Natural and Artificial in the Language of the Malayalam Text

Yuktibhāṣā

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I. INTRODUCTION

In 1832, Charles M. Whish, an Englishman who had worked for the East India Company, presented to the Royal Asiatic Society of Great Britain and Ireland an account[1] of the contents of four palm leaf manuscripts which he had found in the environs of Cochin (or Kochi as it is now known) in coastal central Kerala. One of the four is a work in Malayalam, the language of Kerala, on mathematics and astronomy entitled *Yuktibhāṣā* (abbreviated to *YB* from now on; the others are also mathematical/astronomical texts, by now similarly well-known, in Sanskrit). Thus was brought to the notice of scholarly Europe, perhaps for the first time, the text of which I speak here. Whish's effort seems to have made no impression on historians of mathematics till the 1940s. But the decades since then have seen increasing attention being paid[2] to the quite astonishingly sophisticated mathematics produced by the socalled Kerala school in a relatively brief period of 200 years or so, beginning in the second half of the 14th century CE.

The text of *YB* that I use here is the one that virtually everyone interested in the work relies on directly or indirectly, that given in the annotated Malayalam edition of Rama Varma (Maru) Tampuran and A. R. Akhilesvara Ayyar, published in 1948[3]. It is based on four separate manuscripts which are largely in concordance. Tampuran belonged to a local royal family and was a well-known scholar of Malayalam (and hence of Sanskrit) and of the *śāstra*s, and Ayyar was a schoolteacher with a master's degree in mathematics; in the complementarity of their domains of expertise, a more ideal pair of collaborators is hard to imagine. The main part of the book is 290 printed pages long of which roughly half is commentary. The editing and the commentary are impeccably done, in the best Indian tradition of *bhāṣya*s. Especially noteworthy are the meticulously drawn geometric figures without which the work of those who followed them would have been made greatly more arduous.

The edition of Tampuran and Ayyar covers only Part I of *YB*. Part II contains applications to astronomy and has so far not been published, in any language. The late K. V. Sarma, whose efforts more than of anyone else brought the main texts of Kerala mathematics and astronomy to the attention of the scholarly world, had completed (in association with M. D. Srinivas, M. S. Sriram and K. Ramasubramanian) an English traslation of both parts of *YB* at the time of his death in January 2005, but it is yet to appear. As of now, a faithful presentation of what *YB* contains, in any language other than Malayalam (unfortunately beyond the acquaintance of most interested scholars) does not exist.

II. THE BACKGROUND

Till about 700 CE or so, Kerala formed part of the Tamil linguistic landscape. The rich cultural, especially literary, life of those days is well documented and we also have some knowledge of the polity. There is however no serious evidence of any astronomical or mathematical activity at that time in Tamil Kerala nor indeed in the Tamil country as a whole. Very plausibly, the taste for astronomy and mathematics travelled to Kerala with a series of waves of migration from further north, generally along the coast, of vedic brahmins. Beginning in the 7th – 8th century CE, this influx continued for half a millennium or longer. The early settlers bestowed Aryan legitimacy on local potentates who built temples for them in return, with extensive land grants for the maintenace both of the temples and, in some comfort, of the temple managers. But these brahmins, who came to be known generically as namputiris, brought with them a great deal more than vedic expertise; they were, at least some of them, a learned people, the repositories and practitioners, creators and transmitters of all the classical sciences. Thus it is that they carried to Kerala a strong heritage in mathematics and astronomy, most notably the legacy of Aryabhata. They also brought with them their language of learning, Sanskrit.

But largely independent of northern influences, the old Tamil language of the region had already begun to take on a distinct local personality. On top of this came the impact of Sanskrit, resulting in a rich and versatile new language, Malayalam. By the time of the composition of *YB*, Malayalam had settled down to its definitive form, hardly different from what it is today. Nevertheless, with the exception of *YB* and a few minor texts, Sanskrit continued as the language of mathematical scholarship (and writing) until its final decline and disappearance, as sudden as was its beginning[4].

A number of literary references and inscriptions from this period of transition testify to the existence all over Kerala of educational institutions, ranging from the veda schools attached to temples to veritable quasi-autonomous colleges. Over the first half of the second millennium CE, these schools, together with the academies that flourished under royal patronage, were an integral part of the intellectual vibrancy of the time. It is in this setting that almost all of the literally thousands of scientific manuscripts now lying scattered in private and public collections came to be composed[5].

The earliest known mathematical work which can be unambiguously attributed to Kerala actually predates this golden age. This is a Sanskrit verse commentary by Shankaranarayanan[6] on the first Bhaskara's *Laghubhāskarīyam*. According to the text itself, Shankaranarayanan was the royal astronomer to the Cera emperor Sthanu Ravi Varma – the Ceras had lately consolidated their control over most of Kerala – working in the observatory in the capital Mahodayapuram (the famed Muziris or Musiris, emporium to the world; modern Kodungallur). The text gives the date of composition as 869 CE and refers to the author as hailing from Kollapuri in Paidhyarashtra (Pratishthana or Paithan) which has been identified (see note [5]) very persuasively with modern Kolhapur in Maharashtra. This is rich and rare fare: in one text, we have a precise date and place of composition and a precise provenance for the author or his immediate ancestors.

As far as we know, the emergence of astronomical scholarship in Kerala with Shankaranarayanan was an isolated event. His intellectual ancestry is unknown and no one seems to have carried on after him; in fact there is no sign of any serious mathematical and astronomical activity in Kerala in the next five hundred years. To an extent, this can be blamed on the war and turmoil that visited Kerala from the Tamil country at the turn of the first millennium. By the end of the 11th century, the prosperous Cera kingdom was gone forever, its capital razed, the royal observatory only a memory.

III. MADHAVAN AND HIS FOLLOWERS

In the chaotic aftermath of this '100 year war', the Cera kingdom broke up into a number of principalities, of which the most powerful was that ruled by the Zamorin (samutiri) dynasty from Calicut. This period also saw a renewed influx of brahmins and their rise to a position of social and cultural preeminence. The southern part of the Zamorin's domain became a particularly favoured area for namputiri settlements. It is in this region, on either bank of the river Nila (more commonly known now as the river Bharata), that there emerged, beginning in the second half of the 14th century, a succession of astronomers and mathematicians of truly exceptional quality. They formed a tight-knit group, all linked together by the traditional teacher-disciple bonds. There was at least one father-son pair among them, Paramesvaran and Damodaran. Most were namputiris of one variey or another. According to local tradition, three were native to one village and two taught at the same temple school. Virtually all lived in close geographic proximity and produced their work within a time span of roughly two centuries.

The most original and creative of this brilliant lot, by the acclamation of those who followed him, was also the first: Madhavan, said to be of Sangamagramam[7]. Madhavan has remained a somewhat shadowy figure. We know nothing about his life except that it spanned the second half of the 14th century and the first quarter of the 15th and that the great astronomer Paramesvaran was his disciple. The few manuscript fragments that are attributed to him are primarily astronomical, having little mathematical content of any originality, and it is through the writings of his intellectual heirs that his achievements are known to us. Paramesvaran's student Nilakanthan, in particular, quotes mathematical verses attributed to Madhavan quite liberally in his Sanskrit work *Tantrasaṃgraha*, making it clear that it is meant to be a compendium of his teaching. The overwhelming significance of *YB* lies in the fact that it contains, in chapters 6 and 7, the most thorough exposition we have of the new mathematics that Madhavan created and *Tantrasaṃgraha* summarised. If only to place *YB* itself in context, it is useful to touch on the highlights of Madhavan's legacy.

