Weyl systems, Heisenberg groups and Arithmetic physics *

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1. Introduction. Arithmetic physics, or better, arithmetic quantum theory, is a term that refers to a collection of ideas and partial results, loosely held together, that suggests that there are connections between the worlds of quantum physics and number theory and that onw should try to discover and develop these connections. At one extreme is the modest idea that one should try to formulate some of the mathematical questions arising in quantum theory over fields and rings other than $\mathbf{R}$, such as the field of p–adic numbers $\mathbf{Q}_p$, or the ring of adeles over the rationals $\mathbf{A}(\mathbf{Q})$. The point here is not to try to develop the alternative theories as a substitute for the actual theory or even look for physical interpretations, but rather to look for results that would unify what we already know over $\mathbf{R}$.

The basis for this suggestion is the simple fact that all experimental calculations are essentially discrete and so can be modelled by mathematical structures that are over $\mathbf{Q}$. The theories over $\mathbf{R}$ are thus idealizations that are more convenient than essential and reflect the fact that the field of real numbers is a completion of the field of rational numbers. But there are other completions of the reals, namely the fields $\mathbf{Q}_p$, and it is clear that under suitable circumstances a large finite quantum system may be thought of as an approximation to a system defined over $\mathbf{Q}_p$. If we continue this line of thought further, it becomes necessary to consider *all* the completions of $\mathbf{Q}$, which means working over the ring of adeles $\mathbf{A}(\mathbf{Q})$.

At the other extreme are bold speculations that push forward the hypothesis that the exploration of the structure of quantum theories by replacing $\mathbf{R}$ by $\mathbf{Q}_p$ and $\mathbf{A}(\mathbf{Q})$ is not just a pleasant exercise but is *essential.*. I quote the following remarks of Manin from his beautiful and inspiring paper[1].

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On the fundamental level our world is neither real nor p–adic; it is adelic. For some reasons, reflecting the physical nature of our kind of living matter (e.g. the fact that we are built of massive particles), we tend to project the adelic picture onto its real side. We can equally well spiritually project it upon its non–Archimedean side and calculate most important things arithmetically.

The relation between "real" and "arithmetical" pictures of the world is that of complementarity, like the relation between conjugate observables in quantum mechanics.

2. Quantum systems over finite fields and rings. It is surprising that the idea of considering quantum systems over fields and rings other than $\mathbf{R}$ has a long history and goes back to the origins of quantum theory itself. I shall make some brief remarks about the evoultion of these ideas. It is not my aim to be complete but to show that there has been considerable interest in this theme for a long time.

Weyl (1928c). Indeed, the idea of considering quantum systems in which the configuration space is replaced by a finite abelian group first appears in the famous book[2] of Hermann Weyl. Recall that the quantum theory of Heisenberg prescribes that the mathematical quantities representing the position and momentum of a particle in one dimension should be operators q and p on the Hilbert space of staes of the system satisfying the commutation rules

$$[p, q] = -i\hbar$$

It is in the nature of this relation that the operators q and p are unbounded. Weyl, who preferred to work with bounded rather than unbounded operators, replaced q and p by the unitary groups they generate, and introduced the commutation rules between these unitary operators that are formally equivalent to the Heisenberg commutation rules. Let

$$U(x) = e^{ixq}, \qquad V(\xi) = e^{i\xi p} \qquad (x, \xi \in \mathbf{R})$$

so that U and V are unitary representations of the additive group $\mathbf{R}$. Then we get the commutation rule of Weyl which is formally equivalent to that of Heisenberg,

$$U(x)V(\xi) = e^{-ix\xi}V(\xi)U(x) \qquad (x, \xi \in \mathbf{R})$$

The standard model for these rules is when the HIlbert space is $L^2(\mathbf{R})$ and U, V are defined by

$$(U(x)f)(t) = e^{itx}f(t), \qquad (V(\xi)f)(t) = f(t + \hbar\xi)$$

It is easy to check that the pair (U, V) acts irreducibly on $L^2(\mathbf{R})$. If I am not mistaken, it was Weyl who formulated the uniqueness question associated to the pair of unitary representations satisfying the Weyl commutation rule in the following manner : is it true that such a pair (U, V) of unitary representations, under the further assumption of irreducibility, is equivalent to the pair defined by the standard model. In physical terms, working with the commutation rules alone without referring to any model is called matrix mechanics, while working with the standard model (in which q and p are the usual multiplication and differentiation operators) is called wave mechanics, so that the uniqueness question is really whether matrix and wave mechanics are equivalent.

Weyl did not prove the uniqueness–it was done by Von Neumann[3] and Stone (independently) very soon after– but in his attempts to understand this question Weyl had a very remarkable idea. He realized that his commutation rules could be formulated in much greater generality, in fact *for any pair of abelian groups in duality.* To make precise Weyl's idea let us introduce a definition.

