# 2. THE CONCEPT OF A SUPERMANIFOLD



**2.1. Geometry of physical space.** Someone who is already familiar with the theory of differentiable manifolds or algebraic varieties can be very quickly introduced to the notion of a supermanifold and the concept of supersymmetry. Just as the manifolds and varieties are defined by first starting with local pieces on which the coordinate functions are defined, and then gluing these local pieces together, a supermanifold may be defined as a space on which locally one has coordinates $x^1, \ldots, x^n, \theta^1, \ldots \theta^r$ where the $x^i$ are the usual commuting coordinates and the $\theta^j$, the anticommuting (fermionic) coordinates, with the various sets of local chats being related by transformations of the appropriate smoothness type. Everything is then done exactly as in the classical theory. Supersymmetries are diffeomorphisms of such spaces and these form super Lie groups. One can construct a theory of differentiation and integration on such spaces and write down equations of motions of particles and fields starting from suitable Lagrangians. If one starts with a supersymmetric Lagrangian then one obtains an action of the supersymmetric group on the solutions of the field equations thus defined. The stage on which supersymmetic quantum field theory lives is then a super spacetime, either flat or curved. However, such a treatment, in spite of being very practical and having the advantage of getting into the heart of matters very quickly, does not do full justice either to the understanding of the concepts at a deeper level or to comprehending the boldness of the generalization of conventional geometry that is involved here. In this chapter we shall take a more leisurely and foundational approach. We shall try to look more closely at the evolution of the concept of space as a geometrical object starting from euclid and his plane (and space) and ending with the superspacetimes of the phycisists. This is however a very complicated story with multiple themes and replete with many twists and turns and really too complex to be discussed briefly. Nevertheless the attempt to unravel it will provide (I hope) at least some insight into super geometry at a fundamental level.

We begin with the evolution of geometry. Geometry is perhaps the most ancient part of mathematics. Euclid is its most celebrated expositor and his *Elements* are still the object of great admiration. Euclid's geometry is an idealized distillation of our experience of the world around us. To his successors all of Euclid's axioms except one appeared to be entirely natural. The exception was the famous axiom of parallels. Indeed Euclid himself recognized the special nature of lines in a plane which do not meet; this is clear from the fact that he went as far as he could without the parallel axiom and started using it only when it became absolutely indispensable. One of the crucial places where it is necessary to invoke this axiom is in the proof that the sum of the angles of a triangle is equal to two right angles. One may therefore say that starting from Euclid himself the axiom of parallels was the source of a lot of discomfort and hence the object of intense scrutiny. Already Proclus in the fifth century A. D. was quite sceptical of this axiom and so he might be regarded as one of the earliest figures who thought that an alternative system of geometry was a possibility, or at least that the axiom of parallels should be looked into more closely. One of the first people who started a systematic study of geometry where no assumptions were made about parallel lines was the Italian Jesuit priest Saccheri. Later Legendre made an intense study of the parallel axiom and at one point even thought that he had proved it to be a consequence of the remaining axioms. Eventually he settled for the weaker statement that *the sum of the angles of a triangle is always less than or equal to two right angles, and that the parallel axiom is equivalent to saying that the sum is equal to two right angles; and further, that if this is valid just for one triangle, then it is valid for all triangles*. In retrospect, as we shall see later, this result of Legendre would appear as the definitive formulation of the axiom of parallels that characterizes euclidean geometry, in as much as it describes the fact that euclidean geometry is *flat*.

Eventually this line of thought led to the discovery of noneuclidean geometry by Bolyai and Lobachevsky, although Gauss, as it became clear from his unpublished manuscripts which were discovered after his death, had anticipated them. The discovery of noneuclidean geometry did not end speculations on this subject because it was not at first clear whether the new axioms were self-consistent. However Klein and Beltrami constructed models for noneuclidean geometry *entirely within* the framework of euclidean geometry, from which it followed that noneuclidean geometry was as self-consistent as euclidean geometry. The question of the consistency of euclidean geometry was however not clarified properly till Hilbert came to the scene. He gave the first rigorous presentation of a complete set of axioms of euclidean geometry (using some crucial ideas of Pasch), and proved that its consistency was equivalent to the consistency of arithmetic. What happened after this-the revolution in logic-is quite well-known and is not of concern for us here.

One reason why the discovery of noneuclidean geometry took so long might have been the fact that there was universal belief that euclidean geometry was special because it described the space we live in. Stemming from this uncritical acceptance of the view that the geometry of space is euclidean was the conviction that there was no other geometry. Philosophers like Kant argued that the euclidean nature of space was a fact of nature and the weight of their authority was very powerful. From our perspective we know of course that the question of the geometry of space is of course entirely different from the question of the existence of geometries which are not euclidean. Gauss was the first person who clearly understood the difference between these two questions. In Gauss's Nächlass one can find his computations of the sums of angles of each of the triangles that occured in his triangulation of the Hanover region; and his conclusion was that the sum was always two right angles within the limits of observational errors. Nevertheless, quite early in his scientific career Gauss became convinced of the possibility of constructing noneuclidean geometries, and in fact constructed the theory of parallels, but because of the fact that the general belief in euclidean geometry was deeply ingrained, Gauss decided not to publish his researches in the theory of parallels and the construction of noneuclidean geometries for fear that there would be criticisms of such investigations by people who did not understand these things ("the outcry of the Boeotians").

Riemann took this entire circle of ideas to a completely different level. In his famous inaugural lecture of 1854 he touched on all of the aspects we have mentioned above. He pointed out to start with that a space does not have any structure except that it is a continuum in which points are specified by the values of $n$ coordinates, $n$ being the *dimension* of the space; on such a space one can then impose many geometrical structures. His great insight was that a geometry should be built from the infinitesimal parts. He treated in depth geometries where the distance between pairs of infinitely near points is pythagorean, formulated the central questions about such geometries, and discovered the set of functions, the sectional curvatures, whose vanishing characterized the geometries which are euclidean, namely those whose distance function is pythagorean not only for infinitely near points but even for points which are a finite but small distance apart. If the space is the one we live in, he stated the principle that its geometrical structure could only be determined *empirically*. In fact he stated explicitly that the question of the geometry of physical space does not make sense independently of physical phenomena, i.e., that space has no geometrical structure until we take into account the physical properties of matter in it, and that this structure can be determined only by measurement. Indeed, he went so far as to say that *the physical matter determined the geometrical structure of space.*

Riemann's ideas constituted a profound departure from the perceptions that had prevailed until that time. In fact no less an authority than Newton had asserted that space by itself is an absolute entity endowed with euclidean geometric structure, and built his entire theory of motion and celestial gravitation on that premise. Riemann went completely away from this point of view. Thus, for Riemann, space derived its properties from the matter that occupied it, and that the only question that can be studied is whether the physics of the world made its geometry euclidean. It followed from this that only a mixture of geometry and physics could be tested against experience. For instance measurements of the distance between remote points clearly depend on the assumption that a light ray would travel along shortest paths. This merging of geometry and physics, which is a central and dominating theme of modern physics, may be thus traced back to Riemann's inaugural lecture.