Since before the time of Aryabhata, the one abiding theme of Indian mathematics was the circle, more particularly the relationship between an arc and the corresponding chord, and a variety of questions linked to it. This was entirely in line with the uses to which mathematical reasoning was put in India, first and foremost for the study of the geometry of the celestial sphere and the motion of heavenly bodies in their epicyclical orbits. Aryabhata himself famously recognised the difficulty of giving an exact number for the ratio of the circumference of a circle to its diameter, qualifying his value 3.1416 for π as approximate, *āsanna*. Fundamentally, what Madhavan did was to push the quest for precise values for such ratios to its logical and mathematical limit. (That π was understood by this time to be an irrational number is clear from a well-known passage from Nilakanthan's commentary on the *Āryabhaṭīya*, see below). In the process, he was led, at a technical level, to express such ratios (trigonometric functions) as appropriate power series, infinite series of smaller and smaller terms with none exactly equal to zero. (We know them as the 'Gregory-Leibniz' series for the arctangent and the 'Newton' series for the sine and the cosine, the quotation marks indicating a compromise between current usage and historical accuracy. I shall write these series down as the occasion arises). Madhavan also did some wonderful and practical things with these series: techniques for accelerating their convergence; accurate estimates of the remainder as rational functions when a series is truncated; interpolation formulae for values of trigonometric functions at a point in terms of the values at a neighbouring point; and so on. Most of this is described in detail in *YB*, the interpolation formulae being a notable exception.

But more profoundly, Madhavan arrived at a general method, a philosophy almost, of addressing such questions by passing to the infinitesimally small and then summing the infinite number of the resulting infinitesimal contributions to the relevant geometrical quantity. Stated simply, Madhavan invented calculus, as it applies to circular arcs. Subject only to the limitation to circular arcs, *YB* conveys clearly that the key conceptual step in this was the recognition that local approximation by linear functions (tangents), in other words differentiation, and their subsequent summing up, integration, are converse processes – in essence, the earliest version of what came to be known in its sharp subsequent form as the fundamental theorem of calculus. What is most striking in all this to the present day reader brought up in the mathematical culture of the 19th and 20th centuries is the easy mastery with which the supposed twin demons of the infinite and the infinitesimal are simultaneously tamed.

IV. ABOUT *YB*

K. V. Sarma's date for the composition of *YB*, around 1550-1560 CE, seems now to be generally accepted, as also his attribution of the authorship to Jyeshthadevan[8]. We know a little more about Jyeshthadevan than about Madhavan: he was a disciple of Nilakanthan and Damodaran, both disciples of Madhavan's disciple Paramesvaran and, like Paramesvaran, was connected with the Rama temple (or to a school attached to it) in the village of Alattiyur, just north of the river Nila not far from its mouth. And, most importantly, we have a whole book, extensive and intact, attributed to him[9]. Alattiyur still has memories of his name and of its past mathematical-astronomical glories, but astrology has long since supplanted astronomy as its chief source of pride.

In the context of Indian mathematical writing, *YB* is singular in several respects: i) it is not in Sanskrit verse but in Malayalam (*bhāṣā*) prose, ii) it provides detailed lines of reasoning (*yukti*) and, as a consequence, iii) it is inordinately long. It is also very 'theoretical'; there are few worked examples or numerical illustrations (*udāharaṇams*) unlike say in the work of the second Bhaskara and, despite the strongly geometric methods of proof, no diagrams at all. (Tampuran and Ayyar have included a profusion of them in their commentary and these have been freely and gratefully duplicated in the subsequent literature). It is relevant to ask why it is three-fold singular in this particular way. Given our ignorance of the circumstances of its writing, the answers can only be tentative at best. But, reading the text, it is difficult to escape the conviction that, unlike the traditionally favoured format of *sūtra*s, mnemonic verses serving as an *aide-mémoire* with the reasoning being explained in face-to-face sessions, *YB* was meant to be autonomous, written down from the guru's mouth, to be read, struggled with and, hopefully, mastered in course of time, away from the guru and the classroom. The traditional compact verse format in mathematical writing had at least one function other than that of facilitating (primarily oral) communication and text-preservation; concise expressions are commonly made to stand in for numbers, formulae and even whole collections of ideas and methods, 'packages' to be unzipped in the mind. *YB* does have such packages scattered throughout, of its own and in the form of quoted verses from other sources, but on the whole stands at the opposite extreme. The entire text is in a natural, too natural perhaps, Malayalam, explicit and even verbose, with the Sanskrit technical terms embedded in this matrix. The style is colloquial and often repetitive, especially in its use of connecting words common in conversation but not serving any functional (let alone mathematical) purpose. Altogether, its writing was a startlingly original endeavour. One cannot escape the feeling of reading the faithful notes of a masterly but informal set of lectures – face-to-face presentations – but without the easy present day option of an escape to equations and diagrams[10].

YB claims no originality of content; the first sentence is: "In order to explain[11] all the mathematics useful for the motion of heavenly bodies (*graha*s), according to *Tantrasaṃgraha*[12], I begin by describing the general (or common) mathematical operations such as addition and so on". Three short chapters deal routinely with these basic operations. An even shorter (considering the heavy use made later of the properties of similar triangles) chapter 4 concerns the 'rule of three' (*trairāśikam*). A substantial chapter 5 (together with a long appendix, explicitly stated to be taken over from *Tantrasaṃgraha*, with several numerical examples) is devoted to the method of 'pulverisation' (*kuṭṭākāram*) for the Diophantine solution of an equation of the first degree in two unknowns. This will find its use in Part II of *YB*. As already noted, the meat of the work is in the last two chapters, 6 (the relationship between the circumference and the diameter) and 7 (the theory of chords). Including the commentary, they run to over 200 printed pages.

The inevitable question that arises is how effective the use of an almost totally conversational Malayalam is in conveying the subtle and complex reasoning that this new knowledge is based on and, more generally, what obstacles the reliance on such a natural style of thought and communication places in the way of development of what is after all a highly abstract undertaking. To enable readers to judge for themselves how well these challenges are met, I now present in translation two passages from *YB*, both relating to calculus, before expressing my own views. The first describes the computation of the surface area of a sphere (from chapter 7) and the second is the section in which the integrals of the positive integral powers of a variable are worked out (in chapter 6).

The translation is as literal as I can sensibly make it. The only liberties I have taken are to provide basic punctuation and paragraph breaks, to drop the too-frequent conversational connectives, to supply personal pronouns where English requires them and to add a few brief explanations (not present in the original) or comments [within square brackets]. In particular, I provide no diagrammatic guide to the reasoning (the basic figure which I use to define notation for my own explanation of the passage on integration of powers should not count) – those so inclined will find it instructive, at least as a start, to supply their own figures before looking up Tampuran and Ayyar[3] or Sarasvati Amma[2]. The first passage requires only a minor transformation to make it 'modern'; Sarasvati Amma's version is largely faithful. The second passage holds a few points of mathematical interest which it will be worthwhile to return to later. It is also the part of Kerala calculus that today's scholars, unlike Tampuran and Ayyar, sometimes tend to handle with less than complete fidelity, lapsing occasionally into a reliance on later western developments like the binomial theorem for negative exponents, theorems on limits and their interchangeability with infinite sums etc., in place of what is actually in *YB*. For this reason, in the introductory remarks on this passage and in the few explanations within it, I have made an effort to guard against my use of current notation distorting in any way the material I am trying to convey.

V. THE SURFACE AREA OF A SPHERE

The section headed *golapṛṣṭhakṣetraphalānayanam* gives the proof of the formula: surface area of a sphere = diameter × circumference [of a great circle]. It occurs at the very end of chapter 7 (with only the calculation of the volume of the sphere to follow) though, with the exception of the use of one 'package', the work required is much less demanding than in the proofs of the various trigonometric series earlier in the book. It is probably the case that Madhavan invented the infinitesimal method first for the determination of π via the 'Gregory-Leibniz' series and then found it to be a powerful general tool for settling several other interesting problems[13] (also see below, section I).

