11. Representations of compact Lie groups

11.1. Integration on compact groups. In the simplest examples like $\mathbf{R}^n$ and the torus T^n we have the classical Lebesgue measure which defines a translation invariant integration of functions on the group. The first task in building a theory of representations of Lie groups is to construct a translation invariant measure.

On a manifold M of dimension n integration measures are constructed out of *volume forms* which are n-forms on M. If ω is a volume form and $\omega = wdx_1 \wedge \ldots \wedge dx_n$ in local coordinates, then the corresponding measure μ is defined as follows: for any function g with support inside the open set where the chart (x_i) is valid we have

$$\int g d\mu = \int g(x_1, \ldots, x_n)|w|dx_1 \ldots dx_n$$

The classical change of variable formula in calculus sows that this formula does not depend on the choice of coordinates. For a function g whose support, although compact, is not confined to a chart, one uses a partition of unity to write it as a sum of functions whose supports are contained in charts and then apply the above formula to define the integral as the sum of the integrals of the functions whose supports are confined to charts. If $M = G$ is a Lie group it is obvious that we can define a volume form invariant under all left translations; we start with a nonzero element $\omega_e \in \Lambda^n(T_e(G))$ and define $\omega_x = d\ell_x(\omega_e)$. The corresponding measure is called *Haar measure* or *left Haar measure.* There is a corresponding *right Haar measure* defined by a right translation invariant volume form. The volume forms are unique up to a multiplicative constant. If G is not commutative the right and left invariant forms need not coincide but differ by a *real character*. Thus if the group does not admit a non trivial real character, for example, if it is its own commutator subgroup, or if it is compact, then the left and right invariant Haar measures coincide. We denote a left Haar measure by dx.

The construction of left or right invariant measures on Lie groups is thus extremely simple. What Haar did was to prove that left or right invariant measures exist on *every locally compact group* while Von Neumann showed that such a measure is unique up to a multiplicative constant. In particular, if G is a *compact* group there is a unique measure dx which is both left and right translation invariant and is normalized by

$$\int_G dx = 1.$$

We shall call it *the Haar measure on G.* Thus

$$\int_G u(x)dx = \int_G u(yxz)dx = \int u(x^{-1})dx.$$

From now on we shall assume that G is compact and second countable. The second countability restriction is really not necessary but avoids unnecessary generality. We recall the theorem of Urysohn that any second countable Hausdorff space is metric.We introduce the Hilbert space $\mathcal{H} = L^2(G)$ associated with the measure dx. It is separable when G is second countable.

11.2. Representations of the group. If G is a *finite* group the representation theory of G (over $\mathbf{C}$) is quite well known. The elementary theory of representations of compact groups proceeds along similar lines without much trouble. A *representation* of a compact G is a *continuous* homomorphism L of G into some $\mathrm{GL}(V)$ where V is a complex finite dimensional vector space; if V is a Hilbert space and the $L(x)$ are unitary for all $x \in G$ we say that L is a *unitary* representation. The notions of sub and quotient representations, irreducibility, direct sums of representations, and tensor products of representations, are all as in the finite case. The first crucial point is that we need consider only *unitary* representations.

Theorem 1. *If L is a representation in V, we can find a scalar product on V that makes the representation unitary.*

Proof. Let $(\cdot, \cdot)_0$ be any scalar product for V and define

$$(u, v) = \int_G (L(x)u, L(x)v)_0 dx.$$

It is easy to see that $(\cdot, \cdot)$ is again a scalar product. But it is now invariant under L:

$$\begin{aligned}(L(y)u, L(y)v) &= \int_G (L(x)L(y)y, L(x)L(y)v)_0 dx \\ &= \int_G (L(xy)u, L(xy)v)_0 dx \\ &= \int_G (L(z)u, L(z)v)_0 dz \\ &= (u, v).\end{aligned}$$

Here we have used the invariance of the integral. Thus the $L(y)$ are all unitary with respect to $(\cdot,\cdot)$.

It follows from this that all representations are direct sums of irreducible ones and hence that we need only determine the irreducible ones.