Riemann's lecture was very concise; in fact, as it was addressed to a mostly general audience, there was only one formula in the whole paper. This circumstance, together with the fact that the paper was only published some years after his death, had the consequence that it took a long time for his successors to understand what he had discovered and to find proofs and variants for the results he had stated. The elucidation and development of the purely mathematical part of his themes was the achievement of the Italian school of differential geometers. On the other hand, his ideas and speculations on the structure of space were forgotten completely except for a "solitary echo" in the writings of Clifford[1]. This was entirely natural because most mathematicians and physicists were not concerned with philosophical speculations about the structure of space and Riemann's ideas were unbelievably ahead of his time.

However the whole situation changed abruptly and fantastically in the early decades of the twentieth century when Einstein discovered the theory of relativity. Einstein showed that physical phenomena already required that one should abandon the concept of space and time as objects existing independently by themselves, and that one must take the view that they are rather *phenomenological* objects, i.e., dependent on phenomena. This is just the Riemannian view except that Einstein arrived at it in a completely independent manner and space and time were both included in the picture. It followed from Einstein's analysis that the splitting of space and time was not absolute but depends on the way an observer perceives things around oneself. In particular, only *spacetime*, the totality of physical events taking place, had an intrinsic significance, and that only phenomena could determine what its structure was. Einstein's work showed that spacetime was a differential geometric object of great subtlety, indeed a pseudo Riemannian manifold of signature $(+, -, -, -)$, and its geometry was noneuclidean. The central fact of Einstein's theory was that *gravitation* is just a manifestation of the *Riemannian curvature* of spacetime. Thus there was a complete fusion of geometry and physics as well as a convincing vindication of the Riemannian view.

Einstein's work, which was completed by 1917, introduced curved spacetime only for discussing gravitation. The questions about the curvature of spacetime did not really have any bearing on the other great area of physics that developed in the twentieth century, namely *quantum theory*. This was because gravitational effects were not important in atomic physics due to the smallness of the masses involved, and so the merging of quantum theory and relativity could be done over *flat*, i.e., Minkowskian spacetime. However this situation has gradually changed in recent years. The reason for this change lies in the belief that from a fundamental point of view, the world, whether in the small or in the large, is quantum mechanical, and so one should not have one model of spacetime for gravitation and another for atomic phenomena. Now gravitational phenomena become important for particles of atomic dimensions only in distances of the order of $10^{-33}$ cm, the so-called Planck length, and at such distances the principles of general relativity impose great obstacles to even the measurement of coordinates. Indeed, the calculations that reveal this may be thought of as the real explanations for Riemann's cryptic speculations on the geometry of space in the infinitely small. These ideas led slowly to the realization that radically new models of spacetime were perhaps needed to organize and predict fundamental quantum phenomena at extremely small distances and to unify quantum theory and gravitation. Starting from the 1970's a series of bold hypotheses have been advanced by physicists to the effect that spacetime at extremely small distances is a geometrical object of a type hitherto not investigated. One of these is what is called *superspace.* Together with the idea that the fundamental objects to be investigated are not point particles but extended objects like strings, the physicists have built a new theory, the theory of superstrings, that appears to offer the best chance of unification of all the fundamental forces. In the remaining sections of this chapter I shall look more closely into the first of the ideas mentioned above, that of superspace.

Superspace is just what we call a supermanifold. As I mentioned at the beginning of Chapter 1, there has been no experimental evidence that spacetime has the structure of a supermanifold. Of course we are not speaking of direct evidence but verifications, in collision experiments, of some of the consequences of a super geometric theory of elementary particles (for instance, the finding of the superpartners of known particles). There are reasons to expect however that in the next generation of collision experiments to be conducted by the new LHC (Large Hadron

Collider), being built by CERN and expected to be operational by about 2005, some of these predictions will be verified. However, no matter what happens with these experiments, the idea of superspace has changed story of the structure of space completely, and a return to the older point of view appears unlikely.

I must also mention that an even more radical generalization of space as a geometrical object has been emerging in recent years, namely what people call *noncommutative geometry.* Unlike super geometry, noncommutative geometry is *not localizable* and so one does not have the picture of space as being built out of its smallest pieces. People have studied the structure of physical theories on such spaces but these are even more remote from the physical world than super geometric theories[2].

**Riemann's inaugural talk.** On June 10, 1854, Riemann gave a talk before the Göttingen Faculty that included Gauss, Dedekind, and Weber in the audience. It was the lecture that he had to give in order to regularize his position in the university. It has since become one of the most famous mathematical talks ever given[3]. The title of Riemann's talk was *Über die Hypothesen, welche der geometrie zu Grunde liegen* (*On the hypotheses which lie at the foundations of geometry*). The circumstances surrounding the topic of his lecture were themselves very peculiar. Following accepted convention Riemann submitted a list of three topics from which the Faculty were supposed to choose the one which he would elaborate in his lecture. The topics were listed in decreasing order of preference which was also conventional, and he expected that the Faculty would select the first on his list. But Gauss, who had the decisive voice in such matters choose the last one which was on the foundations of geometry. So, undoubtedly intrigued by what Riemann was going to say on a topic about which he, Gauss, had spent many years thinking, and flouting all tradition, Gauss selected it as the topic of Riemann's lecture. It appears that Riemann was surprised by this turn of events and had to work intensely for a few weeks before his talk was ready. Dedekind has noted that Gauss sat in complete amazement during the lecture, and that when Dedekind, Gauss, and Weber were walking back to the department after the talk, Gauss spoke about his admiration and astonishment of Riemann's work in terms that Dedekind said he had never observed Gauss to use in talking about the work of any mathematician, past or present[4]. If we remember that this talk contained the sketch of the entire theory of what we now call Riemannian geometry, and that this was brought to an essentially finished form in the few weeks prior to his lecture, then we would have no hesitation in regarding this work of Riemann as one of the greatest intellectual feats of all time in mathematics.

In his work on complex function theory he had already discovered that it is necessary to proceed in stages: first one has to start with a space which has just a topological structure on it, and then impose complex structures on this bare framework. For example, on a torus one can have many inequivalent complex structures; this is just a restatement of the fact that there are many inequivalent fields of elliptic functions, parametrized by the quotient of the upper half plane by the modular group. In his talk Riemann started with the concept of what we now call an $n$-dimensional manifold and posed the problem of studying the various geometries that can be defined on them. Riemann was thus aware that on a given manifold there are many possible metric structures, so that the problem of which structure is the one appropriate for physical space required empirical methods for its solution. Now both euclidean and noneuclidean geometry were defined in completely global terms. Riemann initiated the profound new idea that *geometry should be built from the infinitesimal to the global*. He showed that one should start from the form of the function that gave the distance between *infinitesimally near points*, and then to determine distances between finitely separated points by computing the lengths of paths connecting these points and taking the shortest paths. As a special case one has those geometries in which the distance $ds^2$ (called the *metric*) between the points $(x_1, \ldots, x_n)$ and $(x_1 + dx_1, \ldots, x_n + dx_n)$, is given by the pythagorean expression

$$ds^2 = \sum_{i,j} g_{ij}(x_1, \ldots, x_n) dx_i dx_j,$$

where the $g_{ij}$ are functions, *not necessarily constant*, on the underlying space with the property the matrix $(g_{ij})$ is positive definite. Euclidean geometry is characterized by the choice

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2.$$

Riemann also discussed briefly the case

$$ds^4 = F(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$$

where $F$ is a homogeneous polynomial of degree 4. For general not necessarily quadratic $F$ the geometry that one obtains was treated by Finsler and such geometries are nowadays called *Finslerian*[5].

