10. The subgroup–subalgebra correspondence. Homogeneous spaces.

10.1. The concept of a Lie subgroup of a Lie group. We have seen that if G is a Lie group and $H \subset G$ a subgroup which is at the same time a closed submanifold, then H is a Lie group and the inclusion map ι is a morphism such that $d\iota$ is injective. The image of $\mathfrak{h} = \mathrm{Lie}(H)$ in $\mathfrak{g} = \mathrm{Lie}(G)$ is a subalgebra. The natural question is whether every Lie subalgebra of $\mathfrak{g}$ arises in this manner.

Formulated in this manner the answer is negative. Consider the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. Its Lie algebra is $\mathbf{R}^2$ and for any $X \in \mathbf{R}^2$ we can verify that $\exp X$ is the image of X in T^2. Suppose $\mathfrak{h}$ is a line with an *irrational* slope, it is clear that the exponential of it is an one-parameter subgroup of T^2 that winds around the torus in a dense manner and so $\mathfrak{h}$ cannot arise from a closed subgroup of G.

To make the subgroup–subalgebra correspondence bijective it turns out to be sufficient to generalize the notion of a Lie subgroup slightly. We shall say that a subgroup $H \subset G$ is a *Lie subgroup* if the following conditions are satisfied:

1. H is a Lie group and the inclusion map ι is a morphism.
2. The differential $d\iota$ is injective.

It is to be noted that the topology and smooth structure of H need anot be inherited from those of G and that H need not be closed in G.

With this reformulation one can now prove the following theorem.

Theorem 1 (Chevalley). *If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, there is a unique connected Lie subgroup $H \subset G$ such that $d\iota(\mathrm{Lie}(H)) = \mathfrak{h}$. Moreover H is generated by the elements $\exp X (X \in \mathfrak{h})$.*

10.2. Involutive distributions on a manifold. If $\mathfrak{h}$ is one dimensional, we can choose a basis element $X \in \mathfrak{h}$ and integrate the vector field to get the one-parameter group $t \longmapsto \exp tX$ whose topology and Lie structure are given by those of the parameter t. In the general case we use an analogous method. However we need some preparation.

At each point $x \in G$ we have the subspace $\mathfrak{h}_x \subset T_x(G)$ defined as the space of tangent vectors X_x for $X \in \mathfrak{h}$. We have an assignment

$$\mathcal{H} : x \longmapsto \mathfrak{h}_x$$

of tangent spaces with the following properties.

1. $\dim(\mathfrak{h}_x) = d$ is constant for all $x \in G$.
2. The assignment $\mathcal{H}$ is smooth in the following sense: if $x \in G$ we can find vector fields $Z_1, Z_2, \ldots, Z_r$ such that the tangent vectors $(Z_i)_y$ span $\mathfrak{h}_y$ for all y in a neighborhood of x.
3. Suppose $U \subset G$ is open and X, Y are vector fields on U such that $X_y, Y_y \in \mathfrak{h}_y$ for all $y \in U$. Then $[X, Y]_y \in \mathfrak{h}_y$ for all $y \in U$ also.

The conditions 1. and 2. make sense for any assignment

$$\mathcal{L} : x \longmapsto \mathcal{L}_x$$

of tangent spaces on any manifold; $\mathcal{L}$ is then called a *distribution* of rank d. If condition 3. is satisfied, we shall say that $\mathcal{L}$ is *involutive*. Thus $\mathcal{H}$ is an involutive distribution of rank equal to $\dim(\mathfrak{h})$.

The verification of conditions 1. through 3. for $\mathcal{H}$ is easy. 1. is trivial. If $X_1, \ldots X_d$ is a basis for $\mathfrak{h}$, then $(X_i)_y$ span $\mathfrak{h}_y$ for all $y \in G$, proving 2. For 3. we first observe that $[X_i, X_j] = \sum_{q \leq k \leq d} c_{ijk} X_k$ for all $1 \leq i, j \leq d$, as $\mathfrak{h}$ is a subalgebra. It is now easy to show that X and Y can be written as $X = \sum f_i X_i, Y = \sum_i g_i X_i$ wherer f_i, g_i are smooth functions. The property 3. is now obvious.

Let M be a manifold and $\mathcal{L}$ a distribution on M of rank d. As mentioned earlier, we try to find submanifolds $S \subset M$ with the property that at each point $y \in S$, we have

$$T_y(S) = \mathcal{L}_y \qquad (y \in S).$$

Such an S is called an *integral manifold for* $\mathcal{L}$. $\mathcal{L}$ is called *integrable* if through any point of M there is an integral manifold. It is easy to see that integrability of $\mathcal{L}$ implies that $\mathcal{L}$ is involutive. Indeed, in the definition of property 3. above let $y \in U$ and let S be an integral manifold of $\mathcal{L}$ through y. Then X and Y are tangent to S at all of its points, and so $[X, Y]$ is also tangent to S at all of its points. In particular, $[X, Y]_y \in \mathcal{L}_y$. The famous classical theorem of Frobenius asserts now that conversely, if $\mathcal{L}$ is involutive, then $\mathcal{L}$ is integrable.

