Has God made the quantum world p-adic?

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The Dirac Mode

"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected... Non-euclidean geometry and noncommutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and *after* each success in this direction, to try to interpret the new mathematical features in terms of physical entities..."

Remarks

These remarks were made by Dirac in the beginning of his famous paper in which he created the theory of magnetic monopoles.

> P. A. M. Dirac, *Quantized singularities in the electromagnetic field,* Proc. Roy. Soc. Lond., **A133** (1931), 60–72,

Monopoes have not been found to this day but they occur all the time in string theory. Also, to this day, the only proof that electric charge is quantized, i.e., it is always an integral mulitple of a basic charge, remains that of Dirac, who gave it in the above mentioned paper on monopoles, using the existence of monopoles.

Since then people (Schwinger, Schwanziger) have considered particles with both electric and magnetic charges. The Dirac quantization condition now becomes completely arithmetic. It asserts that the symmetry groups of such particles at the quantum level are the groups $\mathrm{Sp}(2n, \mathbf{Z})$. The theory of modular forms thus becomes very important in these theories.

Some comments on the Dirac mode

The Dirac mode is to invent, so to speak, a new mathematical concept or framework first, and then try to find its relevance in the real world, with the expectation that (in a distorted paraphrasing of Dirac) a mathematically beautiful idea must have been adopted by God. Of course the question of what constitutes a beautiful and relevant idea is where physics begins to become an art.

I think this second mode is unique to physics among the natural sciences, being most akin to the mode practiced by the mathematicians. Particle physics, in particular, has thrived on the interplay of these two modes. Among examples of this second approach, one may cite such concepts as

Magnetic monopole

Non-Abelian gauge theory

Supersymmetry

On rare occasions, these two modes can become one and the same, as in the cases of Einstein gravity and the Dirac equation...

Y. Nambu, *Broken Symmetry: Selected Papers of Y. Nambu*, 1995, World Scientific, pp. 383–389, eds. T. Eguchi and K. Nishijima.

The p-adic world

valuations

(i) $|ab|_\infty = |a|_\infty |b|_\infty$

(ii) $|a+b|_\infty \le |a|_\infty + |b|_\infty$.

p-adic absolute value

$$|x|_p = p^{-v} \qquad \text{if } x = p^v \frac{a'}{b'} \text{ where } a', b' \in \mathbf{Z}, (a', b') = 1.$$

properties

(a) $|a+b|_p \le \max(|a|_p, |b|_p)$ (ultrametric inequality)

(b) locally compact, totally disconnected, non archimedean

(c) No other valuations on $\mathbf{Q}$ other than $|\cdot|_p, |\cdot|_\infty$ (Ostrowski's theorem)

product formula

$$\prod_v |x|_v = 1 \ (x \in \mathbf{Q}) \qquad (v \text{ runs through primes } p \text{ and } \infty).$$

Theorem. *There are no other locally compact non discrete fields densely containing* $\mathbf{Q}$ *other than* $\mathbf{R}$ *and the* $\mathbf{Q}_p$*.*

Fourier Transform

Haar measure

$$d(ax) = |a|_v dx \quad (v = p, v = \infty)$$

Fourier transforms of test functions

$\psi : \mathbf{Q}_v \longrightarrow \mathbf{T}$ a non trivial additive character of $\mathbf{Q}_v$

For f any test function on $\mathbf{Q}_v$,

$$\widehat{f}(y) = \int_{\mathbf{Q}_v} f(x)\psi(xy)dx, \quad f(x) = \int_{\mathbf{Q}_v} \widehat{f}(y)\psi(-xy)dy$$

(For $v = \infty$ we take the test functions from the Schwartz space. For $v = p$ we take them to be elements of the Schwartz-Bruhat space, namely, functions which are locally constant and compactly supported. They are thus functions on B_a/B_b where $0 < a < b < \infty$, B_r is the p-adic ball of radius r. The integrals are thus sums, and are generalizations of Gauss sums. Thus the p-adic Fourier transform is arithmetic. The theory is independent of the choice of ψ and one usually makes certain normalized choices. For $\mathbf{R}$ we take $\psi(x) = e^{ix}$ or $e^{2\pi ix}$, depending upon whether one is an analyst or a number theorist!)