Returning to the case when $ds^2$ is a quadratic differential form Riemann emphasized that the structure of the metric depends on the choice of coordinates. For example, euclidean metric takes an entirely different form in polar coordinates. It is natural to call two metrics *equivalent* if one can be obtained from the other by a change of coordinates. Riemann raised the problem of determining invariants of a metric so that two given metrics could be asserted to be equivalent if both of them have the same invariants. For a given metric Riemann introduced its *curvature* which was a quantity depending on $n(n-1)/2$ variables, and asserted that its vanishing is the necessary and sufficient condition for the metric to be euclidean, i.e., to be equivalent to the euclidean one. The curvature at a point depended on the $n(n-1)/2$ planar directions $\pi$ at that point, and given any such $\pi$, it was the *Gaussian curvature* of the infinitesimal slice of the manifold cut out by $\pi$. Obviously, for the euclidean metric, the Gaussian curvature is 0 in all planar directions at all points. Thus Riemann connected his ideas to those of Gauss but at the same generalized Gauss's work to all dimensions; moreover he discovered the central fact in all of geometry that the euclidean geometries are precisely those that are *flat*, namely, their curvature is 0 in all planar directions at all points. The case when this curvature is a constant $\alpha \neq 0$ in all directions at all points was for him the next important case. In this case he found that for each $\alpha$ there was only one geometry whose $ds^2$ can be brought to the form

$$ds^2 = \frac{\sum dx_i^2}{\left[1 + \frac{\alpha}{4}\sum x_i^2\right]^2}$$

in suitable coordinates. The cases $\alpha >, =, < 0$ lead to *elliptic, euclidean* and *hyperbolic* geometries, the hyperbolic case being the noneuclidean geometry of Bolyai–Lobachevsky. People have since discovered other models for the spaces of constant curvature. For instance the noneuclidean plane can be modeled by the upper half plane with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) \qquad (y > 0).$$

This is often called the *Poincaré upper half plane.*

In the last part of his lecture Riemann discussed the problem of physical space, namely the problem of determining the *actual* geometry of *physical space.* He enunciated two bold principles which went completely against the prevailing opinions:

**R1**. Space does not exist independently of phenomena and its structure depends on the extent to which we can observe and predict what happens in the physical world.

**R2**. In its infinitely small parts space may not be accurately described even by the geometrical notions he had developed.

It is highly interesting to read the exact remarks of Riemann and see how prophetic his vision was:

*"Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena ..."*

*An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to ensure that this work is not hindered by too restricted concepts, and that the progress in comprehending the connection of things is not obstructed by traditional prejudices.*

**Einstein and the geometry of spacetime.** It took mathematicians over 50 years to comprehend and develop the ideas of Riemann. The Italian school of geometers, notably Ricci, Bianchi, Levi-Civita, and their collaborators, discovered the tensor calculus and covariant differential calculus in terms of which Riemann's work could be most naturally understood and developed further. The curvature became a covariant tensor of rank 4 and its vanishing was equivalent to the metric being euclidean. The efforts of classical mathematicians (Saccheri, Legendre etc) who tried to understand the parallel axiom, could now be seen as efforts to describe flatness and curvature in terms of the basic constructs of euclid's axioms. In particular, as the deviation from two right angles of the sum of angles of a triangle is proportional to the curvature, its vanishing is the flatness characteristic of euclidean geometry.

Riemann's vision in **R1** became a reality when Einstein discovered the theory of general relativity. However it turned out that spacetime, not space, was the fundamental intrinsic object and that its structure was to be determined by physical phenomena. Thus this was an affirmation of the Riemannian point of view with the proviso that space was to be replaced by spacetime. Einstein's main discoveries were as follows.

- **E1**. Spacetime is a pseudo Riemannian manifold, i.e., its metric $ds^2$ is not euclidean but has the signature $(+, -, ; -, -)$ at each point.
- **E2**. Gravitation is just the physical manifestation of the curvature of spacetime.
- **E3**. Light travels along geodesics.

The metric of spacetime was not euclidean but has the form

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

at each point. This is what is nowadays called a *Lorentzian* structure. Even in the absence of matter the geometry of spacetime could not be asserted to be flat but only *Ricci flat*, i.e., that its Ricci tensor (which can be calculated from the Riemann curvature tensor) is 0. Einstein also suggested ways to put his ideas to test. One of the most famous predictions of his theory was that light rays, traveling along geodesics of the noneuclidean geometry of spacetime, would appear to be bent by the gravitational fields near a star such as the sun. Everyone knows that this was verified during an annular solar eclipse in Sobral off the coast of Brazil in 1919. Subsequently even more precise verifications have been made using radio astronomy. As far as I know however, the data are not accurate enough to decide between Einstein's theory and some alternative ones.

The second of Riemann's themes, which is hinted at in **R2**, lay dormant till the search for a unified field theory at the quantum level forced the physicists to reconsider the structure of spacetime at extremely small distances. One of the ideas to which their efforts led them was that the geometry of spacetime was supersymmetric with the usual coordinates supplemented by several anticommuting (fermionic) ones. This is a model that reflects the highly volatile structure of spacetime in small regions where one can pass back and forth between bosonic and fermionic particles. Modern string theory takes Riemann's vision even further, and replaces the points of spacetime by strings, thereby making the geometry even more noncommutative. However string theory is still very incomplete; no one knows the mathematical structure of a geometry that is string-like at very small distances and approximates Riemannian geometry in the large.

**2.2. The mathematical evolution of the concept of space and its symmetries.** Parallel to the above development of the concept of the geometry of physical space, and in counterpoint to it, was the evolution of the notion of a manifold from the mathematical side. We shall now give a very brief survey of how the concepts of a manifold or space and its symmetries evolved from the *mathematical* point of view.

**Riemann surfaces.** The first truly global types of spaces to emerge were the Riemann surfaces. Riemann's work made it clear that the local complex variable $z$ on such spaces did not have any intrinsic significance and that the really interesting questions were global. However, in Riemann's exposition, the Riemann surfaces generally appeared as a device to make multivalued functions on the complex plane singlevalued. Thus they were viewed as (ramified) coverings of the (extended) complex plane. This obscured to some extent the intrinsic nature of the theory of functions on Riemann surfaces. It was Felix Klein who understood this clearly and emphasized that Riemann surfaces are independent objects and offer the correct context to study complex function theory[6].

The first rigorous description of the concept of Riemann surface is due to Weyl. He formulated for the first time, in his famous book[6] published in 1911, the rigorous notion of a Riemann surface as a complex manifold of dimension 1 with local coordinates which are related on overlapping local domains by biholomorphic transformations. Even today, this is the way we think of not only Riemann surfaces but *all* manifolds.

Weyl's work was the starting point of the view that space is characterized by starting with a topological structure and selecting classes of local coordinates at its points. The nature of the space is then determined by the transformations in the usual affine spaces that connect the various local coordinates. If the connecting transformations are holomorphic (resp. real analytic, smooth, $C^k$), we obtain a holomorphic (resp. real analytic, smooth, $C^k$) manifold. Starting with this axiomatic view it is natural to ask if such abstract spaces could be realized as subspaces of conventional affine or projective spaces. This leads to *imbedding theorems*. Depending on which class of spaces one is interested in, these theorems are associated with Whitney (smooth), Morrey (real analytic), Nash (Riemannian), Kodaira (Kähler), and so on.