Here is the translation.

> Now I narrate that, combining two principles just explained, [namely that] from *piṇḍajyāyogam* can be produced *khaṇḍāntarayogam* and [that] knowing the diameter at one place, we can apply the rule of three (do a *trairāśikam*) as we please, the area of the surface of a sphere will arise.
>
> A uniformly rounded object is called a sphere (*golam*). Through the middle of such a sphere, imagine two circles, one along east-west and the other along south-north. Then imagine circles, one shifted slightly to the south and the other slightly to the north of the east-west circle [the equator]. Their distances from the east-west circle should be the same for all parts (*avayavam*) [the word *samāntaram* for parallel is not used in this section though it is, for straight lines, elsewhere]. Consequently, these two will be slightly smaller than the first (or original) one. Then, starting from these, imagine slightly smaller and smaller circles, all of them at equal distance one from another [i.e., between successive latitudes], so as to end at the south and north edges [the poles]. Their separation along the south-north circle must be seen to be equal. This being so, imagine that the circle-shaped gap between two circles [successive latitudes] is cut at one place, removed and straightened (or spread). Then, of the circles on the two sides of the gap, the larger one will be the base (*bhūmi*) and the smaller one the opposite side (*mukham*) of a trapezium (*samalaṃbacaturaśram*) whose lateral sides (*pārśvabhuja*) will be the separation (*antarālam*) along the south-north circle of [two successive latitude] circles. Now cut out the part outside

the altitude (*laṃbam*) [from an upper vertex to the base], turn it upside down and transfer it to the other side [the opposite edge]. This is a rectangle whose length is half the sum of the base and the oposite edge and whose width is the altitude. In this way, think of all the gaps (*antarālam*) [elsewhere a non-Sanskrit Malayalam word is also used] as rectangles (*āyatacaturaśram*). Their widths are all equal. Lengths have various (or varying) measures (*pramāṇam*). The result of multiplying the length and the width is the area (*kṣetraphalam*). The widths of all being equal, add the lengths of all and multiply by the width. Thus will arise the area of the surface of the sphere. [Nothing so far about the passage to the limit].

Next, the method (*upāyam*) to know how many gaps there are and what their lengths and width are. The radii of the circles that we have imagined above [the latitudes] are half-chords (*ardhajyā*) of a circle whose radius is the radius of the sphere. Hence, multiplying these half-chords by the circumference of the sphere and dividing by the radius of the sphere [i.e., multiplying by 2π] will result in circles [their circumferences to be precise] having the half-chords as radii. These will be the lengths of the rectangles if the chords are taken at the midpoints of the gaps. Multiplying the sum of all half-chords (*ardhajyāyogam*) [by 2π, left unsaid] will result in the total length of all the figures (*kṣetrāyāmam*) [i.e., the sum of the lengths of all the rectangles]. The gap between two arbitrary [the Malayalam phrase is *yāva cilava* = *yāvat tāvat*] circles [an arbitrarily chosen pair of consecutive latitudes is meant] at the south-north circle mentioned earlier is an arc segment (*cāpakhaṇḍam*) of the circumference of the sphere [i.e., of the S-N circle].

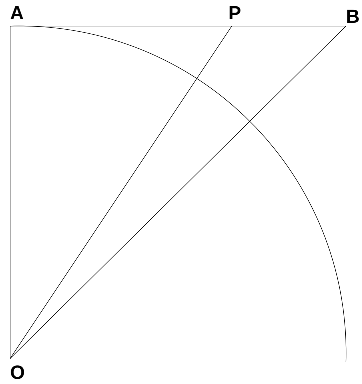
Next, the method of getting the sum of the chords (*jyāyogam*). Multiply the square of the radius by *khandāntarayogam* [I make no attempt at an exact translation of this 'package'. The expression stands for the sum of the second differentials of the half-chords with respect to the arc (or, equivalently, the angle) and has already been treated earlier in the chapter in connection with the series expansions of sine and cosine. The result about to be used below is also derived there.] and divide by the square of the full chord of the arc segment (*cāpakhaṇḍasamastajyāvarggam*). The result is the sum of the half-chords. Multiply this by the width. The width is the chord of the arc segment. The *khaṇḍāntarayogam* is the first chord segment. Because of smallness [of the arc segment] these [the first chord segment and the width] are [both] almost equal to the full chord. These are the multipliers and the square of the full chord is the divisor. But multiplication and division are unnecessary [the square of the full chord cancels from the numerator and denominator]. What remains is the radius. This has to be multiplied by the circumference and divided by the radius. Only the radius will survive. Since we have to get the result for both halves of the sphere, double the radius. Therefore, multiplying the diameter of the sphere by the circumference of the sphere will produce the area of the surface of the sphere.

We have here a straightforward narration in natural spoken Malayalam, understandable literally by any literate person but for the technical terms. By and large, the technical nomenclature is that used in mainstream mathematical writing in Sanskrit over many centuries and is consequently unambiguous. But these linguistic resources are now asked to be at the service of a sophisticated novel mode of reasoning demanded by Madhavan's infinitesimal mathematics, a service they were not designed to provide in full. Added to this burden is the yoke of a foreign tongue, Malayalam, which was not sung, at least mathematically speaking, at the cradle of the author. The strain shows. Even in translation it is evident that, while the first part of the proof (the geometric construction) is simply presented and easy to follow, the transition to the calculus part is marked by an increasing opacity. What the translation does not show is that the Malayalam itself is, apart from being very informally used, quite unrefined in relation to other literary material of the period.

VI. INTEGRALS OF POWERS

Chapter 6 of *YB* is devoted to establishing the validity of the 'Gregory-Leibniz' series, first for the angle $\pi/4$ and then for general angles, and to various improvements on it. Apart from the mathematical content itself, it is noteworthy as the first illustration of the power of the new infinitesimal geometry when combined with the equally new technique of integration or 'summation in the limit', *saṃkalitam*.

The geometry considers a circle of radius r and a square of side $2r$ in which the circle is inscribed, focussing on one octant of the circle as in the diagram below:



The side AB is divided into n equal segments of length d (P is a typical point of this division), with the limit $d \to 0, n \to \infty$ (with $nd = r$ fixed) to be taken subsequently. After a series of steps including the usual clever choices of similar triangles (*trairāśikam*), some simple algebraic identities, a process of iteration and a judicious neglect of terms of order $1/n^2$ (i.e., second order infinitesimals in the limit), the geometry is shown to lead to the result

$$\text{arclength of the octant} = \frac{\pi}{4}r = r \lim \sum_{j=1}^{n} (\frac{1}{n} - \frac{j^2}{n^3} + \frac{j^4}{n^5} - ...),$$

lim meaning the limit described above. Each term within the bracket (including the sum over j and in the limit) is referred to as a *saṃkalitam* which I shall translate as an integral, in anticipation[14]. In addition to the even *saṃkalitam*s contributing to the arc length, *YB* considers also the odd *saṃkalitam*s since the induction procedure requires their evaluation. Thus, ignoring the minus signs, the kth *saṃkalitam* is defined to be

$$I_k := \lim \sum_{j=1}^{n} \frac{rj^k}{n^{k+1}}, \qquad k = 0, 1, 2, ...$$

The introductory paragraph of the section headed *saṃkalitam*s is:

> Here I describe the method of producing the integrals. First the simple integral (*kevalasaṃkalitam*) is described. Then the integral of two equals multiplied together. Then, even though it is not useful here, I describe also integrals of equals multiplied by themselves three, five, etc. times, since they occur in the midst of those which are useful [namely, the even powers].