Rational structures viewed in the various p-adic worlds

Quaternion algebras

For F a field and $a, b \in F^\times$, $(a,b)_F$ is the algebra with generators i, j and relations

$$i^2 = a, \quad j^2 = b, \qquad ij = -ji (= k) \qquad\qquad (*).$$

$$(a,b)_F = F \otimes (a,b)_{\mathbf{Q}} \qquad (a, b \in \mathbf{Q} \subset F)$$

One can view $(a,b)_{\mathbf{Q}}$ $(a, b \in \mathbf{Q}^\times)$ in the various $\mathbf{Q}_v$. We define

$$(a,b)_v = \begin{cases} +1 & \text{if } (a,b)_{\mathbf{Q_v}} \text{ splits} \\ -1 & \text{if } (a,b)_{\mathbf{Q_p}} \simeq \mathbf{H}_v. \end{cases}$$

Here splits refers to the fact that it is isomorphic to the full 2×2 matrix algebra, and $\mathbf{H}_v$ refers to the unique quaternion division algebra over $\mathbf{Q}_v$. We then have

$$\prod_v (a,b)_v = 1.$$

Remarks

1. We can even say more: if we are given numbers $\varepsilon_v = \pm 1$ for all v, we can find a rational quaternion algebra $(a,b)_{\mathbf{Q}}$ such that $(a,b)_v = \varepsilon_v$ for all v if and only if
$$\prod_v \varepsilon_v = 1.$$
This is the same as saying that the number of v-adic worlds in which the algebra remains a division algebra should be *even*, and that these can be specified arbitrarily. Moreover, once these places are specified, the algebra is determined up to isomorphism. The most remarkable thing about these results is that in essence they go back to Gauss and his law of quadratic reciprocity; however they were cast in this form by Hilbert. The symbol
$$(a,b)_p$$
is called the *Hilbert symbol*, a deep generalization of the Legendre symbol. If we try to extend these results from quaternion algebras to arbitrary division algebras with center $\mathbf{Q}$, we get into some of the most beautiful parts of the number theory, in which the results of Artin and Hasse dominate.

2. If we replace the fields $\mathbf{Q}_v$ by much bigger ones, like function fields in several variables, it is an entirely new world of results. Murray Schacher and his collaborators have studied many of these.

Rational algebraic groups

If we take a group like $\mathrm{GL}(n)$ or more generally, an algebraic group G defined over $\mathbf{Q}$ (think $\mathrm{SL}(n), \mathrm{SO}(n), \mathrm{Sp}(n)$), we get the *locally compact groups* $G(\mathbf{Q}_v)$ which can be studied in greta depth. These are examples of v-adic Lie groups:

(a) As topological spaces they are manifolds, i.e., look locally like $\mathbf{Q}_v^n$

(b) The group maps (multiplication and inversion) are morphisms

The structure of the p-adic Lie groups was studied many years ago by Lazard who proved a version of Hilbert's fifth problem for these. The *complex* representation theory of these groups has been intensively studied in recent years, starting with Mautner, Gel'fand-Graev, Bruhat, Harish-Chandra, and then by Langlands and others motivated by arithmetic. The representation theory of p-adic groups over p-adic fields is the theme of the lectures of Peter Schneider.

Originally the motivations for studying groups over p-adic fields were somewhat vague. But they became crystal clear when Langlands discovered that they encode in some way the theory of extensions of the given p-adic field, more precisely, the *representation theory* of the Galois groups of these extensions.

The adelic method

Rational adeles

An adele over $\mathbf{Q}$ is a vector

$$a = (a_\infty, a_2, a_3, a_5, \ldots, a_p, \ldots)$$

where

$$a_\infty \in \mathbf{R}, a_p \in \mathbf{Q}_p, a_p \in \mathbf{Z}_p \text{ for almost all } p.$$

Basic facts:

(a) $\mathbf{A}$ is a locally compact ring

(b) $\mathbf{Q}$ is discrete in $\mathbf{A} and \mathbf{A}/\mathbf{Q}$ is compact

The group $\mathbf{A}/\mathbf{Q}$ is thus a fundamental object associated to the rational field.

Artin/Tate: Harmonic analysis on $\mathbf{A}/\mathbf{Q}$ has deep arithmetic significance.

One can apply this process of thinking to groups such as $\mathrm{GL}(n)$ and rational algebraic groups G. To any such algebraic group we have the pair $G(\mathbf{A}), G(\mathbf{Q})$ where $G(\mathbf{Q})$ is discrete. We thus have the homogeneous space $G(\mathbf{A})/G(\mathbf{Q})$ which is locally compact and has a $G(\mathbf{A})$-action.

Langlands: Harmonic analysis on $\mathrm{GL}(n, \mathbf{A})/\mathrm{GL}(n, \mathbf{Q})$ encodes information about the representation theory of the Galois groups of extensions of $\mathbf{Q}$.

Remarks

When $v = \infty$ we have a differential geometric object while for $v = p$ we get an arithmetic and combinatorial object. If $G = GL(n)$ where $n > 1$, the space $\mathrm{GL}(n, \mathbf{A})/\mathrm{GL}(n, \mathbf{Q})$ is no longer a group.