Theorem 2. *Any representation is a direct sum of irreducible ones.*

Proof. We may assume that the representation is unitary. If V is the representation space and W is an invariant subspace, i.e., a subspace invariant under all $L(x)$, then the unitarity of the $L(x)$ shows that $L^{\perp}$ is also invariant under all $L(x)$. So V splits as the direct sum $W \oplus W^{\perp}$ of subrepresentations and the result follows by induction on $\dim(V)$.

In the case of finite groups representations can be obtained by decomposing the regular representation. We use the same technique here but the problem becomes more interesting because $L^2(G)$ is *infinite dimensional when G is not finite* and it is not obvious *a priori* that it has *finite dimensional* invariant subspaces.

Define the *right regular representation R of G* acting on $\mathcal{H} = L^2(G)$ by

$$(R(x)f)(y) = f(yx) \qquad (x, g \in G).$$

Then, as

$$\int_G |f(y)|^2 dy = \int_G |f(yx)|^2 dy$$

we see that the $R(x)$ are unitary. It is not difficult to see that for any $f \in \mathcal{H}$, the map

$$x \longmapsto R(x)f$$

is continuous from G to $\mathcal{H}$. We wish to prove that there are lots of finite dimensional subspaces invariant under R. If F is one such, the restriction of R to F will give us a finite dimensional unitary representation of G acting on F. The central theorem of the subject is the *Peter-Weyl theorem.* The following is one version of it.

Theorem 1 (Peter–Weyl). *Let $\mathcal{F}$ be the algebraic sum of all finite dimensional subspaces of $\mathcal{H}$ invariant under R. Then $\mathcal{F}$ is dense in $\mathcal{H}$.*

Once this is proved we obtain

Theorem 2. *The finite dimensional irreducible representations of G separate the points of G. More precisely, if $x, y \in G$ with $x \neq y$, there is an irreducible representation L of G such that $L(x) \neq L(y)$.*

Proof. We deduce this from Theorem 1. Suppose this is not true. Then there are $x \neq y$ in G such that every representation L of G satisfies $L(x) = L(y)$. If $t = xy^{-1}$, this means that $t \neq e$ and $L(t) = I$ for every representation L. Hence for any finite dimensional R-invariant subspace F of $\mathcal{H}$ we have $R(t) = I$ on F. Hence $R(t) = I$ on $\mathcal{F}$ and hence, by Theorem 1, $R(t) = I$ on all of $\mathcal{H}$. This is absurd; indeed, select a neighborhood U of e such that U and tU are disjoint and let f be a continuous function on G which is 1 around e and supported within U. Then $R(t)f$ is 1 around t and supported within tU. Clearly $R(t)f \neq f$.

Theorem 3. *If U is a neighborhood of e in G there is a representation L such that $L(x) \neq I$ for all $x \in G \setminus U$. In particular the kernel H of L will be contained in U and we have an imbedding*

$$G/H \hookrightarrow U(N)$$

for some integer $N \geq 1$, so that G/H is a Lie group.

Proof. For each $x \in G \setminus U$ we can find a representation L_x such that $L_x(x) \neq I$, and hence a neighborhood V_x if x such that $L_x(y) \neq I$ for all $y \in V_x$. Take a finite covering $G = \cup_{1 \leq i \leq r} V_{x_i}$ and let $L = \oplus_{1 \leq i \leq r} L_{x_i}$.

Remark. This result is often refereed to as the statement that *any compact group can be approximated by Lie groups which are closed subgroups of unitary groups.*

10.3. Digression on integral operators defined by continuous kernels on a compact space. The idea of Peter and Weyl is to search for finite dimensional subspaces of $\mathcal{H}$ as eigen spaces for integral operators. If the kernels of the integral operators are invariant under right translations then the eigen spaces will be invariant under R.