The term *kevalasaṃkalitam* does not reappear in the section. In this passage it evidently means the integral of the first power, which is referred to in the rest of the section as *mūlasaṃkalitam*. The distinction is with the sequence of repeated k-fold integrals of x (*ādyadvitīyādisaṃkalitam*s); i.e., in modern notation,

$$\int dx \int dx ... \int dx = \frac{x^{k+1}}{(k+1)!}$$

which are taken up further on to generate the factorial denominators in the 'Newton' series. With this potential confusion out of the way, here is the section entitled *mūlasaṃkalitam* on the computation of the integral

$$I_1 = \lim(\frac{r}{n^2} + \frac{2r}{n^2} + ... + \frac{(n-1)r}{n^2} + \frac{r}{n}) = \lim \frac{d}{r}(d + 2d + ... + (n-1)d + r),$$

the second form being the expression with which YB actually works.

> In *mūlasaṃkalitam*, the last side (*bhujā*) [a *bhujā* is the side AP of the right triangle OAP, OP being the corresponding *karṇṇam*; so the last *bhujā* means AB] is equal to the radius of the circle, the one below (or before) that is one segment (*khaṇḍam*) less and the one before that two segments less. Suppose all the sides are equal to the radius. In that case, if the radius is multiplied by the number of sides [i.e., n], that will be the result of the *saṃkalitam*. But here only one side is equal to the radius. Starting from this, the sides of the other smaller and smaller diagonals (*karṇṇam*s) are, in order, one unit (*saṃkhya*) at a time less. Whatever is the number of units the radius is supposed to have, imagine that the number of segments of the side [here it means the full side AB] is the same. That makes it easy to remember. The last but one side will be one unit less. The next shorter one will be two less than the number of units of the radius. The missing part (*aṃśam*), starting with one, will increase one [unit] by one [unit] progressively, the last deficit (*ūnāṃśam*) almost equal to the radius, just one unit less. Now if the deficits are all added, this number (*saṃkhya*) will eventually equal the sum [the word used for this *finite* sum is *saṃkalitam*, as is traditional, see note [14]] of terms starting with one, increasing by one and ending with the radius, less one radius [i.e., it is the original sum minus the radius]. Therefore, multiply the number of units in the radius by one added to

> the number of sides [these two numbers are the same by assumption]. Its half is the *bhujāsaṃkalitam*. *Bhujāsaṃkalitam* means the sum of the sides of all the diagonals.
>
> The smaller the segment, the more accurate (*sūkṣmam*) the result. Therefore imagine that each segment (*bhujākhaṇdam*) [the text has *bhujāsaṃkhya*, which is probably a result of sloppiness somewhere in the transcription] is divided (cut) into atoms (*aṇu*) and then carry out the *saṃkalitam*. For this, if the division is by *parārddham* [any very large number will do; here, specifically, it is 10^{17}], add one to the product of the *bhujā* and *parārddham*, multiply by the radius and halve it. Also, divide by *parārddham*. This is approximately half the square of the radius. To make it a whole number, divide by *parārddham*. Thus, to the extent that the segment is short, only a small part has to be added to the *bhujā* to produce the *saṃkalitam*. [I understand this sentence to mean that the correction to the integral due to the finite length of the segment can be neglected in the limit]. Therefore, adding nothing to the *bhujā*, multiplying it by the radius and halving it will result in the *saṃkalitam* of the extremely finely segmented *bhujā*. This is how half the square of the radius is the accurate *bhujākhaṇḍasaṃkalitam*.

The terminological imprecisions in the passage (for instance the different senses in which the words *bhujā*, *saṃkhya* and even *saṃkalitam* itself are employed) are evident as is the increasing lack of clarity (the *parārddham* business for example), once again, of the writing when it comes to describing the reasoning about the passage to the limit.

After this account of the computation of the integral of x, the next subsection deals with x^2 in some detail. The passage from x to x^2 involves conceptual and technical novelties which then carry over smoothly to the general inductive step, from x^k to x^{k+1}. Here is how the general method of computing the integral of an arbitrary positive power (*saṃkalitānayanasāmānyannyāyam*) is summarised:

> ....To make integrals of higher and higher powers, multiply the particular integral by the radius and remove from it the result divided by the number which is one greater [$I_{k+1} = xI_k - xI_k/(k+1)$ in our notation]. Thus divide the square of the radius by two, the cube by three, the fourth power by four, the fifth power by

five. Thus divide the consecutive powers starting with the first (*ekaikottarasamaghātam*) by [the same] consecutive numbers; the results will be the integrals of powers in increasing order. As the simple integral comes from the square, the integral of the square from the cube, the integral of the cube from the fourth power and so on, the power of an unknown (*rāśi*) when divided by the same number as the power will give the integral of one less power of the unknown. This is the method of producing integrals of all powers.

There are several mathematically interesting reasons for reproducing this brief paragraph, to which I will return shortly. But for the present, let us overlook the familiar prolixity (perhaps it was a class of dull students) and note the clarity with which the theorem itself is stated.

VII. THE LANGUAGE

It bears repetition that, as in the excerpted passages, so in the entire text, no symbols are employed to represent the mathematical objects being manipulated, no formal notation for relations among them and operations on them, no diagrammatic guide to the geometric constructions invoked. The only devices which can be considered as perhaps artificial are, first, the use of common natural (generally Sanskrit-origin) words to denote precisely understood geometric entities: examples include *cāpam, jyā, śaram, bhūmi, bhujā, mukham*, etc; several of these and other similar terms, when translated, have exactly the same significance in European geometry. (The use of natural words as units of an artificial language appropriate for a given discipline is of course a widespread and contemporary practice even in extremely abstract contexts, see the current literature in mathematics and physics). Then there are the phrases whose sense is generally clear from the way they are formed as prescribed in the usual rules of compounding in Sanskrit (and Malayalam), e.g., *samalambacaturaśram*, *vyāsārddhavarggam*, etc. – they have no extra or hidden meaning apart from the literal. Finally, beyond these, there occur certain expressions whose literal meanings are inadequate to convey exactly what they represent or may even be misleading. The most widespread instance of this in *YB* is *trairāśikam*, 'the rule of three', encompassing all properties of a set of four numbers in proportion, and hence all properties of a pair of similar triangles. In the passage on the area of the sphere, the term *piṇḍajyāyogam* is also such a 'package', standing for a certain refined procedure which effectively computes an integral: it means the sum of semichords of arcs of equal length into which a quadrant of a circle is divided, *in the limit of vanishing arc length*. The expression *khaṇḍāntarayogam* stands for an even more subtle procedure – *bhujākhaṇḍam* corresponds to the differential and [*bhujā*]*khaṇḍāntaram* to the second differential of the sine of an angle. (Thus the introductory sentence of the passage on the surface area of the sphere is equivalent to the statement that the second differential of sine is proportional to itself). Probably the best known instance in Kerala mathematics of such esoteric usage is the phrase *jīveparasparam* attributed (in *Tantrasaṃgraha*) to Madhavan and representing the formulae for the sines of the sum and difference of two angles (together with the proof). These are used in *YB* as part of the preparation for the construction of the 'Newton' series for sine and cosine. Unwrapping such 'packages' is essential for the understanding of the *yukti*, but they circumscribe the maximum deviation from the natural use of language that can be found in *YB*.

One other point about the language, or rather the general style of discourse, merits mention. The second section of chapter 7 of *YB* has the heading 'technical terms and definitions'. A demanding reader looking here for precise characterisations of the concepts and terminology necessary for the formulation and proof of the great theorem on the 'Newton' series to follow is likely to be disappointed. The section is neither systematic nor complete, but only a partial account of some of the geometric constructions and the associated nomenclature needed for the proof. In all of *YB*, the terminology itself does not have an invariable significance. To give just one instance, the crucially important word *saṃkalitam* is used to mean both a finite sum as in earlier writing[14] and, in chapter 6, integrals of several different types. In the same chapter 6, the term *yogam* never means an integral, but only a finite sum whereas in chapter 7 we have, for example, *arddhajyāyogam* denoting the finite sum as well as the limiting integral. On the other hand, *saṃkalitam* is never used for the integrals occurring in surface and volume computations and, elsewhere in chapter 7, *khaṇḍāntarayogam* and *khaṇḍāntarasaṃkalitam* are distinguished.