Let $G = \mathrm{SL}(2)$ and take a rational volume form on it using which we compute volumes not only in the real but in all p-adic worlds. In the real case we compute the volume of the usual space

$$\mu_\infty = \mathrm{SO}(2)\backslash\mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z})$$

while in the p-adic worlds we compute the volumes of the *compact groups*

$$\mu_p = \mathrm{SL}(2, \mathbf{Z}_p).$$

It turns out (this is a non trivial computation) that

$$\prod_v \mu_v = 1.$$

More precisely, we find that

$$\mu_\infty = \frac{\pi^2}{6}, \qquad \mu_p = (1 - p^{-2})(Euler!!)$$

For $G = \mathrm{SL}(n)$ the volume at ∞ is $\zeta(2)\zeta(3)\ldots\zeta(n)$.

A general principle

This formula illustrates a general principle: what we are really interested is in computing the volume of the object defined over $\mathbf{R}$; the product formula allows us to compute it equally well if we live in the various p-adic worlds. The basic result is that the adelic volume is 1 which allows us to compute the real volume by arithmetic means. Such an adelic volume can be computed for any semisimple algebraic group defined over $\mathbf{Q}$; it is then known as the *Tamagawa number.* The finiteness of the volumes, which was an open question for a number of years, was first proved by Borel and Harish-Chandra in a famous paper in the late 1950's. The computation of the Tamagawa number for the orthogonal groups is a deep way to encode the discoveries of C. L. Siegel on the number of representations of one quadratic form by another.

These methods work with equal facility whether the ground field is $\mathbf{Q}$ or any algebraic number field (or even function field). This is an illustration of their versatility.

Arithmetic structure of the (quantum) world

Manin (1989)

The world is really adelic. For various reasons, some of which are certainly due to the fact that we (our measuring apparatuses) are built of massive particles, we can see only the real side of this world. However the arithmetic side of the world has equal validity, and we should be able to compute everything of importance from the arithmetic side. The two pictures cannot coexist, and this is something like the uncertainty principle…

Volovich (1987)

Spacetime is non archimedean in the Planck scale because no measurements are possible, and a clue to the structure of the physical world at the Planck scale requires us to understand how the world will behave if we make this non archimedean hypothesis…

Remarks

The most accurate physical theory at present is quantum electrodynamics which deals with the interaction of electrons and positrons (matter) with the electromagnetic field (radiation). This was completed in the early 1950's. Since then the physicists have worked out what is called the *standard model* which allows a unified treatment of electrodynamics with the weak and strong forces. What has not been done is the unification of this with gravity. In attempting to build such unified theories people have encountered deep mathematical and physical obstacles connected with the structure of space time at very short distances and times. There is a fundamental scale called the *Planck scale*, in which it appears certain that no measurements as we can make today can be carried out. The idea of *extended particles* (*strings, branes*) were born in this context. The combination of the hypotheses of Manin and Volovich require us to examine from the mathematical point of view the structures that arise in the description of the quantum world, for instance, in the theory of elementary particles and their fields, in their real as well as the p-adic incarnations, and hence over the adeles as well.

Ingredients for a quantum theory

- Space time X
- A Hilbert space of states $\mathcal{H}$
- Various bundles corresponding to various particles (Dirac, Weyl, etc)
- The symmetry group G (Poincaré, conformal, etc) of X and the bundles
- A unitary representation of G or Lie(G) in $\mathcal{H}$
- Special operators and states in $\mathcal{H}$ corresponding to sections of the bundles.

Remarks

In non relativistic quantum mechanics $X = \mathbf{R}^N \times \mathbf{R}$, G is the Galilean group, and we have the operators for position and momenta, and energy. In relativistic quantum mechanics (single particles) we have $X = \mathbf{R}^4$ as Minkowski space, G is the Poincaré group, and the energy operator. In QFT we have the operators like creation and annihilation, the vacuum state, and the field operators (smeared). If space time is super symmetric, i.e., if there are additional grassman coordinates in addition to the usual ones, G then has to be super symmetric, and the unitary representation of G will have to be super symmetric also. The arithmetic hypotheses then suggest that it is of interest to examine the nature of the structures described by the requirements above when we work over the fields $\mathbf{Q}_p$ and more generally over the adele rings $\mathbf{A}$. One can do this as we explained above in two levels: over a fixed $\mathbf{Q}_p$ (local), or over $\mathbf{A}$ (global); to do the latter it is clear that X and G must be rational.

In this generality the problems that arise are nowhere near any solution of reasonable generality. Already over $\mathbf{R}$, QFT has been rigorously constructed only for space time dimensions 2 and 3, and that too for a very limited class of interactions. The situation over p-adics is even worse: only theories with a cut off have been constructed in a few cases, and the cut off has not been removed. Not even this has been done at the adelic level.