Definition. Let A and B be abelian groups in duality through a bicharacter $(\cdot\ ,\ \cdot)$. then a Weyl system for $A, B)$ is a pair of unitary representations U (of A) and V (of B) such that

$$U(x)V(\xi) = (x, \xi)^{-1}V(\xi)U(x) \qquad ((x, \xi) \in A \times B)$$

Some remarks amplifying this definition are perhaps in order. First of all I have made no topological assumptions on A and B and, as a result, no continuity assumptions on U and V; if however A and B are topological, we shall assume that $(\cdot\ ,\ \cdot)$ is continuous on $A \times B$ and that the unitary representations are continuous. The simplest situation arises when A and B are separable locally compact (abelian) groups but the case when A and B are infinite dimensional is also of great interest in quantum field theory. In fact, already in Dirac's theory of the interaction of matter with the electromagnetic field, the field was expanded as a Fourier series and the Fourier coefficients, which are infinitely many, were regarded as the position coordinates of the field with their time derivatives as the momenta, and were quantized using Heisenberg's prescription but now for infinitely mnay q's and p's.

In this generality neither the existence nor the uniqueness of Weyl systems is obvious. But Weyl considered the case when

$$A = B = \mathbf{Z}_N := \mathbf{Z}/N\mathbf{Z}, \quad (a, b) = e^{2\pi ab/N} \qquad (a, b \in \mathbf{Z})$$

and proved the uniqueness of the Weyl systems. He then indicated in a heuristic manner that when N goes to ∞ one can identify $\mathbf{Z}_N$ with a grid in $\mathbf{R}$ in such a manner that the mesh of the grid goes to 0 and the Weyl system converges in some sense to the standard Weyl system.

Weyl also considered the case when $A = B = \mathbf{Z}_2^N$ and showed that the corresponding quantum system can be identified with that of N spins obeying the Pauli exclusion principle.

Schwinger (1960c). Weyl's ideas were revisited by Julian Schwinger when he examined the foundations of quantum kinematics in a series of beautiful papers in the late 1950's and early 1960's and then expanded on them in a book[4]. Of particular relevance to our point of view are two themes which emerged very clearly from Schwinger's work and which show clearly how he went beyond Weyl in this direction. First he emphasized the fact that the finite systems were of interest in their own way, and not merely as approximations to the continuum systems. Second, he made the approximation process involving Weyl systems much more transparent (although he refrained from giving a general definition). Finally, he suggested that the approximation can be not merely kinematic but dynamic. I shall now explain briefly these contributions of Schwinger.

Classification of finite Weyl systems. Schwinger had already undertaken in [4] a detailed treatment of the kinematics of finite quantum systems and from his point of view the Weyl systems associated to finite abelian groups furnish the most important examples of finite systems. The spectra of the representations U and V define maximal observables and the Weyl commutation rules imply that these are conjugate observables–when one of them is measured with complete accuracy, all values of the other are equally likely. He then noticed that the classification of finite abelian groups gives a classification of finite Weyl systems. In this way he arrived at the principle that the Weyl systems associated to $A = B = \mathbf{Z}_p$ where p

runs over all the primes are the building blocks. Curiously this enumeration is incomplete and one has to include the cases[5] where $A = B = z_{p^r}$ where p is as before a prime but r is any integer ≥ 1.

Approximation of the Weyl system for $A = B = \mathbf{R}$ by that for $A = B = \mathbf{Z}_N$. The idea is to identify $\mathbf{Z}_N$ with a grid in $\mathbf{R}$. This can be done also for $\mathbf{R}^d$ with $\mathbf{Z}_N^d$ as the approximating abelian group for any $d \geq 1$ but we shall treat here only the case $d = 1$. This approximation is also at the foundation of the very useful theory of fast Fourier transforms. Let

$$L_N = \{r\varepsilon \big| r = 0, \pm 1, \pm 2, \ldots, \pm(N-1)/2\} \qquad \varepsilon = \left(\frac{2\pi}{N}\right)^{1/2}$$

where N is an odd integer. The map that sends the equivalence class $[r]$ of r to $r\varepsilon$ is an identification of $\mathbf{Z}_N$ with the grid L_N. The Hilbert space $L^2(\mathbf{Z}_N)$ is imbedded in $L^2(\mathbf{R})$ by sending the delta function at $[r]$ with the function which is the characteristic function of the interval $[(r-1/2)\varepsilon, (r+1/2)\varepsilon]$ multiplied by $\varepsilon^{-1/2}$. Now one can introduce the position operator q_N as the operator of multiplication by the function $[r] \mapsto r\varepsilon$. For the momentum operator p_N Schwinger had the real insight and originality to define it as the *Fourier transform (on the finite group $\mathbf{Z}_N$) of q_N*; actually the approach via Weyl systems shows that this is the only way to define p_N. Notice that p_N is now not a local difference operator on the grid but a global operator, more like a pseudo difference operator if I may use that expression.