Let X be a compact second countable Hausdorff space with a measure μ and let k be a complex valued continuous function on $X \times X$. Then k is the so-called *kernel* of the *integral operator* A_k defined on functions on X by

$$(A_k f)(x) = \int_X k(x,y) f(y) d\mu(y) \quad (x \in X).$$

If $|k(x,y)| \leq C$ then A_k maps $|kk := L^2(X,\mu)$ into itself and

$$||A_k f|| \leq C||f||$$

showing that A_k is a bounded operator on the Hilbert space $L^2(X,\mu)$ (which is separable). Historically these were the first operators in infinite dimensional spaces that were studied because of their very strong analogy with finite dimensional operators defined by matrices. In particular, for these operators with k satisfying a hermitian symmetry condition there are eigen values and the spectral theory is very close to the finite dimensional theory. However not all operators belong to this class. If $X = [0,1]$ with μ Lebesgue measure, the operator A of multiplication by x has no eigen value and its spectral theory is more subtle to formulate. This was first done by Hilbert who proved his famous spectral theorem for bounded self adjoint operators. It was later extended to unbounded operators by Von Neumann under the impetus of Quantum Theory.

The crucial property of the operators A_k is the fact that they are *compact*; more precisely they map bounded sets in $\mathcal{H}$ to sets with compact closure. To see this note first that (recall that X is metric) that the uniform continuity of k implies that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|k(x_1,y) - k(x_2,y)| < \varepsilon \qquad (y \in X, d(x_1,x_2) < \delta)$$

where d is the metric on X. Then

$$|(A_k f)(x_1) - (A_k f)(x_2)| \leq \delta||f||$$

whenever $d(x_1,x_2) < \delta$, *uniformly in* f. In particular, A_k maps bounded sets of $\mathcal{H}$ into *equicontinuous* subsets of $C(X)$, proving the compactness of A_k. Note that A_k is a *smoothing operator*; it takes L^2-functions and converts them into continuous functions. In analogy with matrices we say that k is *hermitian* if

$$k(x, y) = k(y, x)^{\mathrm{conj}}.$$

Then the operator is self adjoint in the Hilbert space sense, i.e.,

$$(A_k f, g) = (f, A_k g) \qquad (f, g \in L^2(X, \mu)).$$

For compact self adjoint operators A in a separable Hilbert space $\mathcal{L}$ we have the following version of the spectral theorem:

Theorem. *All eigen values of A are real and the eigen spaces for the non zero eigen values are finite dimensional. Moreover eigen spaces for distinct eigen values are orthogonal. The eigen values form a sequence (λ_n) tending to 0 as $n \to \infty$. Finally*

$$\mathcal{L} = \mathcal{L}_0 \oplus \oplus_n \mathcal{L}_{\lambda_n}$$

where $\mathcal{L}_\lambda$ is the eigen space for the eigen value λ.

Remark. The finite dimensionality of the $\mathcal{L}_\lambda$ for $\lambda \neq 0$ is seen as follows. On $\mathcal{L}_\lambda$, as A acts as the non zero scalar λ, we see that bounded sets are compact. This cannot happen if $\mathcal{L}_\lambda$ were infinite dimensional. The reality of the eigen values and the mutual orthogonality of the eigen spaces are proved as in the finite dimensional case. So it is a question of showing that there is at least one non zero eigen value if $A \neq 0$. Once this is done, the span of the eigen spaces for the non zero eigen values must be $\mathcal{L}_0^\perp$; otherwise we go to the orthogonal complement which is stable under A and argue that A must be 0 there; if A were not 0, there would be a non zero eigen value for A within this orthogonal complement. The existence of the non zero eigen value is a standard argument and not particularly difficult although it needs some development of Hilbert space theory.