10.3. The local Frobenius theorem. Locally, on any manifold M we can construct distributions which are involutive and integrable as follows. We take coordinates $x^i(1 \leq i \leq m$ on M on an open set U and define $\mathcal{L}_y$

as the span of $\partial/\partial x^i (1 \leq i \leq p)$ for $y \in U$. Then $\mathcal{L}$ is integrable on U since the submanifolds defined by making the $x^i (i \geq p = 1)$ constant are integral manifolds. $\mathcal{L}$ is also obviously involutive. The local frobenius is the assertion that every involutive distribution looks locally like this $\mathcal{L}$. Let M be a smooth manifold with $\dim(M) = m$ and $\mathcal{L}$ an involutive distribution of rank p on M. Then $\mathcal{L}$ is integrable and we have the following precise version of the local Frobenius theorem.

Theorem (Frobenius). *Let $\mathcal{L}$ be an involutive distribution of rank p on a manifold. Then for any $x \in M$, there is an open set U containing x coordinates $(x^i)_{1 \leq i \leq m}$ on U, and an $a > 0$ such that*

$$y \longmapsto (x^1(y), \ldots, x^m(y))$$

is a diffeomorphism of U with the cube (or polydisk if M is a complex manifold) $I_a^m = \{|t^i| < a \forall i\}$, and for each $y \in U$, $\mathcal{L}_y$ is the span of

$$(\partial/\partial x^i)_y, 1 \leq i \leq p.$$

If $y \in U$ we write

$$U[y] = \{z \in U \big| x^j(z) = x^j(y), p+1 \leq j \leq m\},$$

and call it the *slice through y.* We refer to $(U, (x^i), a > 0)$ as *adapted to $\mathcal{L}$*. Now slices are usually only small pieces and our goal is the construction of integrable manifolds which are as big as possible. The technique is the obvious one of piecing together small slices, but as we do it, we shall find that the integrable manifold may return again and again to the same part of M, so that in the end the integrable manifold acquires a topology that may be different from the ambient one. This situation was first analyzed carefully by Chevalley who constructed the global integrable manifolds for the first time. In the succeeding pages we shall give a brief presentation of Chevalley's treatment.

10.4. Immersed and imbedded manifolds. A manifold N is said to be *immersed* in M is (a) $N \subset M$ and (b) the inclusion $i : N \longrightarrow M$ is a morphism having an injective differential for all $n \in N$. Notice that the topology of N is *not required to be induced by M*, i.e., i is not assumed to be a homeomorphism of N onto its image $i(N)$ with its topology induced from M. N is said to be *imbedded*if i is a homeomorphism onto its image in M. From basic manifold theory we know that for N to be immersed the following is necessary and sufficient: if $n \in N$, there are open neighborhoods U in M and V in N of n and a diffeomorphism ψ of U with $I_a^p \times I_a^{m-p}$ such that ψ takes n to $(0,0)$, V to $I_a^p \times \{0\}$, and i to the map $x \longmapsto (x,0)$. For N to be imbedded in M the condition is the same with the extra requirement that $V = U \cap N$.

Proposition 1. *If $N \subset M$ is imbedded and P is any smooth manifold, a map $f(P \longrightarrow N)$ is a morphism if and only if it is a morphism of P to M. If N is only immersed, this is still so provided f is a continuous map of P to N.*

Proof. Fix $p \in P, f(p) = n$. Take $U, V, \psi, a > 0$ as above. Assume N is immersed and that f is a continuous map into N. Then $P_1 = f^{-1}(V)$ is open in P with $f(P_1) \subset V$. It is obvious that if f is a morphism into U it is a morphism into V. In the imbedded case we define $P_1 = f^{-1}(U)$. Once again P_1 is open in P; but now, as $V = U \cap N$, f maps P_1 into V and it is clear that f is a morphism into V.

An immersed manifold is *universally immersed* if for any P, the morphisms $P \longrightarrow N$ are precisley the morphisms $P \longrightarrow M$ with image contained in N. Imbedded manifolds are universally immersed, and some nonimbedded manifolds are also universally imbedded. The classical example is $\mathbf{R}$ immersed in $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ through the map

$$\psi : t \longmapsto [(t, \xi t)]$$

where ξ is an irrational number and $[(a,b)]$ is the image of (a,b) under the natural map $\mathbf{R}^2 \longrightarrow T^2$. The proof that $\mathbf{R}$ is universally immersed in T^2 is left as an exercise.

