4. The concept of a Lie group

4.1. The category of manifolds and the definition of Lie groups.

We have discussed the concept of a manifold-C^r, k-analytic, algebraic. To define the *category* corresponding to each type of manifold we must define what the morphisms are. Since all the types of manifolds were defined as instances of ringed spaces of functions it is enough to define what is a morphism between two such ringed spaces. If $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ are ringed spaces of functions, a morphisn $f(X \longrightarrow Y)$ is a continuous map such that the pull back f^* takes $\mathcal{O}_Y$ to $\mathcal{O}_X$. More precisely $f^* := \varphi \circ f$ is a homomorphism from $\mathcal{O}_Y(V)$ to $\mathcal{O}_X(f^{-1}(U))$ for each open $V \subset Y$. Compositions of morphisms are morphisms and so we have categories of manifolds for each of the type discussed.

All the categories of manifolds defined above admit products. To define a Lie group in one of these categories we take a manifold G which is also a group and require that the group maps are morphisms, namely, the maps

$$\mu : G \times G \longrightarrow G, \quad \mu(x, y) = xy, \qquad i : G \longrightarrow G, \quad i(x) = x^{-1}$$

are morphisms. A morphism between Lie groups is a homomorphism which is a morphism of the underlying manifolds. So we get the corresponding category of Lie groups. If the category is the category of affine algebraic varieties we get the category of *linear algebraic groups*; it is a basic result that the linear algebraic groups are precisely the (Zariski) closed subgroups of some $\mathrm{GL}(n)$.

The simplest example of a Lie group is $\mathrm{GL}(n, k)$. One can view this as a Lie group in all categories. The coordinates are the matrix entries a_{ij} and the group maps are

$$c_{ij} = \sum_k a_{ik} b_{kj}, \qquad a^{ij} = \det((a_{ij}))^{-1} A_{ij}$$

where A_{ij} is the cofactor of a_{ij}. For $k = \mathbf{R}, \mathbf{C}$ we obtain a real or complex analytic Lie group, and for k non-archimedean we obtain a k-analytic Lie group. In the algebraic category we get a linear algebraic group; we can view $\mathrm{GL}(n, k)$ for k algebraically closed as $\mathrm{Spec}(k[\ldots a_{ij}, \ldots \det^{-1}])$. To view it as a closed subgroup of $\mathrm{GL}(n + 1, k)$ we consider the map

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}.$$

It gives an isomorphism with a closed subgroup of $\mathrm{SL}(n+1, k)$.

If $G_i (i = 1, 2)$ are Lie groups, then $G_1 \times G_2$ is also one. This extends obviously to products of an arbitrary number of factors. There is a construction, namely a *semi direct product* which is important. Here G_i are given Lie groups and in addition we are given an action (smooth) of G_1 as *automorphisms* of G_2. We define the semi direct product $G = G_1 \times' G_2$ as follows .The underlying manifold is the product. The group multiplication is defined by

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_1[h_2]g_2).$$

It is easy to verify that this defines a group and that it is a Lie group. The standard examples are when G_2 is a vector space and the action of G_1 is via linear maps.

Problems

1. Find the closed subgroup of $\mathrm{GL}(n+1, k)$ to which $\mathrm{GL}(n.k)$ is isomorphic (in the discussion above) and construct the inverse morphism.

2. Verify that the additive group k^n is a Lie group.

3. Verify that $T^n := \mathbf{R}^n / \mathbf{Z}^n$ is a real analytic Lie group, the *torus* of dimension n.

4. Let L be a *lattice* in $\mathbf{C}^n$, namely an additive subgroup which is discrete with $\mathbf{C}^n/L$ compact; for instance, for $n = 1$ take L as the group generated by 1 and τ where $\mathrm{Im}(\tau) > 0$. Verify that $\mathbf{C}^n/L$ is a compact complex Lie group.

5. Prove the statements regarding semi direct producrs made in the discussion above.

To define further examples of Lie groups the simplest and best method is to look for subgroups of $\mathrm{GL}(n)$. It is a theorem that most (in some sense) Lie groups are subgroups of $\mathrm{GL}(n)$. But for doing this we must know how to define *submanifolds* of manifolds.