**Riemannian and affinely connected manifolds.** In the years following Riemann's epoch-making work the comprehension and dissemination of Riemann's ideas were carried out by Ricci, Levi-Civita, Bianchi, Weyl, and many others. In 1917[7] Weyl introduced a new theme. He noticed that the geometry of a Riemannian manifold is controlled by the notion of parallel transport introduced by Levi-Civita, and realized that this notion could be taken as a basis for geometry without assuming that it arose from a metric. This was the way that the notion of a Riemannian manifold was generalized to an affinely connected manifold, i.e., a manifold *equipped with a connection*. Weyl also introduced another notion, namely that of *conformality*, and discovered that there is a tensor, the so-called Weyl tensor, whose vanishing was equivalent to the space being conformally euclidean.

**Groups of symmetries of space.** Already in euclidean geometry one can see the appearance of transformation groups although only implicitly. For instance, the proof of congruence of two triangles involves moving one triangle so that it falls exactly on the second triangle. This is an example of a *congruent transformation*. In the analytical model of euclidean geometry the congruent transformations are precisely the elements of the group of rigid motions of the euclidean plane, generated by the translations, rotations, and reflections. In the Klein model for noneuclidean geometry the group of congruent transformations is the subgroup of the linear transformations of the projective plane which preserve a circle. It was Klein who put the group theoretic framework in the foreground in his famous Erlangen Programme and established the principle that the structure of a geometry was completely determined by the group of congruent transformations belonging to it.

In the decades following Riemann's work a new theme entered this picture when Sophus Lie began the study of transformations groups which were completely general and acted on *arbitrary* manifolds, even when there was no geometrical structure on the manifolds. Roughly speaking this was a non-linear version of the group of affine transformations on an affine space. What was original with Lie was that the transformations depended on a *finite set of continuous parameters* and so one could, by differentiating with respect to these parameters, study their action *infinitesimally.* In modern terminology, Lie considered *Lie groups* (what else) acting on smooth manifolds. The action of the group thus gave rise to a vector space of *vector fields* on the manifold which formed an algebraic structure, namely a *Lie algebra*, that completely determined the action of the Lie group. Thus Lie did to group actions what Riemann had done for geometry, i.e., made them infinitesimal. No geometrical structure was involved and Lie's researches were based on the theory of differential equations.

Originally Lie wanted to classify all actions of Lie groups on manifolds. But this turned out to be too ambitious and he had to settle for the study of low dimensional cases. But he was more successful with the groups themselves which were viewed as acting on themselves by translations. His work led eventually to the basic theorems of the subject, the so-called fundamental theorems of Lie: namely, that the Lie algebra is an invariant of the group, that it determined the group in a neighborhood of the identity, and that to any Lie algebra one can associate at least a piece of a Lie group near the identity, namely a *local Lie group*, whose associated Lie algebra is the given one. As for the classification problem the first big step was taken by Killing when he classified the *simple Lie groups*, or rather, following Lie's idea, the *simple Lie algebras*, over the *complex numbers*. However the true elucidation of this new theme had to wait for the work of Elie Cartan.

Cartan is universally regarded as the greatest differential geometer of his generation. He took differential geometry to an entirely new level using, among other things, the revolutionary technique of "moving frames". But for our purposes it is his work on Lie groups and their associated homogeneous spaces that is of central importance. Building on the earlier but very incomplete work of Killing, Cartan obtained the rigorous classification of all simple Lie algebras over the complex numbers. He went beyond all of his predecessors by making it clear that one had to work with spaces and group actions *globally*. For instance he established the global version of the so-called third fundamental theorem of Lie, namely the existence of a global Lie group corresponding to a given Lie algebra. Moreover he discovered a remarkable class of Riemannian manifolds on which the simple Lie groups over real numbers acted transitively, the so-called *Riemannian symmetric spaces*. Most of the known examples of *homogeneous spaces* were included in this scheme since they are symmetric spaces. With Cartan's work one could say that a fairly complete idea of space and its symmetries was in place from the differential geometric point of view. Cartan's work provided the foundation on which the modern development of general relativity and cosmology could be carried out.

It was during this epoch that De Rham obtained his fundamental results on the cohomology of a differentiable manifold and its relation to the theory of integration of closed exterior differential forms over submanifolds. Of course this was already familiar in low dimensions where the theory of line and surface integrals, especially the theorems of Green and Stokes, played an important role in classical continuum physics. De Rham's work took these ideas to their proper level of generality and showed how the cohomology is completely determined by the algebra of closed exterior differential forms modulo the exact differential forms. A few years later Hodge went further and showed how, by choosing a Riemannian metric, one can describe all the cohomology by looking at the harmonic forms. Hodge's work led to the deeper understanding of the Maxwell equations and was the precursor of the modern theory of Yang-Mills equations. Hodge also pioneered the study of the topology of algebraic varieties.

**Algebraic geometry.** So far we have been concerned with the evolution of the notion of space and its symmetries *from the point of differential geometry*. But there was, over the same period of time, a parallel development of geometry from the *algebraic* point of view. Algebraic geometry of course is very ancient; since it relies entirely on algebraic operations, it even predates calculus. It underwent a very intensive development in the nineteenth century when first the theory of algebraic curves, and then algebraic surfaces, were developed to a state of perfection. But it was not till the early decades of the twentieth century that the algebraic foundations were clarified and one could formulate the main questions of algebraic geometry with full rigour. This foundational development was mainly due to Zariski and Weil.

One of Riemann's fundamental theorems was that every compact Riemann surface arose as the Riemann surface of some *algebraic function.* It followed from this that there is no difference between the transcendental theory which stressed topology and integration, and the algebraic theory, which used purely algebraic and geometric methods and worked with algebraic curves. The fact that compact Riemann surfaces and nonsingular algebraic curves were one and the same made a great impression on mathematicians and led to the search for a purely algebraic foundation for Riemann's results. The work of Dedekind and Weber started a more algebraic approach to Riemann's theory, one that was more general because it allowed the possibility to study these objects in characteristic $p > 0$. This led to a true revolution in algebraic geometry. A significant generalization of the idea of an algebraic variety occured when Weil, as a basis for his proof of the Riemann hypothesis for algebraic curves of arbitrary genus, developed the theory of abstract algebraic varieties in any characteristic and intersection theory on them. The algebraic approach had greater scope however because it automatically included singular objects also; this had an influence on the analytic theory and led to the development of *analytic spaces*.

In the theory of general algebraic varieties started by Zariski and Weil and continued by Chevalley, no attempt was made to supply any geometric intuition. The effort to bring the geometric aspects of the theory of algebraic varieties more to the foreground, and to make the theory of algebraic varieties resemble the theory of differentiable manifolds more closely, was pioneered by Serre who showed in the 1950's that the theory of algebraic varieties could be developed in a completely geometric fashion imitating the theory of complex manifolds. Serre's work revealed the geometric intuition behind the basic theorems. In particular he showed that one can study the algebraic varieties in any characteristic by the same sheaf theoretic methods that were introduced by him and Henri Cartan in the theory of complex manifolds where they had been phenomenally successful.

