6. Matrix Lie groups

6.1. Definition and the basic theorem. A topological group is called a *matrix Lie group* if it is homeomorphic to a closed subgroup of some $\mathrm{GL}(n, \mathbf{R})$. By Von Neumann's theorem a matrix Lie group is a Lie group/We want to prove the basic theorem that the Lie structure is uniquely determined by the topology. More precisely we want to prove:

Theorem 1. *let $G_i \subset \mathrm{GL}(n_i, \mathbf{R})(i = 1, 2)$ be closed subgroups and $f(G_1 \longrightarrow G_2)$ a continuous homomorphism. Then f is analytic.*

The proof requires several steps and will be given in the following paragraphs.

The special case when $G_1 = \mathbf{R}$. We write $G_1 = \mathbf{R}$ and $G_2 = G$. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Then for any $X \in \mathfrak{g}$ the map

$$h_X : t \longmapsto \exp tX$$

is an analytic homomorphism of $\mathbf{R}$ into G: h_X is analytic, $h_X(0) = I$, and

$$h_X(t + t') = h_X(t)h_X(t') \qquad (t, t' \in \mathbf{R}).$$

We now have

Proposition 1. *Any continuous homomorphism of $\mathbf{R}$ into G is of the form h_X for a unique $X \in \mathrm{Lie}(G)$.*

Proof. We can take G to be $\mathrm{GL}(n, \mathbf{R})$. For if we can prove this special case, then the X we have is automatically in $\mathrm{Lie}(G)$ by definition of $\mathrm{Lie}(G)$. On $\mathrm{GL}(n, \mathbf{R})$ we have the function log defined for elements u with $|u - I| < 1$ by

$$\log u = \sum_{n \geq 1} (-1)^{n-1} \frac{(u - I)^n}{n}.$$

We have the functional equation

$$\log(uv) = \log(u) + \log(v) \qquad (uv = vu)$$

valid if u, v are sufficiently close to I and commute with each other. Suppose now h is a continuous homomorphism of $\mathbf{R}$ into $\mathrm{GL}(n, \mathbf{R})$. Then

$$H(t) = \log h(t)$$

is defined for $|t|$ sufficiently small. Moreover, as the $h(t)$ commute with each other we see that

$$H(t + t') = H(t) + H(t') \qquad (|t|, |t'| << 1).$$

Each entry of the matrix $H(t)$ satisfies this equation and is continuous. The classical argument of Hamel shows now that there is a matrix X such that

$$H(t) = tX$$

for $|t|$ sufficiently small. hence

$$h(t) = h_X(t)$$

for $|t|$ sufficiently small, and so for all t.

Exponential coordinates of the second kind. Let $\{X_1, \ldots, X_m\}$ be a basis for $\mathfrak{g}$. Then we have an analytic map of $\mathbf{R}^m$ into G given by

$$E : (t_1, \ldots, t_m) \longmapsto \exp(t_1 X_1) \ldots \exp(t_m X_m).$$

It follows easily that this map has a bijective differential at the origin $0 = (0, \ldots, 0)$ and so is a diffeomorphism in a neighborhood of 0. In other words every element $x \in G$ sufficiently close to the identity in G can be expressed as a product $\exp(t_1 X_1) \ldots \exp(t_m X_m)$ where the t_i are uniquely determined by x and are analytic functions of x. The t_i are often called *exponential coordinates of the second kind for G.*

Proof of theorem. Let f be a continuous homomorphism of G_1 into G_2. We use exponential coordinates of the second kind for G_1, say, $(t_1, \ldots, t_m)$. In these coordinates the map f becomes

$$(t_1, \ldots, t_m) \longmapsto f(\exp t_1 X_1) \ldots f(\exp t_m X_m).$$

Now for any $i = 1, 2, \ldots, m$, the map $t \longmapsto f(\exp tX_i)$ is a continuous homomorphism of $\mathbf{R}$ into G_2 and so there is $Z_i \in \mathrm{Lie}(G_2)$ such that $f(\exp tX_i) = \exp tZ_i$ for all t. Hence f becomes

$$(t_1, \ldots, t_m) \longmapsto \exp t_1 Z_1) \ldots \exp t_m Z_m).$$

which is obviously analytic.

Uniqueness of the Lie structure. Suppose that $G \subset \mathrm{GL}(n_i, \mathbf{R})$ for $i = 1, 2$ and the identity map of G is a homeomorphism. Then by Theorem 6.1 it is analytic in both directions and hence is an analytic isomorphism. Ths is the uniqueness of the analytic structure.