10.4. Proof of Peter-Weyl theorem. We start with the kernel k which is continuous and hermitian on $G \times G$ and consider the operator A_k. It is immediate that the eigen space for the eigen value λ is invariant under the right translations $R(t)(t \in G)$ if k satisfies

$$k(xt, yt) = k(x, y) \qquad (x, y, t \in G).$$

This is equivalent to saying that for some function u on G,

$$k(x, y) = u(xy^{-1}).$$

If we choose u so that

$$u(t) = u(t^{-1})^{\text{conj}}$$

then k will be hermitian; this is the case of u is real and symmetric, namely,

$$u(t) = u(t^{-1}).$$

The operator A_k is then *convolution* by u:

$$(A_k f)(x) = \int_G u(xy^{-1}) f(y) dy = \int_G u(t) f(t^{-1}x) dt = (u * f)(x)$$

Given any neighborhood U of e we can find such a u whose support is contained within U; indeed, take $V = U \cap U^{-1}$, v to be continuous, supported inside V, and ≥ 0 with $\int_G v dx = 1$ and take

$$u(t) = \frac{1}{2}\left(v(t) + v(t^{-1})\right).$$

The eigen spaces (for non zero eigen values) of all the operators A_k defined by kernels k such that

$$k(x, y) = u(xy^{-1})$$

for real symmetric continuous functions u on G are thus invariant under the right regular representation R. Suppose that their algebraic sum $\mathcal{A}$ is not dense in $L^2(G)$. Let h be a non zero element in the orthogonal complement of $\mathcal{A}$. Then $A_k h = 0$ for all such k. Thus $u * h = 0$ for all u. But if the supports of the sequence (u_n) are contained in open neighborhoods (U_n) of e such that the U_n are decreasing and $\cap_n U_n = \{e\}$, then it is a standard result that

$$u_n * \psi \to \psi \qquad (n \to \infty)$$

for all $\psi \in L^2(G)$. Hence $u_n * h = h \to 0$, showing that $h = 0$. Since $\mathcal{A} \subset \mathcal{L}$ we have the conclusion that $\mathcal{L}$ is dense in $L^2(G)$.

10.5. Existence of faithful representations for a compact Lie group G. Suppose now that G is a Lie group. Theorem 3 above leads to the result that G has a faithful representation. We need a lemma.

Lemma. *Let G be a Lie group. Then G has no small subgroups. More precisely, there is a neighborhood U of e such that if $S \subset U$ is a subgroup, then $S = \{e\}$.*

Proof. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $||\cdot||$ be a norm on $\mathfrak{g}$. Let $\mathfrak{n}_a = \{X \in \mathfrak{g} \mid ||X|| < a\}$. We choose $a > 0$ such that the exponential map is a diffeomorphism of $\mathfrak{n}_a$ onto $G_a := \exp(\mathfrak{n}_a)$. Let $0 < b < a/2$. We claim that $U = G_b$ fulfills the requirements. Let $S \neq \{e\}$ be a subgroup with $S \subset U$. Let $x \neq e$ be an element of S and let $x = \exp X$ for some $X \in \mathfrak{n}_b$. Clearly we can find an integer $r \geq 1$ such that $X, 2X, \ldots, rX$ are all in $\mathfrak{n}_b$, but $(r+1)X \notin \mathfrak{n}_b$. However $(r+1)X = rX + X \in \mathfrak{n}_a$. Now $\exp(r{+}1)X = x^{r+1} \in S \subset G_b$ and so $\exp(r{+}1)X = \exp Y$ for some $Y \in \mathfrak{n}_b$. Thus both Y and $(r{+}1)X$ are in $\mathfrak{n}_a$ and have the same exponential. Hence $Y = (r+1)X$, showing that $(r+1)X \in \mathfrak{n}_b$, a contradiction to the choice of r.

Theorem 1. *Let G be a Lie group. Then G has a faithful unitary representation. Thus for some N, we have $G \hookrightarrow U(N)$.*

Proof. Choose U as in Lemma and find a representation L such that the kernel of L is contained in U. Then this kernel must be trivial and so L must be faithful.

10.6. Variants of the Peter-Weyl theorem. Orthogonality relations. Characters. The orthogonality relations, between the matrix elements and also between the characters, of irreducible representations go over without change in the compact case. If L is an irreducible *unitary representation* of G and we select an ON basis in the representation space, then

$$L(x) = (a_{ij}(x))$$

is a unitary matrix and the $a_{ij}(1 \leq i, j \leq \dim(L))$ are continuous functions on G. They are the so-called *matrix elements* of L. The subspace inside $L^2(G)$ spanned by them is independent of the choice of the ON basis and depends only on the equivalence class ω of L. We write it as $\mathcal{F}(\omega)$.