Such instances can be multiplied. The question is: Is this lack of precision related in some intrinsic way to a reluctance or an inability to transcend the limitations of natural means of communication? Following from this: To what extent has the use of a dominantly natural language impeded the processes of conceptualising and developing the mathematics and then communicating it? The answer to the first question requires a thorough acquaintance with texts contemporary with and anterior to *YB* (all written in the not very natural language of terse mathematical Sanskrit) and is not for me to attempt. As for how well the mathematics was conveyed, if Tampuran and Ayyar could read the work four centuries after it was written and still make perfectly good sense of it – though they confess to having sometimes to guess at the author's train of thought – Jyeshthadevan clearly has not made too bad a job of it.

VIII. THE MATHEMATICS

What, now, are the mathematical ideas and methods, not just the unexpectedly novel results and the clever computations, that are conveyed to us by the imperfect vehicle that is the natural Malayalam of *YB*? The order in which the various series occur in *YB* – first the numerical series for $\pi/4$, then the 'Gregory' series for general angle and then, in a veritable *tour de force*, the two 'Newton' series for sine and cosine – would seem not to be a matter of chance. It is a logical way to proceed if we accept that the motivation behind the search for a series of ever smaller terms whose sum approaches, in the limit, the value $\pi/4$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ....$$

has probably to be sought in the conviction, going back to Aryabhata, that π is an irrational number and therefore cannot be written as the sum of a finite number of fractions. Nilakanthan, Madhavan's true mathematical heir and Jyeshthadevan's teacher, has this comment (in his *Āryabhṭīyabhāṣyam*) explaining why Aryabhata said the value $\pi = \frac{62832}{20000}$ was only *āsanna* and, along the way, defining an irrational number. In loose translation: "Why is an approximate value given here rather than the true (*vāstava*) one? Because it cannot be expressed. A measure which measures the diameter without a remainder cannot measure the circumference without a remainder and *vice versa*. We can only ensure the smallness of the remainder, not its absence". The lack of 'remainderlessness' is naturally accommodated by an infinite series expansion (though of course every infinite series of fractions does not sum up to an irrational number, as Nilakanthan well knew, *cf.* his remarks on convergent geometric series in the same text), in a manner not very different from the way in which infinite continued fractions arise in the use of the Euclidean algorithm.

As described in *YB*, the generalisation from an infinite series representation for the number $\pi/4$ to a power series expansion for an arbitrary angle $\theta(\theta \leq \pi/4)$ in terms of $\tan\theta$:

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - ...$$

involves, technically, only a small step, an elementary application of *trairāśikam*. It has no computational difficulty or significance and helps only marginally in determining a more accurate value for π – replacing $\pi/4$ by $\pi/6$ ($\tan\pi/6 = 1/\sqrt{3}$), for example, improves the convergence slightly. But conceptually it is a giant leap, leaving behind the prop of the irrationality question and towards a notion of what we call today a function. It can be credibly argued that it is this step and the series expansions of $\sin\theta$ and $\cos\theta$ in powers of θ that together mark the advent of calculus, not just as a technique of calculation as in the expansion of $\pi/4$, but as the beginning of a new discipline of analysis. The basic operations of calculus make their appearance as needed in the course of these developments: differentials of the first and second order (which are all that is required for the sine and cosine functions), definite and indefinite integrals, the rule for integration by parts (in computing recursively the integral of a general power), the notion of repeated integrals, the solution of an elementary differential equation, namely $d^2y/dx^2 + y = 0$ for the sine and cosine functions, etc. Though the motivating impulses may not have been the same, it is uncanny to see infinite series ('equations with an infinite number of terms' in Newton's language) playing such a decisive part alongside the basic concepts of differentials and integrals ('fluxions' and 'fluents') also in the early evolution of European calculus and analysis.

The assurance with which *YB* handles these power series should not really be surprising once we recognise their close affinity with the recursive methods used systematically in diverse areas of Indian thought, including earlier mathematical work. The most direct parallel is perhaps with the positional notation for, say, positive integers. In his talk at this Workshop on the positional 'language' for the writing of numbers, John Kadvany[15] highlighted the fact that the positional representation of any positive integer is no more than an abbreviation for a polynomial whose value it is, with coefficients from a finite set of non-negative integers (0 to 9 in the decimal base case) when the variable is fixed at a positive integer (= 10 in the decimal case) – as Kadvany stresses, what makes this representation possible and powerful is the fact that integers can be added and multiplied to get other integers. A set whose elements can be added and subtracted as well as multiplied (but not necessarily divided) subject to the usual rules of addition and multiplication is an example of a ring; the integers, both positive and negative, form a ring of a particularly simple kind. So do polynomials, in the present case in one variable, and this common abstract structure is the key to the positional notation. (When the variable is fixed at an integer, say 10, the uniqueness of the positional representation is assured only if the coefficients are restricted to be less than 10 and the usual carry over rules are imposed; an exact correspondence between integers and their positional representatives requires this to be factored in). Arbitrarily large integers are so representable by polynomials of arbitrarily large degree; there is no upper bound on the integers that can be positionally represented.

The transition from the positional representation of unboundedly large integers to the trigonometric power series now appears natural: substitute for the value of the base used (10 for decimal) the appropriate variable ($\tan\theta$ for 'Gregory', θ for 'Newton') and allow the coefficients to take values in certain determinate fractions, both positive and negative. The set of formal (i.e., ignoring possible nonconvergence) power series with coefficients which are fractions do form a ring just as polynomials do.

Related to this is the point that the coefficients of a power series obviously cannot all be enumerated. They can be specified by, and effectively only by, recursive rules in some variant or another. In *YB* this is done in a number of slightly different but equivalent ways. In the statement of the 'Gregory' series expansion, the first three coefficients are numerically given and then one is instructed to take the general coefficients as the reciprocals of consecutive odd integers with the signs alternating. For the 'Newton' series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ....$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...,$$

the recursive specification of the coefficients is more explicit. The coefficient s_{2k+1} of θ^{2k+1} in the sine series is to be computed as

$$s_{2k+1} = \frac{s_{2k-1}}{(2k)^2 + 2k}.$$

Together with the value $s_1 = 1$, also specified, repeated use of this recursion formula leads to

$$s_{2k+1} = \frac{s_{2k-1}}{2k(2k+1)} = \frac{s_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = ... = \frac{1}{1.2...(2k+1)}.$$

A similar rule is given for the cosine series. It should be noted that the use of recursion is not limited to just the statement of these formulae in an economical form. It is an organic part of the development; the formulae for the coefficients introduce themselves recursively, so to speak, in the very process of their computation. There is, moreover, a unity in the way the coefficients in all these series arise. In the 'Gregory' series they are definite integrals (from 0 to 1 or 0 to $\tan\theta$) of even powers and in the two 'Newton' series they are iterated integrals of even and odd powers respectively, with positive and negative signs alternating. It is accurate to say that the constructive proof of their correctness is itself recursive (though I have to add that the proof is one of the least transparent passages in all of *YB*).

