Euler at 300

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Abstract

It is remarkable that Euler, who preceded Gauss and Riemann, is still very much with us. The ramifications of his work are still not exhausted, three hundred years after his birth. In number theory, in algebraic geometry, in topology, in the calculus of variations, and in analysis, both conceptual and numerical, not to mention mechanics of particles and solid bodies, astronomy, hydrodynamics, and other applied areas, the ideas that he generated are still motivating mathematicians. In this talk, which will require no specialized background, I shall focus on two aspects of his work in analysis–the zeta values, and the theory of summability of divergent series, where he obtained formulae of surpassing beauty, and pointed the way to future work that has proved to be among the most fecund in the history of mathematics.

Topics

- **General remarks**
- **Sum of even powers of reciprocals of integers**
- **Euler sums, Dirichlet series, and cyclotomy**
- **Multizeta values**
- **What is the sum of a divergent series?**
- **Epilogue**

Timeline

1707	Born in Basel, Switzerland, April 15.
1725	Peter the Great and his widow Catherine establish the St. Petersburg Academy of Sciences in St. Petersburg, Russia.
1727	Euler moves to St. Petersburg and becomes an adjunct in mathematics.
1733	Euler takes over the chair in mathematics after Daniel Bernoulli returns to Basel. Gets married and buys a house.
1735	Solves the problem of finding the sum of $\sum_{n\geq 1} \frac{1}{n^2}$ and acquires an international reputation.
1738	Euler loses the vision in his right eye after a serious illness.
1741	Political turmoil in Russia after death of the czarina and the regency. Euler leaves Russia to join the Academy of Sciences in Berlin, Prussia.
1762	Catherine (the Great) II becomes the czarina in Russia and starts the efforts to get Euler back.
1766	Euler returns to St. Petersburg. His eyesight begins to deteriorate.
1771	Euler loses the vision in his left eye also.
1783	Dies in St. Petersburg on September 18, 1783.

Prolific and universal

- TENS OF THOUSANDS OF PAGES OF RESEARCH ARTICLES AND TREATISES
- THE FIRST GREAT TREATISES IN ALGEBRA AND ANALYSIS
- UNIVERSALITY
  - Number Theory
  - Elliptic integrals
  - Calculus of Variations
  - Analysis
  - Mechanics, Astronomy, and Hydrodynamics
- SUNNY TEMPERAMENT
- MAJOR RESULTS IN ALMOST ALL BRANCHES OF PURE AND APPLIED MATHEMATICS
- LOSS OF EYESIGHT, TOTAL TOWARDS THE END (“LESS OF A DISTRACTION”)

Leonhardi Euleri Opera Omnia

- FOUR SERIES WITH SERIES I DEVOTED TO PURE MATHEMATICS

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- IV A. Series quarta A: Commercium Epistolicum (7 vols, 1 in preparation), 2498 pages
- IV B. Series quarta B: Manuscripta (unpublished manuscripts, notes, diaries, etc)

Internet Sources

There is a monumental project at Dartmouth to bring the entire Opera Omnia into the net for universal accessibility. The URL for this is

http://www.math.dartmouth.edu/∼euler/index.html

A letter to Lagrange

I have had all your calculations read to me, concerning the equation $101 = p^2 - 13q^2$, and I am fully persuaded of their validity; but as I am unable to read or write, I must confess that my imagination could not follow the reasons for all the steps you have had to take, nor keep in mind the meaning of all your symbols. It is true that such investigations have formerly been a delight to me and that I have spent much time on them; but now I can only undertake what I can carry out in my head, and often I have to depend on some friend to do the calculations which I have planned...

from Andre Weil's translation

What is the value of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$?

This question occupied a number of major mathematicians like John Bernoulli and came to be known as the *Basel problem.* Since the error committed by stopping after n terms lies between $1/n$ and $1/(n+1)$, a million terms have to be added to get the value accurate to 6 decimals.

Euler first found a transformation of the series that allowed him to calculate it with accuracy up to 6 decimals. Then in 1735 he found the exact value. The first proof was open to many objections, but knowing he was right, he persevered till he got a proof after 10 years that would satisy the most exacting of his critics.