The foundations of classical algebraic geometry developed up to this time turned out to be entirely adequate to develop the theory of groups that acted on the algebraic varieties. This was done by Chevalley in the 1950's. One of Chevalley's aims was to determine the projective varieties that admitted a *transitive* action by an *affine* algebraic group, and classify both the spaces and groups that are related in this manner. This comes down to the classification of all *simple* algebraic groups. Chevalley discovered that this was essentially the same as the Cartan-Killing classification of simple Lie algebras over $\mathbf{C}$, except that the classification of simple algebraic groups could be carried out over an algebraically closed field of arbitrary characteristic, directly working with the groups and not through their Lie algebras. This meant that his proofs were new even for the complex case of Cartan and Killing. The standard model of a projective variety with a transitive affine group of automorphisms is the Grassmannian or a flag manifold, and the corresponding group is $\mathrm{SL}(n)$. Chevalley's work went even beyond the classification. He discovered that a simple group is actually an object defined over $\mathbf{Z}$, the ring of integers; for instance, if we start with a complex simple Lie algebra $\mathfrak{g}$ and consider the group $G$ of automorphisms of $\mathfrak{g}$, $G$ is defined by polynomial equations with integer coefficients as a subgroup of $\mathrm{GL}(\mathfrak{g})$. So the classification yields simple groups over *any finite field*, the so-called finite groups of Lie type. It was by this method that Chevalley constructed new simple finite groups. This development led eventually to the classification of finite simple groups.

The theory of Serre was however not the end of the story. Dominating the landscape of algebraic geometry at that time (in the 1950's) was a set of conjectures that had been made by Weil in 1949. The conjectures related in an audacious manner the generating function of the number of points of a smooth projective variety over a finite fields and its extensions with the complex cohomology of the same variety viewed as a smooth complex projective manifold (this is only a rough description). For this purpose what was needed was a cohomology theory in *characteristic zero* of varieties defined over fields of *any characteristic*. Serre's theory furnished only a cohomology over the same field as the one over which the varieties were defined, and so was inadequate to attack the problem posed by the Weil conjectures. It was Grothendieck who developed a new and more profound view of algebraic geometry and developed a framework in which a cohomology in characteristic zero could be constructed for varieties defined over any characteristic. The conjectures of Weil were proved to be true by Deligne who combined the Grothendieck perspective with some profound ideas of his own.

Grothendieck's work started out in an unbelievably modest way as a series of remarks on the paper of Serre that had pioneered the sheaf theoretic ideas in algebraic geometry. Grothendieck had the audacious idea that the effectiveness of Serre's methods would be enormously enhanced if one associates to *any commutative ring with unit* a geometric object, called its *spectrum*, such that the elements of the ring could be viewed as functions on it. A conspicuous feature of Grothendieck's approach was its emphasis on generality and the consequent use of the functorial and categorical points of view. He invented the notion of a *scheme* in this process as the most general algebraic geometric object that can be constructed, and developed algebraic geometry in a setting in which *all* problems of classical geometry could be formulated and solved. He did this in a monumental series of papers called *Elements*, written in collaboration with Dieudonne, which changed the entire landscape of algebraic geometry. The Grothendieck approach initiated a view of algebraic geometry wherein the algebra and geometry were completely united. By fusing geometry and algebra he brought number theory into the picture, thereby making available for the first time a systematic geometric view of arithmetic problems. The Grothendieck perspective has played a fundamental role in all modern developments since then: in Deligne's solution of the Weil conjectures, in Faltings's solution of the Mordell conjecture, and so on.

One might therefore say that by the 1960's the long evolution of the concept of space had reached its final stage. Space was an object built by gluing local pieces, and depending on what one chooses as local models, one obtained a space which is either smooth and differential geometric or analytic or algebraic[8].

**The physicists.** However, in the 1970's, the physicists added a new chapter to this story which had seemed to have ended with the schemes of Grothendieck and the analytic spaces. In their quest for a unified field theory of elementary particles and the fundamental forces, the physicists discovered that the Fermi-Bose symmetries that were part of quantum field theory could actually be seen classically if one worked with a suitable a generalization of classical manifolds. Their ideas created spaces in which the coordinate functions depended not only on the usual coordinates but also on a certain number of *anticommuting* variables, called the *odd variables.* These odd coordinates would, on quantization, produce fermions obeying the Pauli exclusion principle, so that they may be called *fermionic coordinates*. Physicists like Salam and Strathdee, Wess and Zumino, Ferrara and many others played a decisive role in these developments. They called these *superspaces* and developed a complete theory including classical field theory on them together with their quantizations. Inspired by these developments the mathematicians created the general theory of these geometric objects, the *supermanifolds*, that had been constructed informally by hand by the physicists. The most interesting aspect of supermanifolds is that the local coordinate rings are generated over the usual commutative rings by *Grassmann variables*, i.e., variables $\xi^k$ such that ${\xi^k}^2 = 0$ and $\xi^k\xi^\ell = -\xi^\ell\xi^k (k \neq \ell)$. These have always zero numerical values but play a fundamental role in determining the geometry of the space. Thus the supermanifolds resemble the Grothendieck schemes in the sense that the local rings contain nilpotent elements. They are however more general on the one hand, since the local rings are not commutative but supercommutative, and more specialized than the schemes in the sense that they are smooth.

The mathematical physicist Berezin was a pioneer in the creation of superalgebra and super geometry as distinct disciplines in mathematics. He emphasized super algebraic methods and invented the notion of the *superdeterminant*, nowadays called the *Berezenian.* He made the first attempts in constructing the theory of supermanifolds and super Lie groups and emphasized that this is a new branch of geometry and analysis. Berezin's ideas were further developed by Kostant, Leites, Bernstein, and others who gave expositions of the theory of supermanifolds and their symmetries, namely the super Lie groups. Kac classified the simple Lie super algebras and their finite dimensional representations. Manin, in his book introduced the general notion of a *superscheme.* A wide ranging perspective on super geometry and its symmetries was given by Deligne and Morgan as a part of the volume on Quantum Field theory and Strings[9].

**2.3. Geometry and algebra.** The idea that geometry can be described in algebraic terms is very old and goes back to Descartes. In the nineteenth century it was applied to projective geometry and led to the result that projective geometry, initially described by undefined objects called points, line, planes, and so on, and the incidence relations between them, is just the geometry of subspaces of a vector space over some division ring. However for what we are discussing it is more appropriate to start with the work of Hilbert on algebraic geometry. Hilbert showed in his famous theorem of zeros that an affine algebraic variety, i.e., a subset of complex euclidean space $\mathbf{C}^n$ given as the set of zeros of a collection of polynomials, could be recovered as the set of homomorphisms of the algebra $A = \mathbf{C}[X_1, \ldots, X_n]/I$ where $I$ is the ideal of polynomials that vanish on the set. In functional analysis this theme of recovering the space from the algebra of functions on it was discovered by Stone and Gel'fand in two different contexts. Stone showed that if $B$ is a Boolean algebra, the space of all maximal filters of $B$ can be given a canonical topology in which it becomes a totally disconnected compact Hausdorff space $X(B)$, and the Boolean algebra of subsets of $X(B)$ which are both open and closed is canonically isomorphic to $B$. Gel'fand showed that any compact Hausdorff space $X$ can be recovered from the algebra $C(X)$ of complex valued continuous functions on it as the space of homomorphisms of $C(X)$ into $\mathbf{C}$:

$$X \approx \text{ Hom } (C(X), \mathbf{C}).$$

Inspired by the work of Norbert Wiener on Fourier transforms, Gel'fand introduced the concept of a commutative *Banach algebra* (with unit) and showed that if we associate to any such algebra $A$ its *spectrum*, namely, the set

$$X(A) := \text{Spec}(A) = \text{Hom}(A, \mathbf{C})$$

then the evaluation map

$$a \longmapsto \widehat{a}, \quad \widehat{a}(\xi) = \xi(a) \qquad (a \in A, \xi \in X(A))$$

gives a representation of $A$ as an algebra of continuous functions on $X(A)$ where $X(A)$ is equipped with the compact Hausdorff weak topology. The map

$$a \longmapsto \widehat{a},$$

the so-called *Gel'fand transform*; it generalizes the Fourier transform. It is an isomorphism with $C(X(A))$ if and only if $A$ has a star-structure defined by a conjugate linear involutive automorphism $a \mapsto a^*$ with the property that $||aa^*|| = ||a||^2$. We can thus introduce the following general heuristic principle:

• **Hilbert–Gel'fand Principle**

The geometric structure of a space can be recovered from the commutative algebra of functions on it.