Schwinger gave a treatment of the behaviour of this approximation more detailed than that Weyl and even suggested that the states of the continuum system be restricted to those for which this approximation procedure is uniform in some sense. But there was also a suggestion that this approximation was also good *dynamically*, namely, that if we take a reasonably simple Hamiltonian such as the oscillator,

$$H = 1/2(p^2 + q^2)$$

then the corresponding dynamical group can be approximated very well by the finite Hamiltonian

$$H_N = 1/2)p_N^2 + q_N^2)$$

for large N. Numerical calculations[6] showed that this is correct and that the approximation is unexpectedly good even for relatively small values of

N, and a very strong dynamical limit theorem[7] can be proved for Hamiltonians

$$H + 1/2(p^2 + V(q))$$

where the potential V goes to ∞ when $|q| \to \infty$.

Beltrametti (1971), Nambu (1987). The idea that one should try to examine the possibilities for doing physics over fields other than the real or complex fields is an old one. The fields enter at at least two places–one when we decide to build space time as a vector space over this field, and second when we introduce the carrier space of all the values of physical fields and functions. In view of the well known divergences that occur in the conventional models of space time it is an attractive idea to examine what the possibilities for a quantum field theory are when finite fields, rings, and other algebraic structures are allowed to replace the field of real and complex numbers. One of the earlier tretaments of this question of the microstructure of space time goes back to Beltrametti [8]; there are earlier treatments of similar questions[9]. Nambu [10] also examined this question more recently but so far there has been no systematic effort to develop a quantum field theory in such a context.

Weil (1961). The most profound discussion of Weyl systems over locally compact abelian groups is due to Andre Weil. In a pair of epoch-making papers[11] he examined Weyl systems when A is any locally compact abelian group and $B = \widehat{A}$ is its dual group. Weil considered the case when A is a finite dimensional vector space over a local field (e.g. a p–adic field) or a free module over the ring adeles over a global field such as a field of algebraic numbers. He applied his theory to reinterpret the Siehel theory of quadratic forms over global fields and Weil's work may be regarded as the quantum theory of a quadratic form.

3. Convergence of Weyl systems. The remarks made above on the approximation of Weyl systems over $\mathbf{R}$ by those of $\mathbf{Z}_N$ suggest that it is desirable to have a formal definition of convergence of Weyl systems. This is not difficult to do[5] and then one has the following theorem[12].

Theorem. *Let A be any separable locally compact abelian group. then there is a sequence of finite abelian groups A_N such that the Weyl system associated to $(A, \widehat{A})$ is the limit of the Weyl systems associated to $(A_N, \widehat{A_N})$.*

For instance the Weyl systems associated to the field $\mathbf{Q}_p$ of p–adic numbers is the limit of Weyl systems associated to $\mathbf{Z}_{p^r}$.

4. Quantization and Schrödinger theory over nonarchimedean fields. The first question is whether we can view quantum theory over nonarchimedean fields from the point of view of deformation quantization. The simplest situation is the following. let K be a local field of characteristic different from 2 and let $X = K \times K$. We write $S(X)$ for the Schwartz–Bruhat space of X, namely the space of locally constant functions with complex values on X. There is no structure of a Poisson algebra on $S(X)$ but at least there is the structure of an associative algebra on $S(X)$, namely the one coming from pointwise multiplication. One can ask at least whether this algebra has nontrivial, for instance, nonabelian, deformations. The answer is no, at least if one interprets defomations in the usual formal sense. However, if one asks whether there is a topoloigical space T and a point $t_0 \in T$ such that there are associative algebra structures $f, g \mapsto f \cdot_t g$ on $S(X)$ for each $t \in T$ which are nonabelian, such that

(a) as $t \to 0, f \cdot_t g \to fg$

(b) $f \cdot_{t_0} g = fg$

Then the answer is yes. In fact, the Moyal–Weyl formula for $*$–product on $S(\mathbf{R}^2)$, the Schwartz space of $\mathbf{R}^2$ makes sense over $S(X)$ and defines a family of associative algebra structures parametrized by K having the properties described above. It would be of interest to examine if such $*$–products can be defined for the spaces $S(X)$ for other manifolds over K[13].