The story

- DILOGARITHM AND $\zeta(2) = 1.644944\ldots$

- A RECKLESS APPLICATION OF NEWTON'S THEOREM:

$$\zeta(2) = \pi^2/6, \qquad \zeta(4) = \frac{\pi^4}{90}$$

- OBJECTIONS

- TRIUMPH: INFINITE PRODUCT FOR SINE

Dilog and $\zeta(2)$

- DEFINITION:

$$\mathrm{Li}_2(x) = \int_0^x \frac{-\log(1-t)}{t} dt = \int_{x>t_1>t_2>0} \frac{dt_1 dt_2}{t_1(1-t_2)}$$

- SERIES:

$$\mathrm{Li}_2(x) = \sum_n \frac{x^n}{n^2}, \qquad \mathrm{Li}_2(1) = \zeta(2).$$

- FUNCTIONAL EQUATION:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = -\log x \log(1-x) + \mathrm{Li}_2(1)$$

- EULER'S FORMULA:

$$\zeta(2) = (\log 2)^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2.2^n}$$

$$\log 2 = -\log\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{n.2^n}.$$

- The geometric nature of the terms allows an accurate computation of the series from a small number of terms.

$$\zeta(2) = 1.644934\ldots.$$

Proof of functional equation

Start with

$$\zeta(2) = \int_0^1 \frac{-\log(1-x)}{x} dx.$$

We split the integration from 0 to u and u to 1. In the second integral we change x to $1-x$ to get

$$\zeta(2) = \int_0^u \frac{-\log(1-x)}{x} dx + \int_0^{1-u} \frac{-\log x}{1-x} dx = I_1 + I_2.$$

To evaluate I_2 we integrate by parts first to get

$$I_2 = \log x \log(1-x)\Big|_0^{1-u} + \int_0^{1-u} \frac{-\log(1-x)}{x} dx$$

Since $\log x \log(1-x) \sim -x \log x \to 0$ as $x \to 0$,

$$\zeta(2) = \log u \log(1-u) + \sum_{n=1}^{\infty} \frac{u^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1-u)^n}{n^2}.$$

$$\mathrm{Li}_2(u) + \mathrm{Li}_2(1-u) = -\log u \log(1-u) + \mathrm{Li}_2(1)$$

Evaluation

- "*So much work has been done on the series* $\zeta(n)$ *that it seems hardly likely that anything new about them may still turn up... I, too, in spite of repeated efforts, could achieve nothing more than approximate values for their sums... Now, however, quite unexpectedly, I have found an elegant formula for* $\zeta(2)$, *depending on the quadrature of a circle* $\big[$*i.e., upon* $\pi\big]$ ".

  from Andre Weil's translation

- In a letter to Daniel Bernoulli he communicated his formulae

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$$

- He calculated (laboriously, as he admitted himself), the values of

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\ldots$$

  for quite a few even values of k, certainly up to $k=12$, often using his summation formula.

Newton's theorem applied to power series

- From Newton

$$1-\alpha s+\beta s^2-\ldots\pm\rho s^k=\left(1-\frac{s}{a}\right)\left(1-\frac{s}{b}\right)\ldots\left(1-\frac{s}{r}\right)$$

$$\frac{1}{a}+\frac{1}{b}+\ldots+\frac{1}{r}=\alpha,\qquad \frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{r^2}=\alpha^2-2\beta$$

$$1-\sin s=1-s+\frac{s^3}{6}-\frac{s^5}{120}+\ldots$$

which vanishes *doubly* at

$$\frac{\pi}{2},-\frac{3\pi}{2},\frac{5\pi}{2},\ldots$$

we get (with $\alpha=1,\beta=0$ etc)

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\text{(LEIBNIZ)}$$
$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=(1-2^{-2})\zeta(2)$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$$

Objections

- ARE THERE COMPLEX ROOTS OF $1 - \sin s$?
- SINCE $e^s(1 - \sin s)$ HAS THE SAME ROOTS, WHY COULD NOT ONE WORK WITH IT?
- NEWTON FOR POWER SERIES OK?
- (POST RIEMANN) THE LEIBNIZ SERIES CONVERGES ONLY CONDITIONALLY

Infinite product for the sine

Euler himself was aware of the shortcomings of his proof. It took him close to ten years before he was able to get a proof that satisfied everybody, even us.