As examples of this correspondence between spaces and the algebras of functions on it we mention the following:

> Compact Hausdorff spaces $\simeq$ commutative Banach $*$-algebras
>
> Affine algebraic varieties over $\mathbf{C} \simeq$ finitely generated algebras over $\mathbf{C}$ with no nonzero nilpotents
>
> Compact Riemann surfaces $\simeq$ finitely generated fields over $\mathbf{C}$ with transcendence degree 1.

However two important aspects of this correspondence would have to be pointed out before we can use it systematically. First, the representation of the elements of the algebra as functions on the spectrum in the general case is not one-to-one. There may be elements which are nonzero and yet go to 0 in the representation. Thus, already in both the Hilbert and Gel'fand settings, any element $a$ such that $a^r = 0$ for some integer $r > 1$, i.e., a *nilpotent* element, necessarily goes to 0 under any homomorphism into any field or even any ring with no zero divisors, and so its representing function is 0. For instance, $\mathbf{C}[X, Y]/(X)$ is the ring $\mathbf{C}[Y]$ of (polynomial) functions on the line $X = 0$ in the $XY$-plane, but the map

$$\mathbf{C}[X, Y]/(X^2) \longrightarrow \mathbf{C}[X, Y]/(X) \longrightarrow \mathbf{C}[Y]$$

gives the representation of elements of $\mathbf{C}[X, Y]/(X^2)$ as functions on the line $X = 0$ in which the element $X$, which is nonzero but whose square is 0, goes to 0. In the Grothendieck theory this phenomenon is not ignored because it contains the mechanism to treat certain fundamental aspects (for instance infinitesimal) of the representing space. In the example above, $\mathbf{C}[X, Y]/(X^2)$ is the ring of functions on the *double line* $X^2 = 0$ in the $XY$-plane. The double line is a distinctive geometric object; indeed, when we try to describe the various degenerate forms of a conic, one of the possibilities is a double line. In the Hilbert theory this idea leads to the principle that all algebras of the form $A = \mathbf{C}[X_1, \ldots, X_n]/I$ where $I$ is *any* ideal, describe geometric objects; if $I$ is not equal to its own radical, there will be elements $p$ such that $p \notin I$ but $p^n \in I$ for some integer $n \geq 2$, so that such $p$ define nilpotent elements of $A$. Grothendieck's great insight was to realize that the full force of this correspondence between affine algebraic varieties and commutative rings can be realized only if the notions of an affine variety and functions on it are enlarged so as to make the correspondence between affine varieties and commutative rings with unit bijective, so that the following principle can be relentlessly enforced:

• **Grothendieck Principle**

*Any* commutative ring $A$ is essentially the ring of functions on some space $X$. The ring is allowed to have nilpotents whose numerical values are 0 but which play an essential role in determining the geometric structure. The functions on $X$ may have to take their values in fields which differ from point to point.

This space, called the *spectrum* of $A$ and denoted by $X(A) = \text{Spec }(A)$, is a much more bizarre object than in the Hilbert or Gel'fand theories, and we shall not elaborate on it any further at this time. It is simply the set of all prime ideals of $A$, given the Zariski topology. The ring $A$ can be localized and so one has a sheaf of rings on $X(A)$. Thus $X(A)$ come with a structure which allows one to consider them as objects in a category, the category of *affine schemes*, and although the objects themselves are very far from intuition, the entire category has very nice properties. This is one of the reasons why the Grothendieck schemes work so well[10].

The second aspect of the concept of a manifold or scheme that one has to keep in mind is that it can be *localized.* This is the idea that space should be built up from its smallest parts, and is done, as mentioned above, by investing space with a sheaf of rings on it. Thus space acquires its characteristic features from the sheaf of rings we put on it, appropriately called the *structure sheaf.* The small parts of space are then described by local models. In differential geometry the local models are $\mathbf{R}^n$ or $\mathbf{C}^n$, while in algebraic geometry they are affine schemes which are spectra of commutative rings. The general manifold is then obtained by gluing these local models. The gluing data come from the requirement that when we glue two models, we should establish a correspondence between the structure sheafs on the parts which are to be identified. The end result is then a premanifold or a *prescheme*; the notions of *smooth manifolds* or *schemes* are then obtained by adding a suitable *separation condition.* In the case of manifolds this is just the condition that the underlying topological space is Hausdorff; for a prescheme $X$ this is the condition that $X$ is closed in $X \times X$. The gluing process is indispensable because some of the most important geometrical objects are projective or compact and so cannot be described by a single set of coordinates. The geometrical objects thus defined together with the maps between them for a category. One of the most important properties of this category is that products exist.

Clearly the Grothendieck scheme (or prescheme) is an object very far from the classical notion of an algebraic variety over the complex numbers, or even the notion of an algebraic variety in the sense of Serre. It is an index of the genius of Grothendieck that he saw the profound advantages of using the schemes even though at first sight they are rather unappetizing.

To conclude this brief discussion and as a simple illustration let us consider the case of affine varieties over an algebraically closed field $\mathbf{K}$ and ignore the complications coming from nilpotent elements of the structure sheaf. The correspondence here is between Zariski closed subsets of affine space $k^n$ and finitely generated algebras over $k$ which are reduced in the sense that they have no nonzero nilpotents. In this category products exist. Because of this one can define algebraic groups $G$ over $k$ in the usual manner. In terms of the coordinate rings the maps of multiplication, inverse, and the unit element have to be interpreted in terms of the corresponding $k$-algebra. Thus the $k$-algebra $A = A(G)$ has a *comultiplication* which is a morphism

$$\Delta : A \longrightarrow A \otimes A,$$

a *coinverse*

$$\Sigma : A \longrightarrow A,$$

and a *counit*,

$$\Omega : A \longrightarrow k,$$

all of which are related by diagrams that dualize the associative law and the properties of the inverse and the unit element. The result is that $A$ is a commutative *Hopf algebra*. Thus the category of algebraic groups over $k$ corresponds to the category of commutative Hopf algebras. For instance the Hopf algebra corresponding to $\mathrm{GL}(n, k)$ is

$$A = k[a_{ij}, \det{}^{-1}]$$

with

$$\begin{aligned} \Delta : a_{ij} &\longmapsto \sum_r a_{ir} \otimes a_{rj} \\ \Sigma : a_{ij} &\longmapsto a^{ij} \\ \Omega : a_{ij} &\longmapsto \delta_{ij}. \end{aligned}$$

The theory of Serre varieties provides a fully adequate framework for the theory of algebraic groups and their homogeneous spaces.