Perhaps the best illustration from *YB* of the use of recursion to establish the validity of an infinite sequence of propositions is the short passage on the calculation of the general *saṃkalitam* given in translation earlier (section VI) together with the two sections of *YB* which precede it. The untranslated (for reasons of space) sections give a recursive proof for the correctness of the values of the integrals of powers. First I_2 is computed in terms of I_1, then I_3 in terms of I_2, both in detail. This is followed by the statement that I_4 can be computed from I_3 and, generally, I_{k+1} from I_k in the same way. It is perfectly obvious that the recursive method cannot work if all values of k, even and odd, are not considered together, even though the series for $\pi/4$ requires I_k to be determined only for even k. Once the first step in the recursion, I_2 from I_1, is done, generalisation is straighforward, involving as the common basic step a recognisable finite form of integration by parts. Indeed, a hint that the recursive nature of the computation was fully understood is available in the choice of the term *mūlasaṃkalitam* for I_1, the root *saṃkalitam* from which all others are generated.

How close does all this come to the method of mathematical induction as it is understood today? A modern formulation of the principle of induction, essentialy deductive, will go something like: The truth of an infinite sequence of propositions $P_k, k = 1, 2, \ldots$ is established if i) P_1 is true and if ii) P_k is true implies P_{k+1} is true for all k. Lacking the necessary symbolic aptitude, Madhavan and his followers could not possibly have expressed their computation-oriented reasoning in such abstract and elegant language. But for one not versed in logical niceties, it is difficult to see that their use of recursion, not only to generate an infinite sequence of true mathematical statements but also to prove them, is in any fundamental way different, once allowance is made for the general preference in India for constructive as opposed to deductive proofs.

From a modern perspective, there are of course other issues raised by *YB*'s handling of infinite series, chief among them the question of convergence. Not surprisingly, *YB* is not concerned in any serious way with this question. It is striking however that all the series occurring in *YB* are convergent for the considered range of values of the variable. In particular, the geometry of the derivation of the 'Gregory' series works as it is described only for angles in the first octant which, together with the fourth, fifth and eighth octants, is also the range in which the series converges.

IX. THE GENEROSITY OF LANGUAGE, THE POWER OF ABSTRACTION

Reading *YB* from today's vantage point, it is possible to imagine several directions along which the development of calculus as depicted in it could have been done with greater generality, without straining the abundant conceptual and computational resources it draws upon. To try to identify these roads not taken and to speculate on why they were not taken is, in spite of the risks, a temptation not easily resisted. I shall limit myself to one particular instance of a missed turn which, in my view, clearly brings out the possible role of the language in which the mathematics was 'done', as distinct from how it was communicated, in inhibiting the requisite degree of abstraction and generalisation.

But first a remark, not directly related to language, on an issue *YB* does not concern itself with, that of the irrationality of π. At first sight this is a surprising omission, given the central position occupied by the geometry of the circle in *YB*'s calculus, especially if, as I have suggested above, the infinite series expansions grew out of the search for a method of 'controlling' an irrational number like π. Also as we have seen, the person who asserted this irrationality was none other than Nilakanthan, Jyeshthadevan's teacher and the author of *Tantrasaṃgraha*, the source book for *YB*. One reason for the silence may be technical, that a *yukti* for the assertion was beyond the computational methods at the command of the Kerala school – after all, it was only in 1761 that in Europe Johann Heinrich Lambert proved Nilakanthan's conjecture (without of course knowing it as such). But a more likely explanation may lie in the philosophical underpinning of Indian science as a whole throughout its long history. Indian mathematicians, unlike the Greeks, seem never to have come to grips with proofs of the irrationality of any number, not of the far easier case of $\sqrt{2}$ nor of $\sqrt{10}$, often used as an approximation for π at the time of Aryabhata. Almost certainly, this failure has to do with the necessity of having to use *reductio ad absurdum* methods in any such putative proof. For the Greeks, proof by contradiction, involving as it does the principle of the excluded middle, was legitimised by the authority of Aristotle. But in India the view that P and not$P$ are complementary and mutually exclusive was explicitly rejected at least by some and from at least the time of the Buddha[16]. Subsequently, reasoning by contradiction (*tarka*) was much debated but generally not accorded full admissibility[17]. Like all scholars, mathematicians would have had a good grounding in other sciences as well (this certainly was the case in Kerala at the relevant time[18]), and it is almost inconceivable that they did not know of the reservations regarding recourse to *tarka* as a means of establishing a logical truth – *YB* contains not a single instance of it; elsewhere in India, it is not till a hundred years later that we come across its first hesitant and insecure use in mathematics[19]. The fact remains that this disdain denied Indian mathematics the use of a powerful proof device. Contrast this with Newton's free use of it in the *Principia* (Book 1) whose reliance on infinitesimal geometry (but with diagrams supplied), especially of similar triangles, is otherwise very reminiscent of the Kerala techniques.

The missed turn I wish to focus on concerns the generalisation of Madhavan's series expansion of $\sin\theta$ in powers of θ with numerical coefficients ('around $\theta = 0$') to an expansion around some nonzero value of θ, namely an expansion of $\sin(\phi + \theta)$ in powers of θ with coefficients depending on ϕ. This would amount to a generalisation of the Maclaurin series for the sine function (and the cosine function) to the corresponding Taylor series. Since such expansions of general (sufficiently 'good') functions f:

$$f(x + y) = f(x) + y\frac{df}{dx} + \frac{y^2}{2!}\frac{d^2f}{dx^2} + ...$$

are relatively early landmarks in European calculus, the question whether Madhavan's interpolation formula can be thought of as giving the first few terms of the Taylor series of the sine function has recently provoked some debate[20]. It may or may not be a satisfactory interpolation, but it certainly cannot be thought of as the beginning of the Taylor expansion – there is a numerical mismatch in the coefficient of the cubic term.

If I were time-transported to the 15th-16th century as a worthy member of the Kerala community of mathematicians I could have pointed out (in *YB*'s irresistible prose style) that, combining i) *jīveparasparam* (the formula for $\sin(\phi + \theta)$), ii) *jyānayanam* (the method of determining an arbitrary chord to arbitrary accuracy, i.e. Madhavan's sine and cosine series) and iii) *bhujākhaṇḍam* and *bhujākhaṇḍāntaram* (the first and second differentials of sine and cosine), the (correct) Taylor series for the sine function will result. In detail,

$$\begin{aligned}\sin(\phi+\theta) &= \sin\phi\cos\theta+\cos\phi\sin\theta \\ &= \sin\phi(1-\frac{\theta^2}{2!}+...)+\cos\phi(\theta-\frac{\theta^3}{3!}+...) \\ &= \sin\phi+\theta\cos\phi-\frac{\theta^2}{2!}\sin\phi-\frac{\theta^3}{3!}\cos\phi+...\end{aligned}$$

This is computationally trivial and, by truncation, produces an interpolation formula to any order in θ. Furthermore, *all* derivatives of the sine function are determined by *khaṇḍāntaram* (as also I would have pointed out):

$$\frac{d^{2n+1}\sin\phi}{d\phi^{2n+1}}=(-1)^n\cos\phi, \qquad \frac{d^{2n}\sin\phi}{d\phi^{2n}}=(-1)^n\sin\phi$$

and similarly of the cosine function. So Madhavan's expansion, when shifted from $\theta=0$ to some (any) other point on the circle, together with a realisation that, abstractly, a circle (a 'uniformly round' figure) does not distinguish one radius from another, is exactly the Taylor series.

At issue here is not whether calculus was invented in India. That question is already answered in the affirmative in the material available in *YB*: though in a very different mathematical language, it is unmistakeably the calculus of Newton and Leibniz, but applied *only to functions corresponding to arcs of the circle*, $y=(1-x^2)^{\frac{1}{2}}$ or, parametrically, $x=\cos\theta$, $y=\sin\theta$. What Kerala mathematics did not have was an appreciation of the great generality of the concepts and methods it had in hand and deployed so effectively; what was missed is the power of abstraction.