- INFINITE PRODUCT FOR SINE

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

- METHOD OF PROOF

$$q_n(z) := \lim_{n\to\infty} \frac{\left(1+\frac{iz}{n}\right)^n - \left(1-\frac{iz}{n}\right)^n}{2iz}$$

$$q_n(z) = \prod_{k=1}^{p} \left(1 - \frac{z^2}{n^2}\frac{1+\cos\frac{2k\pi}{n}}{1-\cos\frac{2k\pi}{n}}\right) \quad (n = 2p+1)$$

$$\frac{\sin z}{z} = \lim_{n\to\infty} q_n(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

Remarks

- Nowadays it has become customary to establish this and the product formula for cosine by complex analysis, with the partial fraction for the cotangent as the starting point, and using periodicity and Liouville's theorem. Euler's proof is elementary, beautiful, and direct, and needs only the use of uniform convergence to justify the last step, an elementary application of the so-called M-test of Weierstrass.

- The partial fraction for the cotangent follows by logarithmic differentiation of the product for sine, as observed by Daniel Bernoulli.

- The product is absolutely convergent and so extending Newton becomes trivial.

- There are no complex roots for the sine.

The sums $1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \dots$

- THE OCCURRENCE OF BERNOULLI NUMBERS

  The value

$$\zeta(12) = \frac{691}{6825 \times 93555}\pi^{12}$$

  must have suggested to him that the Bernoulli numbers are lurking around the evaluation of $\zeta(2k)$.

- LOGARITHMIC DIFFERENTIATION OF THE PRODUCT FOR SINE AND PARTIAL FRACTION FOR COTANGENT

$$\pi \cot \pi s - \frac{1}{s} = \sum_{n=1}^{\infty} \left(\frac{1}{n+s} - \frac{1}{n-s} \right)$$

- REPEATED DIFFERENTIATION GIVES

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!}\pi^{2k}$$

Concluding remarks

Here is an alternative proof for evaluating $\zeta(2)$ given by Euler a few years afterwards. The method does not extend to $\zeta(4)$ etc, in spite of efforts by Euler.

Start with

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

to get first

$$\arcsin x = x + \sum_{k\geq 1} \frac{1.3.\ldots.2k-1}{2.4.\ldots 2k} x^{2k+1}.$$

Then use integration by parts to get

$$\frac{1}{2}(\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt$$

Replace $\arcsin t$ in the integrand by its power series and integrate term by term from 0 to 1. We get

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \ldots + \frac{1}{(2k+1)^2} + \ldots$$

Euler sums and cyclotomic numbers

- Euler evaluated a number of sums of the form

$$\sum_{n\in\mathbf{Z}} \frac{h(n)}{n^r}$$

where $h(n)$ is a function *periodic* with a small period, especially a character mod N for some small N. As examples one can mention

$$\frac{2\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\chi(n)}{n}, \qquad \frac{\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

where

$$\chi(n) = \begin{cases} +1 & \text{if } n \equiv 1 \bmod 3 \\ -1 & \text{if } n \equiv -1 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

- The general sum is $g\pi^r$ where g is a cyclotomic number. Such sums finally came into their own in Dirichlet's work on L-functions and his interpretation of some of them as class numbers.

Multi zeta values

Thirty years after his discovery of the zeta values Euler wrote a paper where he introduced the double zeta values (with a minor variant here)

$$\zeta(a,b) = 1 + \frac{1}{2^a} + \frac{1}{3^a}\left(1 + \frac{1}{2^b}\right) + \frac{1}{4^a}\left(1 + \frac{1}{2^b} + \frac{1}{3^b}\right) + \ldots$$

These can be generalized to the so-called *multi zeta values*:

$$\zeta(s_1, s_2, \ldots, s_r) = \sum_{n_1 > n_2 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots n_r^{s_r}}$$

He obtained relations like

- $\zeta(2,1) = \zeta(3)$
- $2\zeta(\lambda-1,1) = (\lambda-1)\zeta(\lambda) - \sum_{2\le q\le \lambda-2} \zeta(\lambda-q)\zeta(q)$
- Recently there has been great interest in these numbers and the relations between them.