10.5. The global Chevalley-Frobenius theorem. Let $\mathcal{L}$ be an involutive distribution on M. A *leaf* is a connected manifold L which is immersed in M with $T_y(L) = \mathcal{L}_y$ for all $y \in L$, i.e., L is an integral manifold for $\mathcal{L}$. A *slice* is an imbedded leaf. A *maximal leaf* is a leaf L with the following property: if L_1 is another leaf with $L \cap L_1 \neq \emptyset$, then $L_1 \subset L$, and L_1 is an open submanifold of L. The topology and smooth structure of a slice are uniquely determined by M. Any leaf is a union of slices which are open in it and so the topology and smooth structure on a leaf are uniquely determined. The point is that these may not coincide with the structures inherited from M.

Global Frobenius theorem. *Through every point x of M passes a unique maximal leaf. Any two maximal leaves are either identical or disjoint. All leaves are universally immersed.*

The idea of the proof is very simple. We shall introduce the *leaf topology* for M for which the leaves are form a basis. The leaf topology is finer than the given topology of M. The maximal leaves are the connected components of M in the leaf topology. The proofs are quite simple and follow from the local structure of involutive distributions. The only subtle point is to prove that the connected components in the leaf topology are second countable. Once this is done, the connected components are leaves and are obviously maximal.

10.6. The Lie subgroup corresponding to a Lie subalgebra. Let us return to the context of a Lie group G with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. We introduced the involutive distribution $\mathcal{H}$ earlier and so we have the maximal leafs for it.

Theorem 1. *The leaf through e is the unique Lie subgroup with Lie algebra $\mathfrak{h}$.*

10.7. Homogeneous spaces Let G be a Lie group and H a closed Lie subgroup. Then G acts transitively on G/H. The fundamental question in the theory of Lie groups is to view G/H as a smooth manifold (smooth being in the C^∞, real analytic, or complex analytic categories) on which the natural action of G is morphic.

Let
$$X = G/H, \qquad \pi : g \longmapsto gH \quad (g \in G),$$
and let us give X the quotient topology; this means that $U \subset X$ is open if and only if $\pi^{-1}(U)$ is open. We assert that π is an open map; indeed, if V is open in G, $\pi^{-1}(\pi(V)) = VH = \bigcup_{\xi \in H} V\xi$ is open. We also observe the fact that X is Hausdorff if and only if H is closed (this is true when G is only locally compact). To see this, suppose first that H is closed and let $g, k \in G$ be such that $\pi(g) \neq \pi(k)$. Then $g^{-1}k \notin H$ and so there is open $V \subset G$ containing $g^{-1}k$ such that $V \cap H = \emptyset$. By continuity there are open neighborhoods V_2, V_3 of g, k respectively such that $V_2^{-1}V_3 \cap H = \emptyset$.

This means that for $x \in V_2, y \in V_3, x^{-1}y \notin H$ or $\pi(x) \neq \pi(y)$. Thus $\pi(V_2)$ and $\pi(V_3)$ are disjoint open neigborhoods of $\pi(g), \pi(k)$ respectively. In the reverse direction, if $x \notin H$, select an open neighborhood X_1 of $\pi(x)$ not containing $\pi(1)$; then $\pi^{-1}(X_1)$ is an open neighborhood of x disjoint from H.

Theorem. *Let H be a closed Lie subgroup of G. Then there is a unique structure of a smooth manifold on $X = G/H$ such that the natural map $\pi : g \longmapsto gH$ of G onto X is a submersion. The action of G on X is morphic.*

Comments on the proof of the theorem. Let $\mathcal{O}_G$ be the structure sheaf of G. It is natural to start with the sheaf $\mathcal{O}_X$ where, for U an open subset of X,

$$f \in \mathcal{O}_X(U) \Leftrightarrow f \circ \pi \in \mathcal{O}_G(\pi^{-1}(U)).$$

It is trivial to verify that the action of G on X is morphic. The point to show is that X becomes a smooth manifold when equipped with this structure with the property that π is a submersion. The uniqueness of the smooth structure under the requirement that π is a submersion is a standard fact.

Let us introduce some standard terminology. Let M be a smooth manifold and N, P submanifolds. Suppose $m \in N \cap P$. We say that N and P *meet transversally at m* if the tangent spaces $T_m(N), T_m(P)$ are complementary, i.e.,

$$T_m(M) = T_m(N) \oplus T_m(P).$$

In this case we have of course $\dim(M) = \dim(N) + \dim(P)$. The key to the proof is the following lemma.