**2.4. A brief look ahead.** To go over to the super category one has to replace systematically all the algebras that occur on the classical theory by algebras that have a $\mathbf{Z}_2$-grading, namely *super algebras*. To study supervarieties one then replaces sheaves of commutative algebras by sheaves of supercommutative algebras. Here the *supercommutative algebras* are those for which any two elements either commute or anticommute according as one of them is even or both of them are odd. Just as commutative rings determine geometric objects supercommutative rings determine super geometric objects. We give a brief run over the themes that will occupy us in the remaining chapters.

**Super linear algebra.** A *super vector space* $V$, is nothing but a vector space over the field $k$ which is graded by $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$, namely,

$$V = V_0 \oplus V_1.$$

The elements of $V_0$ (resp. $V_1$) are called *even* (resp. *odd*). Morphisms between super vector spaces are linear maps that preserve the parity, where the parity function $p$ is 1 on $V_1$ and 0 on $V_0$. A *super algebra* is an algebra $A$ with unit (which is necessarily even) such that the multiplication map $A \otimes A \longrightarrow A$ is a morphism, i.e., $p(ab) = p(a) + p(b)$ for all $a, b \in A$. Here and everywhere else any relation between linear objects in which the parity function appears the elements are assumed to be homogeneous (that is, either even or odd) and the validity for nonhomogeneous elements is extended by linearity. As an example we mention the definition of supercommutative algebras: a super algebra $A$ is *supercommutative* if

$$ab = (-1)^{p(a)p(b)} ba \qquad (a, b \in A).$$

This differs from the definition of a commutative algebra in the sign factor which appears. This is a special case of what is called the *rule of signs* in super algebra: *whenever two elements are interchanged in a classical relation a minus sign appears if both elements are odd.* The simplest example is the exterior algebra $\Lambda(U)$ of an ordinary vector space $U$. It is graded by $\mathbf{Z}$ (degree) but becomes a super algebra if we introduce the coarser $\mathbf{Z}_2$-grading where an element is even or odd if its degree is even or odd. $\Lambda(U)$ is a supercommutative algebra. Linear super algebra can be developed in almost complete analogy with linear algebra but there are a few interesting differences. Among the most important are the notions of *supertrace* and *superdeterminant.* The superdeterminant is nowadays called the *Berezenian*, named after Berezin who discovered it. If $A$ is a supercommutative $k$-algebra and

$$R = \begin{pmatrix} L & M \\ N & P \end{pmatrix} \qquad (L, P \text{ even}, \; M, N \text{ odd})$$

where the entries of the matrices are from $A$, then

$$\begin{aligned} \mathrm{str}(R) &= \mathrm{tr}(L) - \mathrm{tr}(P) \\ \mathrm{Ber}(R) &= \det(L)\det(I - MP^{-1}N)\det(P)^{-1} \end{aligned}$$

where $\mathrm{Ber}(R)$ is the Berezinian of $R$. Unlike the classical determinant, the Berezinian is defined only when $R$ is invertible, which is equivalent to the invertibility of $L$ and $P$ as matrices from the commutative $k$-algebra $A_0$, but has the important property that

$$\mathrm{Ber}(RR') = \mathrm{Ber}(R)\mathrm{Ber}(R')$$

while for the supertrace we have

$$\mathrm{str}(RR') = \mathrm{str}(R'R).$$

By an *exterior algebra over a commutative k-algebra A* ($k$ a field of characteristic 0) we mean the algebra $A[\theta_1, \ldots, \theta_q]$ generated over $A$ by elements

$$\theta_1, \ldots, \theta_q$$

with the relations

$$\theta_j^2 = 0, \qquad \theta_i \theta_j = -\theta_j \theta_i \ (i \neq j).$$

Exterior algebras are supercommutative. It must be remembered however that when we view an exterior algebra as a super algebra, its $\mathbf{Z}$-grading is to be forgotten and only the coarser grading by $\mathbf{Z}_2$ into even and odd elements should be retained. In particular they admit automorphisms which *do not preserve the original* $\mathbf{Z}$*-degree.* Thus for

$$A = k[\theta_1, \ldots, \theta_r] \qquad (\theta_i \theta_j + \theta_j \theta_i = 0),$$

the map

$$\theta_1 \mapsto \theta_1 + \theta_1 \theta_2, \theta_i \mapsto \theta_i (i > 1)$$

extends to an automorphism of $A$ that does not preserve the original $\mathbf{Z}$-grading. The existence of such automorphisms is the factor that invests super geometry with its distinctive flavour.

**Supermanifolds.** The concept of a smooth supermanifold, say over $\mathbf{R}$, is now easy to define. A supermanifold $X$ is just an ordinary manifold such that on sufficiently small open subsets $U$ of it the super coordinate ring $R(U)$ is isomorphic to a supercommutative exterior algebra of the form $C^\infty(U)[\theta_1, \ldots, \theta_q]$. The integer $q$ is independent of $U$ and if $p$ is the usual dimension of $X$, its dimension as a supermanifold is $p|q$. However this is not the same as an exterior bundle over the ordinary manifold $X$; for instance, the supermanifold $\mathbf{R}^{1|2}$ has the coordinate rings $C^\infty(U)[\theta_1, \theta_2]$ but the map

$$t, \theta_1, \theta_2 \longmapsto t + \theta_1 \theta_2, \theta_1, \theta_2$$

defines a superdiffeomorphism of the supermanifold but not of an exterior bundle over $\mathbf{R}$. If $U$ is an open set in $\mathbf{R}^p$, then $U^{p|q}$ is the supermanifold whose coordinate rings are $C^\infty(U)[\theta_1, \ldots, \theta_q]$. Replacing the smooth functions by real analytic or complex analytic manifolds we have the concept of a real analytic or a complex analytic supermanifold. Unfortunately it is not possible to define supermanifolds of class $C^k$ for finite $k$ because one needs the full Taylor expansion to make sense of morphisms like the one defined above. If we replace these special exterior algebras by more general supercommuting rings we obtain the concept of a *superscheme* which generalizes the concept of a scheme.

A brief comparison between manifolds and supermanifolds is useful. The coordinate rings on manifolds are commutative, those on a supermanifold are supercommutative. However *as the odd elements of any exterior algebra are always nilpotent*, the concept of a supermanifold is closer to that of a scheme than that of a manifold. So the techniques of studying supermanifolds are variants of those used in the study of schemes, and so more sophisticated than the corresponding ones in the theory of manifolds.

**Super Lie groups.** A *super Lie group* is a group object in the category of supermanifolds. An *affine super algebraic group* is a group object in the category of affine supervarieties. In analogy with the classical case these are the supervarieties whose coordinate algebras are *super Hopf algebras.* Here are some examples:

$\mathbf{R}^{1|1}$ : The group law is given (symbolically) by

$$(t^1, \theta^1)\cdot(t^2, \theta^2) = (t^1 + t^2 + \theta^1\theta^2, \theta^1 + \theta^2).$$

$\mathrm{GL}(p|q)$ : Symbolically this is the group of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the entries are treated as coordinates, those of $A$ and $D$ being even and those of $B$ and $C$ odd. The group law is just matrix multiplication.

It may be puzzling that the group law is given so informally in the above examples. The simplest way to interpret them is to stay in the algebraic rather than the smooth category and view the formulae as defining the automorphisms of the corresponding exterior algebras. Actually one can use the same symbolic description in all cases by utilizing the notion of *functors of points*. The idea is that any object $M$ in the category under discussion is determined completely by the functor that associates to any object $N$ the set $\mathrm{Hom}(N, M)$; the elements of $\mathrm{Hom}(N, M)$ are called the $N$-points of $M$. Viewed in this manner one can say that affine supergroups are functors from the category of supercommutative rings to the category of groups, which are representable by a supercommutative Hopf algebra.