Summing divergent series

- HOW CAN ONE ASSIGN A VALUE FOR A DIVERGING SERIES?

- EULER AND ABEL SUMMABILITY

- THE SUMMATION OF THE FACTORIAL SERIES

  $1 - 1! + 2! - 3! + 4! - 5! + \ldots\ldots = 0.596347362123\ldots$

- BOREL SUMMABILITY

- FOURIER SERIES FOR $\delta(x)$

$$\sum_{n=-\infty}^{\infty} e^{2in\pi x} = 0 \qquad (0 < x < 2\pi)$$

- FUNCTIONAL EQUATION FOR ZETA AND SOME L-SERIES

Euler on summing divergent series

In *De seriebus divergentibus*, communicated in 1755 and published in 1760, Euler says:

Notable enough, however, are the controversies over the series 1-1+1-1+1-1+... whose sum was given by Leibniz as 1/2, although others disagree. No one has yet assigned another value to that sum, and so the controversy turns on the question whether the series of this type have a certain sum. Understanding of this question is to be sought in the word "sum"; this idea, if thus conceived–namely the sum of a series is said to be that quantity to which it is brought closer as more terms of the series are taken–has relevance only for convergent series, and we should in general give up this idea of sum for divergent series. Wherefore, those who thus define a sum cannot be blamed if they claim they are unable to assign a sum to a series. On the other hand, as series in analysis arise from the expansion of fractions or irrational quantities or even transcendentals, it will in turn be permissible in calculation to substitute in place of such a series that quantity out of whose development it is produced. For this reason, if we employ this definition of sum, that is to say, the sum of a series is that quantity which generates the series, all doubts with respect to divergent series vanish and no further controversy remains on this score, in as much as this definition is applicable equally to convergent or divergent series.

Accordingly, Leibniz, without any hesitation, accepted for the series 1-1+1-1+1-1+. . ., the sum 1/2, which arises out of the expansion of the fraction 1/1+1, and for the series 1-2+3-4+5-6+. . ., the sum 1/4, which arises out of the expansion of the formula $1/(1+1)^2$. In a similar way a decision for all divergent series will be reached, where always a closed formula from whose expansion the series arises should be investigated. However, it can happen very often that this formula itself is difficult to find, as here where the author treats an exceptional example, that divergent series par excellence 1-1+2-6+24-120+720-5040+. . ., which is Wallis' hypergeometric series, set out with alternating signs; this series, in whatever formula it finds its origin and however much this formula is valid, is seen to be determinable by only the deepest study of higher Analysis. Finally, after various attempts, the author by a wholly singular method using continued fractions found that the sum of this series is about 0.596347362123, and in this decimal fraction the error does not affect even the last digit. Then he proceeds to other similar series of wider application and he explains how to assign them a sum in the same way, where the word "sum" has that meaning which he has here established and by which all controversies are cut off.

from the translation by E. J. Barbeau and P. J. Leah

Actually Euler had discussed his ideas about divergent series much earlier, in correspondence with Goldbach and Nicolaus Bernoulli I. In a letter to Goldbach written on August 7, 1745, discussing his attempts to sum the divergent series

$$1 - 1! + 2! - 3! + \dots$$

Euler has this to say (free translation from German):

> ... I believe that every series should be assigned a certain value. However, to account for all the difficulties that have been pointed out in this connection, this value should not be denoted by the name sum, because usually this word is connected with the notion that a sum has been obtained by a real summation: this idea however is not applicable to "seriebus divergentibus"...

Earlier in the same year he had written a letter to Nicolaus Bernoulli I [6] on July 17, 1745, in which he had discussed in some detail his method of summing the series

$$1 - 1! + 2! - 3! + \dots + (-1)^n n! + \dots.$$

The glimpse into Euler's views on divergent series provided by these and other letters is quite remarkable. Indeed, Hardy, while discussing Euler's remarks in his letter to Goldbach mentioned above, says that

> ... this is language which might almost have been used by Cesàro or Borel.