Theorem 1. *We have the orthogonality relations*

$$(a_{ij}, a_{k\ell}) = \begin{cases} 0 & \textit{if } (i,j) \neq (k,\ell) \\ \frac{1}{\dim(L)} & \textit{if } (i,j) = (k,\ell). \end{cases}$$

Moreover if ω, ω' are two distinct classes of irreducible representations, then

$$\mathcal{F}(\omega) \perp \mathcal{F}(\omega').$$

Finally, we have the orthogonal decomposition

$$L^2(G) = \oplus_\omega \mathcal{F}(\omega).$$

For any representation L of G, irreducible or not, we define its *character* Θ_L by

$$\Theta_L(x) = \mathrm{tr}(L(x)) \qquad (x \in G).$$

If L is irreducible, its character is an element of $\mathcal{F}(\omega)$. It is a *class function* on G, namely it is constant on the conjugacy classes of G:

$$\Theta_L(xyx^{-1}) = \Theta(y) \qquad (x, y \in G).$$

The character Θ_L depends only on the equivalence class ω of L and determines L up to equivalence and we have

Theorem 2. *The irreducible characters Θ_ω satisfy the orthogonality relations*

$$(\Theta_\omega, \Theta_{\omega'}) = \delta_{\omega\omega'}.$$

Moreover the (Θ_ω) form an ON basis for the subspace of $L^2(G)$ of class functions.

In both Theorem 1 and Theorem 2 the last statements are called *completeness* of the irreducible representations. The completeness at the L^2-level can be sharpened. Namely

Theorem 3. *The algebraic sum of the $\mathcal{F}(\omega)$ is precisely $\mathcal{F}$, is a subalgebra of $C(G)$ closed under complex conjugation, and is dense in $C(G)$ in the uniform topology. The linear span of the Θ_ω is also an algebra closed* *under complex conjugation and is dense in the uniform topology of the subspace of $C(G)$ of class functions.*

Sketch of proof. For the first statement it is enough to prove that if S is a finite dimensional subspace of $L^2(G)$ invariant under the right regular representation R and on which R acts as a representation in the equivalence class ω, then $S \subset \mathcal{F}(\omega)$. Let (s_i) be an ON basis for S. Then

$$s_j(xy) = \sum_i a_{ij}(y)s_i(x)$$

for each y for almost all x. By Fubini this relation is then true for almost all (x, y) and hence for some x_0 for almost all y. If λ is left translation by x_0^{-1} we then have

$$\lambda(s_j) = \sum_i s_i(x_0)a_{ij}$$

in $L^2(G)$ and so s_j is continuous and is in $\mathcal{F}(\omega)$. That $\mathcal{F}$ is an algebra closed under complex conjugation is immediate by considering conjugates and tensor products of representations. It separates the points of G by the Peter-Weyl theorem and so is dense in $C(G)$ by the Stone-Weierstrass theorem. To prove the second part we need to know that Θ_ω is the unique element (up to a scalar multiple) of $\mathcal{F}(\omega)$ invariant under all inner automorphisms, i.e., the unique class function. We have a projection operator P from $C(G)$ to $C(G)_{\mathrm{cl}}$ given by

$$(Pf)(y) = \int_G f(xyx^{-1})dx \qquad (y \in G).$$

Suppose now f is in $C(G)_{\mathrm{cl}}$. Then there is a sequence (g_n) from $\mathcal{F}$ such that $g_n \to f$ in $C(G)$. So $Pg_n \to Pf = f$ in $C(G)$. But Pg_n is in the linear span of the Θ_ω.

Suppose that G is Lie group. We then have a *faithful representation.* Let $L_1, \dots L_n$ be its irreducible constituents.

Theorem 4. *Let G be a Lie group. The irreducible representations obtained by decomposing the tensor products*

$$M_1 \otimes M_2 \otimes \ldots \otimes M_r$$

where each M_j is either an L_i or is equivalent to a conjugate of some L_i, exhaust all the irreducibles of G.

