|^2 \neq |\phi_1|^2 + |\phi_2|^2.$$

Path integral

The probability for a particle to go from spacetime point A to a spacetime point B, is

$$|\phi|^2$$

where ϕ is given by the *Feynman integral*

$$\phi = \int_{A \to B} \exp\{iS(\gamma)/\hbar\}.$$

Here the integration is over the space of *all* paths γ from A to B and $S(\gamma)$ is the *classical action* for the path γ. Also $\hbar$ is Planck's constant divided by 2π.

In the domain where $\hbar$ is very small, the *method of stationary phase* shows that the main contributions to the integral come from the paths for which the action S is *stationary*, i.e., *the classical paths*. Thus we recover the *classical action principle*.

Warning! This is only a *heuristic* formula. To make it rigorous requires considerable effort.

Rigorous meaning of the path integral

Because of the factor i in the exponent of the path integral the Feynman integral, even under a very generous interpretation, is *highly oscillatory*. A *toy model* is the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{-ix^2/2\}dx.$$

If we change i to σ where $\sigma > 0$, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{-\sigma x^2/2\}dx = \frac{1}{\sqrt{\sigma}}.$$

The function of σ on the right is analytic on $\mathrm{Re}(\sigma) > 0$ and continuous on $\mathrm{Re}(\sigma) \geq 0$ and so we can evaluate at i to get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{-ix^2/2\}dx = \frac{1-i}{\sqrt{2}}.$$

Discussion

Wiener measure In practice, what is done is to change over to a *genuine probability measure*, called the *Wiener measure*, which is a sort of *Gaussian measure* on the path space. It models *Brownian Motion.* One then evaluates the integrals with respect to the Wiener measure and then make an *analytic continuation* of the result.

Sum over histories The path integral and methods of calculating it, have revolutionized modern physics and mathematics. It is briefly called the *sum over histories* and has been used by **Stephen Hawking** and others to construct a remarkable theory of *quantum black holes*. In mathematics, in addition to its applications to various applied areas, it has led to alternative views and proofs of such landmark results as the *Index Theorem* of **Atiyah-Singer**.

REFERENCES

[1]] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.

[2] V. S. Varadarajan, *Geometry of Quantum Theory*, Springer, 1984, 2006.

[3] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

[4] K. R. Parthasarathy, *Introduction to Quantum Stochastic Calculus*, Birkhäuser, 1992.

[5] R. P. Feynman, *The concept of Probability in Quantum Mechanic*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1951, 533–541.

[6] R. P. Feynman, *Spacetime approach to non-relativistic quantum mechanics*, Reviews of Modern Physics, 20(1948), 367–387.

[7] B. Simon, *Functional Integration and Quantum Physics*, Academic press, 1979.