Lemma 1. *Suppose that we can find a submanifold W of G such that*

a. $1 \in W, \quad W \cap wH = \{w\}$ *for all* $w \in W$

b. W *and* wH *meet transversally at* w *for all* $w \in W$.

Then $(X, \mathcal{O}_X)$ is a smooth manifold, π is a submersion, and the action of G on X is morphic.

Proof. We set up the map

$$\psi : W \times H \longrightarrow G, \qquad \psi(w, \xi) = w\xi.$$

The image of $d\psi_{w,1}$ is the span of $T_w(H)$ and $T_w(W)$, and so $d\psi_{w,1}$ is bijective. If R_η is right translation by $\eta \in H$, we have $R_\eta \psi = \psi(\mathrm{id} \times R_\eta)$ so that

$$(dR_\eta)_w d\psi_{w,1} = d\psi_{w,\eta} d(\mathrm{id} \times R_\eta)_{w,1}$$

showing that $d\psi$ is bijective everywhere on $W \times H$. Moreover ψ is also bijective; for, if $w\eta = w'\eta'$, then $w' \in W \cap wH$ so that $w' = w$ and thence $\eta = \eta'$. Hence $U = WH$ is open in G and ψ is a diffeomorphism. In particular $\pi(W)$ is open in X. It follows from this that for any open subset V of W, a smooth function f on VH is right H-invariant if and only if $(f \circ \psi)(w, \xi)$ is dependent only on w and is a smooth function of w on V. In other words the restriction to $\pi(W)$ of the sheaf $\mathcal{O}_X$ is isomorphic to $\mathcal{O}_W$. This proves that $\pi(W)$ is a smooth manifold and that π is a submersion from WH onto $\pi(W)$. Using the G-action on X the rest of the lemma is clear.

To finish the proof of the theorem it is enough to construct such a W. Let $\mathfrak{h} = \mathrm{Lie}(H)$ and let $\mathcal{L}$ be the involutive distribution on G defined by $\mathfrak{h}$. The maximal leaves of $\mathcal{L}$ are H and the cosets gH. Let $(U, (x^i), a > 0)$ be adapted to $\mathcal{L}$ with $a \in U$ and $x^i(1) = 0$. For $0 < b \leq a$ write U_b for the preimage of the cube T_b^m under the map $y \longmapsto (x^1(y), \ldots, x^m(y))$. Since H is closed we can find an open subset T of G such that $U[1] = T \cap H$ and hence we can find a_1 such that $0 < a_1 \leq a$ and $U_{a_1} \cap H = U_{a_1}[1]$.

Lemma 2. *We can find b with $0 < b \leq a_1$ such that for all $y \in U_b, U_b \cap yH = U_b[y]$. In particular $W = \{y \in U_b \big| x^1(y) = \ldots = x^p(y) = 0\}$ has the properties described in Lemma 1.*

Proof. Select $a_2, 0 < a_2 < a_1$ such that $U_{a_2}U_{a_2} \subset U_{a_1}$ and $a_3, 0 < a_3 < a_2$ such that $U_{a_3}^{-1}U_{a_3} \subset U_{a_2}$. We claim that $b = a_3$ has the required property. Let $y \in U_b$. Since yH is the maximal leaf containing y it is clear that $U_b[y] \subset yH$ and so we need only show that $U_b \cap yH \subset U_b[y]$. Now $U_{a_2} \cap H = U_{a_2}[1]$ and so

$$U_{a_3} \cap yH = y(y^{-1}U_{a_3} \cap H) \subset y(U_{a_2} \cap H) = yU_{a_2}[1].$$

Now, $yU_{a_2}[1]$ is a leaf contained in $U_{a_2}U_{a_2} \subset U_{a_1}$ and so $x^j (p+1 \leq j \leq m)$ are constant on it, showing that they are also constant on $U_{a_3} \cap yH$. Hence $U_{a_3} \cap yH \subset U_{a_3}[y]$. This proves the lemma.

The proof of the theorem is now clear. From the proof we have the following corollary which asserts that the fibration of G over G/H as a principal H-bundle is locally trivial.

Corollary 3. *If H is a closed Lie subgroup of G, there is a connected open neighborhood S of $\pi(1)$ and a map $\gamma : S \longrightarrow G$ such that*

a. $\gamma(\pi(1)) = 1, \qquad \pi \circ \gamma = \mathrm{id}$

b. $\gamma(S)H$ *is an open subset of* G *and the map*

$$\psi : S \times H \longrightarrow \gamma(S)H, \qquad \psi(s, \xi) = \gamma(s)\xi$$

is a diffeomorphism commuting with right translations by elements of H.