4.2. Submanifolds. Let $(X, \mathcal{O}_X)$ be a ringed space of functions. If $Y \subset X$ is open, then $V \longmapsto \mathcal{O}_X(V)$ is a ringed space of functions on Y; we refer to Y as the open subspace of X. Suppose now Y is only closed. Then there is a natural ringed space on Y defined by X which we denote by $\mathcal{O}_Y$. We view Y as a topological space with the topology inherited from X. If $V \subset Y$ is open in Y and $f(V \longrightarrow k)$ is a function, then $f \in \mathcal{O}_Y(V)$

if and only if the following is satisfied: for each $y \in V$, there is an open U in X such that

$$y \in U, \quad U \cap Y \subset V, \quad \exists g \in \mathcal{O}_X(U) \text{ such that } g\big|_{U \cap Y} = f\big|_{U \cap V}.$$

In other words, $f \in \mathcal{O}_Y(V)$ if and only if f is, locally on V, the restriction of a function from $\mathcal{O}_X$. The local nature of this definition implies that $\mathcal{O}_Y$ is a sheaf and so $(Y, \mathcal{O}_Y)$ is a ringed space of functions; we call it the *ringed subspace* of X defined by Y.

This construction can be slightly generalized. By a *locally closed* subset of X we mean a subset Y such that $Y \subset Z \subset X$ where Z is open and Y is closed in Z. Then Y becomes a ringed space if we first go from X to Z (open subspace) and then from Z to Y (closed subspace).

The ringed space Y above has the following universal property:

Theorem 1. *We have (1) The identity map $I(Y \longrightarrow X)$ is a morphism (2) If Z is a ringed space of functions and $f(Z \longrightarrow Y)$ is a morphism, then f is a morphism of Z ito X; conversely, a morphism of Z into X which has image contained in Y is a morphism of Z into Y.*

Even when X s a manifold it is seldom the case that Y is a manifold. There is a simple way to ensure this. The following theorem is classical.

Theorem. 2 *Let X be a manifold and let $Y \subset X$. Suppose that for each $y \in U$ there is an open neighborhood U of y in X and functions $f_1, f_2, \ldots, f_p \in \mathcal{O}_X(U)$ such that (a) $U \cap Y = \{x \subset U \mid f_1(x) = \ldots = f_p(x) = 0\}$ (b) $df_1, \ldots, df_p$ are linearly independent at y. Then Y is locally closed and $(Y, \mathcal{O}_Y)$ is a manifold of dimension $\dim(X) - p$.*

Proof(Sketch). The result is local and so we may use coordinates. We may assume that $y = 0 \in k^n$ and $f - 1, \ldots, f_p$ functions in $\mathcal{O}_{k^n}(U)$ with the required properties. The hypothesis means that the matrix

$$\left(\frac{\partial f_i}{\partial x_j} \right)_{1 \le i \le p, 1 \le j \le n}$$

has rank p at 0. By permuting the coordinates x_i we may assume that the matrix

$$\left(\frac{\partial f_i}{\partial x_j} \right)_{1 \le i \le p, 1 \le j \le p}$$

3

is invertible at 0. If we define

$$y_i = f_i(1 \leq i \leq p), \quad y_i = x_i(p+1 \leq i \leq n)$$

we have

$$\left(\frac{\partial y_i}{\partial x_j}\right)_{1 \leq i,j \leq n}$$

is invertible at 0. Hence the (y_j) form a coordinate system at 0 in which Y appears near 0 as the locus where the $y_i(1 \leq i \leq p)$ are 0. The theorem is now obvious.

Remark. The Jacobian criterion used in the proof above for $(y_j)_{1 \leq j \leq n}$ to form a system of local coordinates is the basic result in Calculus of several variables and is valid in the C^1 category. It is also valid in the k-analytic category when k is a complete field with an absolute value (see Serre's Springer Lecture Notes for a proof or see problem below).

Problems

1. Prove that Y is locally closed if and only if it is open in its closure and equivalently, if and only if each point of Y has an open neighborhood U in X such that $U \cap Y$ is closed in U. Deduce that te submanifolds defined above are locally closed in X.

2. Prove the Jacobian criterion for $(y_j)_{1 \leq j \leq n}$ to be a local coordinate system in the k-analytic category.

3. If Z is a manifold and f is a morphism of Z into an X above such that the image of f is contained in Y, then f is a morphism into Y. Together with the fact that the identity map $Y \longrightarrow X)$ is m morphism, this universal property characterizes the manifold structure defined on Y.

4. Prove that if X is a topological group and Y is a locally open subgroup, then Y is already closed in X.(*Hint*: Replace X by V^{cl} to assume Y dense open in X. Prove that Y is closed.)

4.3. Subgroups of Lie groups as Lie groups. We have

Theorem 1. *Let G be a Lie group and and H a closed subgroup which defines a submanifold. Then H is a Lie group.*

4

Proof. Let I_H be the identity map of H into G. Let μ_H be the multiplication $H \times H \longrightarrow H$. To prove that this is a morphism it is enough to show that it is a morphism into G. But it is the composition $\mu_G \circ I_H \times I_H$ and so is indeed a morpism. Similarly we prove that ι_H, the map $h \longmapsto h^{-1}$ is a morphism of H into itself.