To fully appreciate the extent of this power, we only need to call to mind how mathematics (and, to a lesser extent, the other exact sciences) has evolved over the past few centuries, driven as it is by an increasing emphasis on the structural properties of the sets of objects being studied, to the disadvantage of predominantly calculational techniques. When mathematical objects belonging to a set are characterised by the operations that can be carried out on them and the rules the operations have to obey (the relations amomg them) we have, effectively, defined those objects in terms of the structure of the set. An object has those properties and only those that come from its belonging to a structurally defined set, bringing with it the freedom to transcend the particular circumstances in which it may first have presented itself. Thus we may wish to study the set of transformations that take a 3-dimensional cube into itself and may soon realise that the resultant of any two such symmetry transformations is also a symmetry transformation, their product, uniquely determined by the first two taken in order; that every symmetry can be undone by another, its inverse, also uniquely determined; etc., in short that it belongs to a class of sets called groups. Already at this stage we have gone through two levels of abstraction. At the first level, we have written down the multiplication table – the specification of the resultant of each ordered pair of symmetry transformations – for the group of symmetries of the cube. With this table in hand, we know the group fully: there is no proposition concerning such transformations, no matter how complicated or subtle, whose truth cannot be decided by the table alone, without reference to the cube. It follows that all groups of transformations of any 'physical' entity whatever, indeed all groups not even necessarily of transformations, are abstractly the same group if they have the same multiplication table. At the second, deeper, level, we would recognise that, though the group of symmetries of a cube and of, say, a regular tetrahedron are not the same, i.e., do not have the same multiplication table, not even the same number of elements, they still share a common abstract structure, the structure of a group. We would then proceed to elaborate the (essentially syntactic) properties that *all* groups must have; in short we would make a general theory of groups.

It is obvious that I could have made my point more concisely and elegantly, in fact more powerfully, if I had taken recourse to the language that developed to accommodate the abstract structural point of view, a language primarily algebraic: symbols for the various groups under consideration and for their elements, a notation for the group operations, an economical statement of the relations, etc., the language in which theorems/'truth' will be presented/expressed. Everyone now knows that group structures occur in virtually all sciences, that modern physics in particular has been revolutionised by a systematic exploitation of the formal, syntactic, understanding we have acquired of these structures. As for mathematics, and staying with groups, I only mention the recently completed classification of all finite simple groups (the recursive building blocks of all finite groups) as a general illustration of the generative power inherent in the artificial language of algebra. Without that as a vehicle of thought and communication, it seems inconceivable that we could have reduced mathematical reasoning and creativity to an application of "rules without meaning" so successfully.

Rules without meaning but not without purpose. Decisions about what abstract (syntactic) structures are likely to prove 'interesting' and even judgements about what results are 'deep' or 'beautiful' have still to be made semantically, by reference to the contexts in which questions arise and are resolved. The tension between problem-solving and theory-making, the virtual inseparability of content and language, is arguably the main source of the extarordinarily wide scope and power of mathematics as it is practised now (and is perhaps also the reason for the "unreasonable effectiveness of mathematics in the physical sciences"). Once the creative process of discovery is thus algebraised and put on autopilot as it were, there is no limit to what is discoverable – algebra is truly, infinitely, generous.

Moving back from these generalities to the Kerala of the 15th and 16th centuries, one cannot of course be dogmatic about the extent to which progress towards greater abstraction and hence greater generality was inhibited by the spurning of symbolic methods. But on the evidence of *YB*, it cannot be doubted that obvious general points of view were overlooked or neglected. As the episode of the Taylor series makes clear, an intuitive feeling for the symmetry of the circle (the corresponding group being the group of *all* rotations around the centre, which is not a finite group but still, structurally, a group) did not translate into the precise understanding that this symmetry denies a privileged role to any particular diameter, even if it is called the east-west direction. I like to think that if only they could have been persuaded to designate a diameter as the line AB or $\alpha\beta$ or a Malayalam equivalent thereof, the Kerala mathematicians might have gone on eventually to a more general, less concrete, appreciation of their own achievement.

What is puzzling in all this, at least to a non-expert, is that abstraction and symbolic representations of objects and their relationships were not strangers to Indian thought. Grammar (first and foremost), prosody and the classification of meters, cosmogonic speculation, systematisation of the rules of reasoning and several other areas of intellectual inquiry come to mind as having been greatly enriched by a fundamentally structural point of view. In mathematics, the positional notation for numbers is itself a triumph of structure over clumsy description; as we have seen, it is pure syntax. Bhaskara the second, who lived not more than three centuries before the efflorescence of mathematics in Kerala and whose work was known and cherished there, was a true algebraist; his *Bījagaṇita* proposes the use of the names of colours to symbolically represent variables in equations, as distinct from numerical values for them, and states the rules for manipulating them, exactly as if they were numbers. (In a strange echo, modern physics employs the term chromodynamics for the interactions of quarks of different 'colours'). Bhaskara's algebraic legacy seems to have been better nourished in the Arabic lands than among his own mathematical heirs in India. Perhaps, at the time and place we are talking about, mathematics was seen as no more than a handmaiden of astronomy, with no high intellectual standing of its own. But then, what else but intellectual curiosity can have inspired a Nilakanthan to cogitate on the irrationality of π or a Madhavan to compute it to thirteen decimal places?

Close to three hundred years after Madhavan, Newton and Leibniz laid the foundations of calculus as we know it, in the form and language that we follow, more or less faithfully, today. That came very soon after the first integrations were performed in Europe, by Cavalieri, Fermat and a host of others (perhaps only coincidentally, the integral of x^2). The context in which this happened was vastly different from the one in which the heroes of our narrative perfected their infinitesimal geometry of the circle. Descartes had already algebraised geometry[21] and both Newton and Leibniz were well versed in the Cartesian method and philosophy. The first ideas about a general function of a variable were taking shape and the first correspondences between functions and the simplest curves, the conic sections, beginning to be understood. The Newton-Leibniz calculus reflects this great opening up, especially in the willingness and ability to handle more general functions. (In the case of Newton, the Newton of the *Principia* in particular, a degree of generality could not have been avoided; the position of a mass-particle as a function of time is *a priori* unknown, being determined, in terms of a given force, only after the equations of motion are solved. In contrast, with the early exception of Aryabhata with his belief in a spinning earth and ideas on relative motion, Indian theories of the motion of celestial bodies were almost entirely phenomenological (in the modern physicist's usage of this term), paying little heed to possible underlying causes). It is in this fundamental respect, in the recognition that the infinitesimal method is of universal applicability, that the European calculus of the late 17th century – even while admitting that an acceptable foundation for it was not laid till the middle of the 19th – can be seen to have gone far beyond Madhavan's pathbreaking achievements. If history were different and granted them the time, could Madhavan's followers have taken their own road one day to this high ground? Going by what *YB* tells us, we must remain sceptical. For one thing, there is the much-discussed difference in the Europian and Indian approaches to doing mathematics, summarised in the catch words 'deductive' or 'axiomatic' and 'constructive' or 'computational'. Our look at (a tiny sample of) *YB*'s contents provides, I think, enough evidence that a more severe obstacle was the aversion to a spare and refined language and the consequent absence of a structural or general point of view.

But what history brought was the Portuguese incursion into Kerala, signalling another period of strife and disorder. The resistance to the invaders was led by the Zamorin who, in his other traditional role, was overlord and protector of the temples and patron of learning. The delta of the river Nila saw much violence and bloodshed during the very time in which, half a day's walk away, Nilakanthan, Jyeshthadevan and others were teaching and writing down their new mathematics in their temple villages. All the great texts date from this turbulent 16th century and with its end mathematics in Kerala also went into terminal decline.

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Notes

[1] "On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four *śāstras* Tantrasamgraha, Yuktibhasha, Caranapaddhati and Sadratnamala", published in the *Transactions of the Society* **3**(1835), p.509.