Sketch of proof. If there is some irreducible L which is not in the collection defined in the theorem, its matrix elements will be orthogonal to the matrix elements of all the irreducibles in the collection. Now the *linear span* of these matrix elements is an *algebra* closed under complex conjugation and separates the points of G. It is thus dense in $C(G)$. This is a contradiction.

Remark. The *classical groups* $\mathrm{SU}(n), \mathrm{SO}(n), \mathrm{Sp}(n)$ are *defined* as subgroups of unitary groups and so Theorem 4 implies that their irreducible representations cal all be obtained by decomposing the tensor algebra over the defining representation. This problem is a beautiful one and leads to a beautiful theory. It was first carried out by Hermann Weyl in his great classic *The Classical groups*: *their invariants and representations.*

10.7. Determination of the irreducible characters. If a compact group G is *concretely* given, such as one of the classical groups, there arises the problem of *explicit determination of all the irreducible characters.* This problem was first solved by Hermann Weyl in a series of papers in the mid 1920's that many regard as his greatest work. I cannot go into it here but give an idea of it by discussing the group SU(2). Let $G = \mathrm{SU}(2)$. If Θ is any character, we have

$$\int_G \Theta\Theta^{\mathrm{conj}} dx = 1 \Longleftrightarrow \Theta \text{ is an irreducible character} \qquad (*).$$

Weyl's method is to use this result in the determination of all the irreducible characters.

In the first place, let

$$u(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Let T be the diagonal subgroup of all the $u(\theta)$:

$$T = \big\{u(\theta)\big\}.$$

Then by the spectral theorem any element of G is conjugate to an element of T and so to determine a class function it is enough to know its restriction to T. Moreover, the intersection of a conjugacy class of G with T consists precisely of $u(\theta)$ and $u(-\theta)$ for some θ. Hence the restriction of a class function to T is *symmetric*, i.e., invariant under $\theta \longrightarrow -\theta$:

$$f(u(\theta)) = f(u(-\theta)).$$

If L is any representation of G, irreducible or not, the restriction of L to T can be decomposed as a direct sum of characters of T. Hence the character Θ of L has the following form

$$\Theta(u(\theta)) = \sum_n c_n e^{in\theta}$$

where the c_n are integers ≥ 0 and the sum is finite. In order to make use of the relation $(*)$ it is now necessary to reduce the integration to one on T. This is done by the famous *Weyl integration formula.* Let

$$\Delta(u(\theta)) = e^{i\theta} - e^{-i\theta}.$$

Then, for any continuous class function f on G, we have

$$\int f(x)dx = \frac{1}{2}\int_T f(u(\theta))\Delta(u(\theta))\Delta(u(\theta))^{\mathrm{conj}} d\theta.$$

Here we write $d\theta$ for the Lebesgue measure on T normalized to give measure 1 for T. Let

$$\varphi = (\Theta\big|_T)\Delta.$$

Then Φ is *skew symmetric*, is a finite Fourier series with integer coefficients, and $(*)$ becomes, by the integration formula,

$$\int_T \Phi\Phi^{\mathrm{conj}} d\theta = 2 \qquad (**)$$

for irreducible characters. Now the functions

$$\varphi_n := e^{i(n+1)\theta} - e^{-i(n+1)\theta} \qquad (n = 0, 1, 2, \ldots)$$

are skew symmetric, and any finite Fourier series which is skew symmetric and has integer coefficients is *an integral linear combination of the* φ_n. Hence

$$\Phi = \sum_n d_n \varphi_n$$

where the sum is finite and the d_n are *uniquely determined integers.* Now the φ_n are orthogonal in $L^2(T)$ and

$$\int \varphi_n \varphi_n^{\mathrm{conj}} d\theta = 2$$

It is then immediate that

$$\sum_n d_n^2 = 1$$

showing that

$$(\Theta\big|_T) = \pm\varphi_n$$

for some n. Hence

$$\Theta(u(\theta)) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

Here we have chosen the plus sign because the value of Θ at e must be a positive integer. This is the famous *Weyl character formula.* We must have that for *integers* $n \geq 0$ the right side of the formula *must represent an irreducible character.* For if some n is missing, the function defined above on T, which is a symmetric finite Fourier series, extends uniquely to G as a class function, and that class function ill be orthogonal to all irreducible characters (by the integration formula again!), which is impossible. By taking the limit as $\theta \to 0$ we see that