Abel summability

- Euler's favourite summation procedure which has come to be called *Abel summability*:

$$\sum_{n=0}^{\infty} a_n = s(A) \Leftrightarrow \lim_{x \to 1-0} \sum_{n=0}^{\infty} a_n x^n = s$$

  Leibniz's series sums to $1/2$ by this procedure:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \ldots = \frac{1}{2} \ (A)$$

  Similarly

$$\sum_{n=0}^{\infty} (-1)^n \binom{n+k-1}{k-1} = \frac{1}{2^k} \ (A)$$

- WHEN DOES AN ABEL SUMMABLE SERIES CONVERGE?

  If $a_n = O\left(\frac{1}{n}\right)$ (LITTLEWOOD)

- Wiener released the theory from its artificial limitations and made it a branch of L^1-harmonic analysis which eventually led to the theory of commutative Banach algebras (GEL'FAND).

Functional equation for zeta

One hundred years before Riemann Euler conjectured the functional equation for the zeta function, and verified it for all integer and some rational values. With

$$\eta(s) = (1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \dots$$

he obtained the relation

$$\frac{\eta(1-s)}{\eta(s)} = -\frac{2^s - 1}{2^{s-1} - 1}\frac{\cos\frac{s\pi}{2}}{\pi^s}\Gamma(s)$$

He verified it for all integer values of s and for some fractional values by numerical computation, conjecturing that it should be true for all s. If s is an even positive integer, the series at $1 - s$ is

$$1^m - 2^m + 3^m - \dots$$

where m is a positive odd integer, and he used Abel summation to sum it.

The factorial series

- Euler called

$$\sum_{n=0}^{\infty}(-1)^n n! x^n = 1 - 1!x + 2!x^2 - 3!x^3 + 4!x^4 \ldots$$

  the divergent series par excellence. It does not converge anywhere and so the usual methods of summability fail. The sum as a *formal power series* satisfies Euler's differential equation

$$x^2\frac{dg}{dx} + g = x, \qquad g(x) \sim e^{1/x}\int_0^x \frac{1}{t}e^{-1/t}dt$$

  The integral is *asymptotic* to the factorial series and its value was computed by Euler using numerical integration:

$$\sum_{n=0}^{\infty}(-1)^n n! \sim g(1) = 0.5963\ldots$$

- The theory of solutions to such *irregular singular differential equations* would come into their own in the nineteenth century with the work of many, most notably of Poincaré who made precise the notion of asymptotic series and analytic solutions that are asymptotic to the formal solutions.

Continued fraction for the integral

- He also obtained a continued fraction for g:

$$g(x) = \frac{1}{1+}\frac{x}{1+}\frac{x}{1+}\frac{2x}{1+}\frac{2x}{1+}\frac{3x}{1+}\frac{3x}{1+} \text{ etc} \sim \sum_{n\geq 0}(-1)^n n! x^n.$$

$$\sum_{n=0}^{\infty}(-1)^n n! = 0.596347362123\ldots$$

- He summed not only the factorial series but a whole class of similar series by using these methods.

- Eventually Borel discovered the general ideas of Borel summation that could bring these and other such series summed by Euler into the general theory. These methods and their generalizations to what are called *multisummability methods* have proved tremendously powerful in quantum field theory and dynamical systems!

Smeared summation

- If $(a_n)_{n\in\mathbf{Z}}$ is a sequence of complex numbers which may not go to 0 sufficiently fast, how to make sense of the Fourier series

  $$\sum_{n=-\infty}^{\infty} a_n e^{inx} \quad ?$$

  In modern distribution theory a la Laurent Schwartz, the sum is a *generalized function* $f(x)$:

  $$\int \varphi(x) f(x) dx = \sum_{n=-\infty}^{\infty} a_n \int \varphi(x) e^{inx} dx$$

  $$\sum_{n=-\infty}^{\infty} e^{inx} = \delta(x)$$

  Euler proved that

  $$\sum_{n=-\infty}^{\infty} e^{inx} = 0 \qquad (0 < x < 2\pi)$$

The sum $\sum_{n=-\infty}^{\infty} e^{inx}$

Euler used the Abel method of course. Actually it is better behaved: a Cesaro 1-summation would be sufficient. Let

$$s_N(x) = \sum_{n=-N}^{n=N} e^{inx}$$

$$\sigma_N(x) = \frac{s_0(x) + s_1(x) + \ldots s_N(x)}{N+1}.$$

Then

$$s_N(x) = \frac{\varepsilon^{N+1} - \varepsilon^{-N}}{\varepsilon - 1} \qquad \varepsilon = e^{ix}.$$