[2] As far as I can tell, Whish's baton was first picked up by K. Mukunda Marar and C. T. Rajagopal, "On the Hindu quadrature of the circle", *J. Royal Asiatic Soc.(Bombay branch)* **20**(1944), p.65. Of the numerous subsequent publications, two are worthy of special mention: T. A. Sarasvati Amma's thesis (Madras University) later published as *Geometry in Ancient and Medieval India* (1979; second revised edition 1999), Motilal Banarsidass, Delhi; and George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics* (1992), Penguin Books, London. The first is largely faithful to the methods actually employed by the Kerala mathematicians. Joseph's book has brought the story to a large readership. There are a fair number of good websites that discuss the work of the Kerala school and there are also some which are not fully reliable.

[3] Rama Varma (Maru) Tampuran and A. R. Akhilesvara Ayyar, *Yuktibhāṣā* (Part I - General Mathematics), with *vyākhyā* (1948), Mangalodayam Ltd., Trissivaperur.

[4] The question of *why YB* was composed in Malayalam is an intriguing one, not addressed

here.

[5] This extremely concise sketch of the medieval history of Kerala is an inadequate summary of the pioneering studies of Ilamkulam (or Elamkulam) P. N. Kunjan Pilla. Some aspects of his work have recently been questioned; despite this, his research provides the one coherent overall picture we have of the history of Kerala, especially of the time relevant to us. Ilamkulam wrote in Malayalam, but there is a volume of English translations of a selection of his essays: *Studies in Kerala History* (1970), National Book Stall, Kottayam. He has the added distinction of having been the first to seriously read Shankaranarayanan's commentary on the *Laghubhāskarīyam*, thereby establishing a first direct link between astronomical activities in Kerala and elsewhere in India.

[6] I have chosen to transcribe all names as they are conventionally written in Malayalam. Diacritical marks are supplied, generally, only for technical terms, also as they are written in Malayalam script in *YB* (though they are of Sanskrit origin). I have ignored the Malayalam distinction between the short and long *e* and *o* as well as the two *l* s.

[7] It is possible to make the case that Madhavan or his family was a recent arrival from up north. He is said to have been an emprantiri (brahmins recently arrived from coastal Karnataka, an established staging area on the migrant route). There is no known Sangamagramam ('the village at the confluence') in Kerala, if we set aside some fanciful etymology of place names. Temples to Sangamesvara (Siva) stand at several confluences of rivers or streams in northern Karnataka and southern Maharashtra.

[8] K. V. Sarma and S. Hariharan, "Yuktibhāṣā of Jyeṣṭadeva", *Ind. J. Hist. of Sci.* **26(2)**(1991), p.185. The date suggested in this article is about 1530 CE but Sarma later tended to favour a slightly later dating. Once it is accepted that *YB* is an account of Madhavan's mathematics, transmitted *via* Nilakanthan and others, a sharp date is of little consequence except to historiographers.

[9] K. V. Sarma's view ("Aryabhata and Aryabhatan Astronomy: Antecedents, Status and Development" in *Proceedings of the International Seminar and Colloquium on 1500 Years of Aryabhateeyam* (2002), Kerala Sastra Sahitya Parishad, Kochi) that Jyeshthadevan also authored *Dṛkkaraṇam*, an astronomical chronicle datable to 1607, apparently on the basis of hearsay gathered by Whish, is difficult to reconcile with the accepted date of *YB* and the dramatically different linguistic styles of the two texts.

[10] An independent confirmation of Sarma's view (see note [9]) will strengthen the case for

someone other than Jyeshthadevan, a disciple for example, doing the actual writing down of the material. Edward Stadum, who was present at the Workshop, later used the phrase 'literary calculus' to describe *YB*. In my mind I can see Jyeshthadevan teaching under a coconut tree, with a sign above saying 'calculus spoken'.

[11] The Malayalam verb used is *colluka* which can mean state, relate, narrate, speak, describe, explain, etc. and is employed in all these senses in *YB*.

[12] There are other roughly contemporaneous texts, though none in Malayalam, based on Nilakanthan's *Tantrasaṃgraha*, for example Sankara Variyar's *Yuktidīpika*. In many ways, not least in the breadth of his interests, Nilakanthan appears to have been the real inheritor of Madhavan's mantle.

[13] Bhaskara II arrived at the correct formulae for the surface area and volume of the sphere, but by seminumerical methods which the inventor of the infinitesimal method must surely have scoffed at.

[14] Anterior to the Kerala work, *saṃkalita* is the standard Indian term for the sum of any (finite) series. According to B. Datta and A. N. Singh, "Use of series in India", *Ind. J. Hist. of Sci.* **28(2)**(1993), p.103, this usage goes back at least to the (ambiguously dated) Bakhshali manuscript.

[15] John Kadvany, "Positional Notation and Linguistic Recursion", lecture at the Workshop.

[16] Scepticism about the validity and/or usefulness of the rule of the excluded middle was expressed by the Buddha himself, according to the Pali canon. The famous list of questions on which the Buddha declined to take a position includes some concerning the nature of the physical world, e.g.: Does the universe have a finite extent or not or neither or both? The great congress (3rd century BCE) which saw the parting of the Mahasanghikas from the orthodoxy following a ferocious debate, and whose proceedings are recorded in the *Kathāvattu*, not only made references to such 'multi-valued' propositions from the Buddha's teachings, but also discussed rules of debate and disputations.

[17] In Indian philosophy, the term *tarka* signifies "some sort of *reductio ad absurdum* where an appeal to some absurdity or absurd consequence is made in order to lend an indirect support to a positive thesis [by showing] in fact that the opposite thesis leads to absurdities" (Bimal Krishna Matilal, *Perception: An Essay on Classical Indian Theories of Knowledge*(1986), Oxford University Press, New Delhi, p.79). The widespread reluctance to accept *tarka* as "a means leading to a positive piece of knowledge" is also discussed here.

[18] For an examination of the epistemological roots of Kerala mathematics, from the writings of its most thoughtful and articulate representative (Nilakanthan), see Roddam Narasimha, "The *Yukti* of Classical Indian Astronomical Science", to appear. It would seem that the study of the impact of philosophical systems on mathematical thinking in the Indian context is still in its infancy. I thank Frits Staal for a correspondence on the possible links between *tarka* and *reductio ad absurdum* (or their absence!).

[19] M. D. Srinivas, "Proofs in Indian Mathematics" in *Contributions to the History of Indian Mathematics*, G. Emch, R. Sridharan and M. D. Srinivas, Eds. (2005), Hindustan Book Agency, New Delhi.

[20] A good starting point for following the debate regarding Madhavan's 'wrong' interpolation formula is Kim Plofker, "Relations between Approximations to the Sine in Kerala Mathematics", in Emch, Sridharan and Srinivas, cited above. The article is also a valuable step-by-step guide to the computation of the interpolation formula as given in Sankara Variyar's *Yuktidīpika*. *YB* itself does not mention the topic.

[21] Two and a half centuries before Descartes, Nicole Oresme (or Nicolas d'Oresme) had taken the first step to liberate space from its Euclidean literalness and give it a more metaphorical role by plotting on a plane the graph of a 'function' of a 'variable'. I owe my acquaintance with Oresme's work to David Mumford, who is also responsible for the characterisation above of its significance. The Indian view of geometry is, if anything, more down-to-earth than the Greek – compare i)the use of the terms *bhūmi* and base for the 'horizontal' line in a polygon in the two traditions, ii) *YB*'s instruction to 'drop a plumbline' with drawing (or dropping!) a perpendicular, not necessarily to a base, and several other similar instances. Notable coincidence: Oresme and Madhavan (if we accept the conventional chronolgy) lived at about the same time, the former being the senior by twenty or thirty years.