$$\Theta(e) = (n+1).$$

We thus have

Theorem 1 (Weyl character and dimension formulae). *The irreducible characters of $G = \mathrm{SU}(2)$ are precisely all the functions whose restrictions to T are given by*

$$\Theta_n(u(\theta)) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

The corresponding representation has dimension $n+1$.

Remark 1. It is an interesting question to describe explicitly the representation whose character is Θ_n. From the action of G on $\mathbf{C}^2$ we get a representation of G on the *symmetric tensors of degree* n *over* $\mathbf{C}^2$. If $\{e_1, e_2\}$ is the standard basis for $\mathbf{C}^2$,

$$e_1^r e_2^{n-r} \quad (0 \leq r \leq n)$$

form a basis for the space of symmetric tensors, and the character of this representation is easily computed to be Θ_n.

Remark 2. In his great papers already referred to, Weyl obtained the character and dimension formulae for irreducible representations of *all compact Lie groups* G*.* The role of T is played by a *maximal torus* of G, and the role of the symmetry $\theta \longrightarrow -\theta$ is played by a finite group W known as the *Weyl group.* There is a skew symmetric function analogous to Δ in the general case and the integration formula contains this function in exactly the same way. For instance, if $G = \mathrm{SU}(n+1)$, T is the diagonal subgroup, W is the permutation group in n letters acting by permutation of the diagonal entries, and

$$\Delta(\operatorname{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_{n+1}}\right)) = \prod_{i<j}(e^{i\theta_j} - e^{-i\theta_j}).$$

The general case requires the development of the *structure theory* for compact Lie groups which Weyl obtained from the theory of *complex semi simple Lie algebras* that was developed by E. Cartan.

Remark 3. The physicists became interested in unitary representations because they were the means of expressing the *covariance* of a quantum system. The need to go beyond compact groups became clear since the space-time symmetry groups are non-compact, for example the Poincaré or the Galilean groups. In 1939 Wigner obtained a classification of free elementary particles in terms of their mass and spin by determining the unitary irreducible representations of the Poincaré group. Then in 1943, Gel'fand and Raikov proved that any locally compact group had enough irreducible unitary representations to separate its points. The physicist Bargmann, then Gel'fand and Naimark, and finally Harish-Chandra, began the deep study of unitary representations of the semi simple Lie groups. Harish-Chandra finally obtained a character formula for certain irreducible representations of a semi simple Lie group which was an exact analogue of the Weyl character formula. The *Harish-Chandra character formula* proved to be as decisive for the representation theory of and harmonic analysis on all semi simple Lie groups as the Weyl formula was for compact Lie groups.

In the above discussion I have referred to the characters of representations. As long as the representation is finite dimensional this is the usual character. However, for a real simple Lie group which is not compact, typically all unitary irreducible representations are *infinite dimensional* and it is not clear how to define the character of such a representation. Harish-Chandra was able to define a notion of character for infinite dimensional unitary irreducible representations of *any* semi simple Lie group, nowadays called *the Harish-Chandra character.* If $\mathcal{H}$ is the Hilbert space of the representation and (e_n) is an ON basis for $\mathcal{H}$, and π is the representation, the series

$$\sum_n (\pi(x)e_n, e_n),$$

although hopelessly divergent if considered pointwise, *nevertheless converges as a series of distributions (in the sense of Laurent Schwartz).* This distribution is the Harish-Chandra character of π. Harish-Chandra proved the fantastic result that this distribution is a *locally integrable function which is analytic at te generic points of the group.* Thus one can speak of character formulae and he showed that the most fundamental irreducible unitary representations of a semi simple Lie group are determined by a formula which is an almost exact analogue of the Weyl character formula. This is what I have referred to as the Harish-Chandra character formula.