Since

$$1 + z + z^2 + \ldots + z^N = O(1) \qquad (z = \varepsilon, \varepsilon^{-1} \neq 1)$$

we have

$$\sigma_N(x) \to 0 \qquad (N \to \infty).$$

But

$$\sigma_N(0) = N + 1 \to \infty \qquad (N \to \infty).$$

The correct result is

$$\sum_{n=-\infty}^{\infty} e^{inx} = \delta(x)$$

which is the Parceval-Plancherel formula for the circle. Its modern incarnations are the Plancherel formulae of Weyl, Gel'fand, and Harish-Chandra.

Euler products

- PRODUCT FORMULA FOR ZETA AND THE INFINITUDE OF THE SET OF PRIMES

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

$$\prod_p \left(1 - \frac{1}{p}\right) = 0$$

- MORE GENERAL PRODUCTS AND SUMS CORRESPONDING TO CHARACTERS χ MOD N (FOR SMALL N)

$$L(s : \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{\left(1 - \frac{\chi(p)}{p^s}\right)}$$

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \ldots = L(s : \chi)$$

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \text{ mod } 4 \\ -1 & \text{if } n \equiv -1 \text{ mod } 4 \\ 0 & \text{if } n \equiv 0, 2 \text{ mod } 4 \end{cases}$$

Comments

- Nearly 100 years after Euler's work, Dirichlet took up these themes and introduced Euler products for *arbitrary complex characters* mod N for *all* moduli N. As is well known he used them, especially their behaviour at $s = 1$, to prove the infinitude of primes in each residue call mod N that is prime to N.

- The generalizations to nonabelian characters and the true significance of such generalizations for Galois theory, due to Artin, Weil, Langlands, and many others lead one directly to the modern era, namely the *Langlands program.* This relates representations of the Galois groups of Galois extensions of number fields and function fields to the unitary representations of reductive groups (to be precise, the groups of points of the reductive groups over local fields and adele rings)

Functional equation for zeta

- One hundred years before Riemann, Euler obtained the functional equation for the zeta function:

  $$\eta(s) = (1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \ldots \quad (s > 0)$$

  which satisfies

  $$\frac{\eta(1-s)}{\eta(s)} = -\frac{2^s - 1}{2^{s-1} - 1} \frac{\cos\frac{s\pi}{2}}{\pi^s} \Gamma(s)$$

  Euler wrote m in place of s and so wrote this as

  $$\frac{1 - 2^{m-1} + \text{etc}}{1 - 2^{-m} + \text{etc}} = -\frac{1.2.3.\ldots(m-1)(2^m - 1)\cos\frac{m\pi}{2}}{(2^{m-1} - 1)\pi^m}.$$

  He had discovered and worked out many properties of the Gamma function, including the identity

  $$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin s\pi}.$$

  Using what we now call Abel summation he verified the functional equation for integral values of s, conjectured its validity for all values of s, and numerically verified it for many values of s, namely $s = i + \frac{1}{2}$ where $i = 1, 2, 3, \ldots$.

Functional equation for $L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \dots$

- Euler always worked with alternating series for ease of numerical computation, hence with $\eta(s)$ rather than $\zeta(s)$. In addition he always considered the companion to ζ, namely the L-function

$$L(s) = L(s : \chi) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \dots$$

- For L, Euler obtained the functional equation (in the same sense as for the zeta)

$$\frac{L(1-s)}{L(s)} = \frac{2^s \Gamma(s)}{\pi^s} \sin \frac{s\pi}{2}$$

Epilogue

There can be really no epilogue to Euler as long as his themes are still flourishing. His universal genius and prodigious creativity are things that are wondrous and fill our minds with awe. His ideas on divergent series led to the modern theories of summation of divergent series and integrals, eliminated the fear of dealing with them, and at the hands of physicists, have produced very powerful predictive tools such as the *Feynman path integral formula*. His theory of zeta values is still unfinished, as the true structure of the zeta values at odd positive integers remains elusive. Perhaps only Ramanujan in the modern era came close to Euler in discovering formulae of such surpassing beauty in the classical framework.