Thus $\mathbf{R}^{1|1}$ corresponds to the functor which associates to any supercommuting ring $R$ the group of all elements $(t^1, \theta^1)$ where $t^1 \in R_0$ and $\theta^1 \in R_1$, the multiplication being exactly the one given above. Similarly, the functor corresponding to $\mathrm{GL}(p|q)$ associates to any supercommuting ring $R$ the group of all block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the entries of $A$ and $D$ are even elements of $R$ and those of $B$ and $C$ are odd elements of $R$; the group law is just matrix multiplication. This group is denoted by $\mathrm{GL}(p|q)(R)$. If one wants to view these as super Lie groups in the smooth category, the functors go from the category of smooth supermanifolds to the category of groups. For instance, the functor defining the super Lie group $\mathbf{R}^{1|1}$ takes any supermanifold $T$ to the group of all $(t, \theta^1, \theta^2)$ where $t, \theta^1, \theta^2$ are global sections of $\mathcal{O}_T$ with $t$ even and $\theta^i$ odd. Similarly $\mathrm{GL}(p|q)$ is defined by the functor that takes $T$ to the group $\mathrm{GL}(p|q)(R(T))$ where $R(T)$ is the supercommutative ring of global sections of $T$. The concept of the functor of points shows why we can manipulate the odd variables as if they are numerical coordinates. This is exactly what is done by the physicists and so the language of functor of points is precisely the one that is closest to the intuitive way of working with these objects that one finds in the physics literature.

**Super spacetimes.** Minkowski spacetime is the manifold $\mathbf{R}^4$ equipped with the action of the Poincaré group. To obtain *super spacetimes* one extends the abelian Lie algebra of translations by a Lie super algebra whose odd part is what is called the *Majorana spinor module*, a module for the Lorentz group which is spinorial, real, and irreducible. This is denoted by $\mathbf{M}^{4|4}$. The super Poincaré group is the super Lie group of automorphisms of this supermanifold. Physicists call this *rigid supersymmetry* because the affine character of spacetime is preserved in this model. For *supergravity* one needs to construct local supersymmetries. Since the group involved is the group of diffeomorphisms which is infinite dimensional, this is a much deeper affair.

Once super spacetimes are introduced one can begin the study of Lagrangian field theories on super spaces and their quantized versions. Following the classical picture this leads to supersymmetric Lagrangian field theories. They will lead to superfield equations which can be interpreted as the equations for corresponding superparticles. A superfield equation gives rise to several ordinary field equations which define a *multiplet* of particles. These developments of super field theory lead to the predictions of susy quantum field theory.

## REFERENCES

[1] See Weyl's book in reference[4] of Chapter 1, p. 97. The passage from Clifford's writings is quoted in more detail in the book

M. Monastyrsky, *Riemann, Topology, and Physics*, Birkhäuser, 1987, p. 29.

The Clifford quotation is in

W. K. Clifford, *On the space-theory of matter*, Mathematical Papers, Chelsea, 1968, p. 21.

Clifford's deep interest in the structure of space is evident from the fact that he prepared a translation of Riemann's inaugural address. It is now part of his Mathematical Papers. In the paper (really an abstract only) mentioned above, here is what he says:

*Riemann has shewn that as there are different kinds of lines and surfaces*, *so there are different kinds of spaces of three dimensions*; *and that we can only find out by experience to which of these kinds the space in which we live belongs. In particular, the axioms of plane geometry are true within the limit of experment on the surface of a sheet of paper, and yet we know that the sheet is really covered with a number of small ridges and furrows, upon which* (*the total curvature not being zero*) *these axioms are not true. Similarly*, *he says although the axioms of solid geometry are true within the limits of experiment for finite poretions of our space*, *yet we have no reason to conclude that they are true for very small portions*; *and if any help can be got thereby for the explanation of physical phenomena*, *we may have reason to conclude that they are not true for very small portions of space.*

*I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold in fact*

(1) *That small portions of space* are *in fact analogous to little hills on a surface which is on the average flat*; *namely, that the ordinary laws of geometry are not valid in them.*

(2) *That this property of being curved or distorted is continually being passed on from one portion of space to another after ther manner of a wave.*

(3) *That this variation of the curvature of space is what really happens in that phenomenon which we call the* motion of matter, *whether ponderable or etherial.*

(4) *That in the physical world nothing else takes place but this variation*, *subject* (*possibly*) *to the law of continuity.*

*I am endeavouring in a general way to explain the laws of double refraction on this hypothesis*, *but have not yet arrived at any results sufficiently decisive to be communicated.*

[2] Noncommutative geometry has had many impulses which are too complicated to go into here. For some insights refer to the books of Manin and Connes:

A. Connes, *Noncommutative Geometry*, Academic Press, 1990.

Yu. Manin, *Topics in Noncommutative Geometry*, Princeton University Press, 1991.

Yu. Manin, *Quantum groups and Noncommutative Geometry*, Montreal University, 1988.

Noncommutative geometry is closely tied up with quantum groups. See for example

V. Chari and A. Presseley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[3] Riemann's talk is included in his Collected works:

*Bernhard Riemann: Collected Papers*, R. Narasimhan, (ed.), Springer Verlag, 1990.

There are translations of Riemann's talk in English. See Clifford[1] and Spivak[4]. In addition, the Collected Papers contains Weyl's annotated remarks on Riemann's talk.

[4] See pp. 132–134 of

M. Spivak, *Differential geometry*, Vol II, Publish or Perish Inc, 1979.

[5] See Spivak[4], p. 205.

[6] See Weyl's book, Preface, p. VII:

H. Weyl, *The Concept of a Riemann Surface*, Addison-Wesley, 1955.

See also Klein's little monograph

F. Klein, *On Riemann's Theory of Algebraic Functions and their Integrals*, Dover, 1963.

[7] Weyl's paper on pure infinitesimal geometry was the first place where the concept of a connection was freed of its metric origins and the concept of a manifold with an affine connection was introduced axiomatically. See

H. Weyl, *Reine Infinitesimalgeometrie*, Math. Zeit., 2(1918), p. 384; Gesammelte Abhandlungen, Band II, Springer-Verlag, p. 1.

[8] For a wideranging treatment of the historical evoution of algebraic geometry there is no better source than the book of Dieudonne:

J. Dieudonne, *Cours de Géométrie Algébrique*, Vol. 1, *Apercu historique sur le développement de la géométrie algébrique*, Presses Universitaires de France, 1974.

9 For all the references see[24–28] of Chapter 1.

Berezin was a pioneer in super geometry and super analysis. For an account of his work and papers related to his interests see

*Contemporary Mathematical Physics: F.A. Berezin Memorial Volume*, R. L. Dobrushin, R. A. Minlos, M. A. Shubin, and A. M. Vershik, (Eds.), AMS Translations, Ser. 2, Vol. 175, 177, 1996.

10 See Deligne's brief survey of Grothendieck's work in

P. Deligne, *Quelques idées maîtresses de l'œuvre de A. Grothendieck*, in *Mateériaux pour L'Histoire des Mathématiques au XX$^{e}$ Siècle*, Soc. Math. Fr, 1998, p. 11.