Example: SO(n). Let $G = \mathrm{SO}(n) \subset k^{n^2}$. The matrix entries a_{ij} are coordinates and let $A = (a_{ij})$. Write

$$F = F^T = A^T A - I = (f_{ij}), \quad f_{ij} = f_{ji} = \sum_r a_{ri} a_{rj} - \delta_{ij}.$$

Then G is the set of points where the f_{ij} vanish and it is enough to show that the $(1/2)n(n+1)$ differentials

$$df_{ij} (i \leq j)$$

are linearly independent at all points of G. We shall prove this at all points g which are invertible. We must show that for a fixed invertible g, if $c_{ij} (i \leq j)$ are constants,

$$\sum_{i \leq j} c_{ij} (df_{ij})_g = 0 \implies c_{ij} = 0.$$

This can be rewritten in matrix notation as

$$\mathrm{Tr}(S dF_g) = 0 \implies S = 0, \qquad S = S^T$$

where S is the symmetric constant matrix

$$S = (s_{ij}), \quad s_{ij} = s_{ji}, \quad s_{ij} = \begin{cases} c_{ij} & \text{if } i = j \\ (1/2)c_{ij} & \text{if } i < j \end{cases}$$

Now

$$\begin{aligned} \mathrm{Tr}(S dF_g) &= \mathrm{Tr}(S(dA^T g + g^T dA)) \\ &= \mathrm{Tr}(g^T dAS) + \mathrm{Tr}(Sg^T dA) \\ &= 2\mathrm{Tr}(dASg^T). \end{aligned}$$

Since the coordinate differentials dA are linearly independent everywhere we conclude that $Sg^T = 0$ and hence, as g is invertible, $S = 0$.

Example: $SL(n)$. Here there is only one equation, namely $f(A) = \det(A) - 1 = 0$. It is thus a question of showing that df is not zero at any $g \in SL(n)$. Once again we shall show that $df_g \neq 0$ at any g which is invertible. We have

$$\frac{\partial f}{\partial a_{ij}} = A_{ij}$$

where A_{ij} is the cofactor of a_{ij}; certainly at each invertible g some cofactor is not zero.

Example: $U(n)$. Here the field is $\mathbf{C}$ but as complex conjugations are involved, we have to view $U(n)$ as a real Lie group. The coordinates are the $\mathrm{Re}(a_{ij})$ and $\mathrm{Im}(a_{ij})$.

We proceed as before. Write X^* for $\bar{X}^T$. Let

$$F = F^* = A^*A - I = (f_{ij}).$$

Notice that $f_{sr} = \bar{f}_{rs}$; in particular, f_{rr} are real. Write $f_{rs} = u_{rs} + iv_{rs}$. We wish to prove that $df_{rr}, du_{rs}, dv_{rs}(r < s)$ are linearly independent at any invertible g. If $c_{rr}, a_{rs}, b_{rs}(r < s)$ are real constants, we have

$$\sum_r c_{rr}df_{rr} + \sum_{r<s}(a_{rs}du_{rs} + b_{rs}dv_{rs}) = \mathrm{Re}(\mathrm{Tr}(EdF))$$

where $E = E^* = (e_{rs})$ is hermitian and given by

$$e_{rs} = \begin{cases} c_{rr} & \text{if } r = s \\ (1/2)(a_{rs} - ib_{rs}) & \text{if } r < s \\ (1/2)(a_{sr} + ib_{sr}) & \text{if } r > s. \end{cases}$$

We shall see presently that $\mathrm{Tr}(EdF)$ is real. As before, at the point g,

$$\begin{aligned} \mathrm{Tr}(EdF_g) &= \mathrm{Tr}(EdA^*g + g^*dAE) \\ &= \mathrm{Tr}(g^*dAE)^{\mathrm{conj}}) + \mathrm{Tr}(g^*dAE) \\ &= 2\mathrm{Re}(\mathrm{Tr}(g^*dAE)) \\ &= 2\mathrm{Re}(\mathrm{Tr}(dAEg^*)). \end{aligned}$$

This shows that $\mathrm{Tr}(EdF_g)$ is real and is given by

$$\mathrm{Tr}(EdF_g) = 2\mathrm{Tr}dXC) - 2\mathrm{Tr}(dYD)$$

6

where $Eg^* = C + iD$ (C, D real) and $A = X + iY$. Since dX, dY are linearly independent everywhere, we must have $C = D = 0$, hence $Eg^* = 0$, so that $E = 0$.

Problems

1. Verify that $\mathrm{Sp}(2n, k)$ is a Lie group.
2. Verify that $\mathrm{SU}(n)$ is a real Lie group.