Let D be a division ring which is finite dimensional and central over K. Let V be a vector space of dimension $n < \infty$ over D with a norm $\cdot|$ which is homogeneous and satisfies the ultrametric norm inequality

$$|u + v| \leq \max(|u|, |v|) \qquad (u, v \in V)$$

Using a nontrivial additive character on D one can define a Fourier transform $\mathcal{F}$ on the Schwartz–Bruhat space $\mathcal{H} = L^2(V)$. One can define a family of (pseudodifferential!) operators on $\mathcal{H}$ by

$$-\Delta_b = \mathcal{F} M_{|x|^b} \mathcal{F}^{-1} \qquad (b > 0)$$

where $M_{|x|^b}$ is the operator of multiplication by $|x|^b$. Notice that if $D = \mathbf{R}$ and $b = 2$, then $-\Delta$ conincides with minus the usual Laplacian. The Hamiltonians on $\mathcal{H}$ are then

$$H = \Delta_b + V$$

where V is a Borel function. The theory of Kato goes through without difficulties and allows us to view H as an essentially self adjoint operator on the Schwartz–Bruhat space o$S(V)$f V under suitable condiitons on V, for instance if V is locally L^2 and bounded at infinity, or if, after an affine transformation V is a function of $< n$ variables of the type described just now.

It is now possible to prove that we can obtain a path integral representation for the propagator of the dynamical group generated by this Hamiltonian. One has to go to imaginary time for getting a rigorous measure on path space. The probability measure on the path space is not Wiener measure now but an appropriate measure whose finite dimensional densities can be explicitly described. They are not gaussian but have fourier transforms of the form

$$\varphi_{t_1,b}\varphi_{t_2-t_1,b}\ldots\varphi_{t_N-t_{N-1},b} \qquad (0 < t_1 < t_2 < \ldots < t_N)$$

where

$$\varphi_{t,b} = e^{-|u|^b}$$

There is an additional feature that the measure is not defined on the space of continuous maps from $[0,\infty)$ to V but on the space of maps which are right continuous and have limits from the left everywhere[13].

5. The Segal–Shale–Weil representation. In the remainder of this note I shall discuss one of the deepest aspects of Weyl systems, namely the construction of the so called Segal–Shale–Weil representation and its consequences. To begin with let A, B be a pair of abelian groups in duality and let (U, V) be a Weyl system for this pair. Write

$$G = A \times B$$

and define

$$W(a,b) = U(a)V(b) \qquad ((a,b) \in G)$$

Then W satisfies the relations

$$W(a,b)W(a',b') = m((a,b),(a',b'))W(a+a',b+b')$$

where

$$m((a,b),(a',b')) = (a',b)$$

This means that W, although not a representation of G, is a *projective unitary representation* with multiplier m. It is a standard idea in the theory of projective representations to go over to an extension of G defined in such a way that W corresponds to a unitary representation of this extension. More precisely, let

$$E_m = G \times T$$

where T is the multiplicative group of complex numbers of absolute value 1. Then E_m becomes a group if we define a multiplication on it by the rule

$$(x,t)(x',t') = (xx', tt'm(x,x')) \qquad (x, x' \in G, t, t' \in T)$$

and we have the exact sequence

$$0 \longrightarrow T \longrightarrow E_m \longrightarrow G \longrightarrow 0$$

where the maps

$$T \longrightarrow E_m, \quad E_m \longrightarrow G$$

are respectively

$$t \longrightarrow (1,t), \quad (x,t) \longrightarrow x$$

It is an easy consequence of the fact that A and B are in duality that T is the center of E_m. Thus E_m is a central extension of G by T with the additional property that T is precisely the center of E_m. One can then verify at once that the map

$$W_m : (x,t) \longmapsto tW(x) \qquad (x \in G, t \in T)$$

is a unitary representation of E_m with the property that

$$W_m(1,t) = t, \quad W_m(x,1) = W(x)$$

The correspondence

$$W \longleftrightarrow W_m$$

is a bijection between the set of unitary representations of E_m which restrict to the character $t \mapsto t$ on T and projective unitary representations of G with multiplier m. In case A and B are topological and $(\cdot, \cdot)$ is continuous, we give to E_m the product topology and the maps in the exact sequence above are continuous. The group E_m is said to be the *Heisenberg group* associated to (G, m). The uniqueness theorem of Weyl systems is then the statement that upto unitary equivalence there is only one irreducible representation of E_m which restricts on T to the character $t \longmapsto t$. It was proved by Von Neumann[14] and Stone[15] independently that for $A = B = \mathbf{R}^d$ with the usual duality

$$(a, b) = e^{i2\pi a \cdot b)}$$

the uniqueness theorem is valid. This was later extended to all separable locally compact abelian A with $B = \widehat{A}$ by Mackey[16]. For a detailed discussion of this result see[17].

Let us now assume that the uniqueness theorem is valid for the pair A, B. Let $\mathcal{H}$ be the Hilbert space on which the corresponding irreducible unitary representation W_m of E_m acts. Let

$$\mathbf{A}_m = \text{ Aut }^0(E_m)$$

be the group of all automorphisms of E_m that restrict on T to the identity.

REFERENCES