

# **Indian Mathematics**

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# Abstract

In these two lectures I shall talk about some Indian mathematicians and their work. I have chosen two examples: one from the 7<sup>th</sup> century, **Brahmagupta**, and the other, **Ramanujan**, from the 20<sup>th</sup> century. Both of these are very fascinating figures, and their histories illustrate various aspects of mathematics in ancient and modern times. In a very real sense their works are still relevant to the mathematics of today.

## Some great ancient Indian figures of Science

Varahamihira (505–587)

Brahmagupta (598-670)

Bhaskara II (1114–1185)

## The modern era

Ramanujan, S (1887–1920)

Raman, C. V (1888–1970)

Mahalanobis, P. C (1893–1972)

Harish-Chandra (1923–1983)

Bhaskara represents the peak of mathematical and astronomical knowledge in the 12th century. He reached an understanding of calculus, astronomy, the number systems, and solving equations, which were not to be achieved anywhere else in the world for several centuries...(*Wikipedia*).

Indian science languished after that, the British colonial occupation did not help, but in the 19<sup>th</sup> century there was a renaissance of arts and sciences, and Indian Science eventually reached a level comparable to western science.

## BRAHMAGUPTA (598–670c)

### Some quotations of Brahmagupta

*As the sun eclipses the stars by its brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more, if he solves them.*

Quoted in F Cajori, *A History of Mathematics*

*A person who can, within a year, solve  $x^2 - 92y^2 = 1$ , is a mathematician.*

## Life

Brahmagupta was born in 598 CE in Bhinmal city in the state of Rajasthan of northwest India. He was the head of the astronomical observatory at Ujjain, and during his tenure there wrote four texts on mathematics and astronomy.

Ujjain is in northwestern India and was the location of a great astronomical observatory. It was place of great tradition in mathematics and astronomy, lasting several centuries. Varahamihira was a predecessor to Brahmagupta in that observatory.

The *Brahmasphutasiddhanta* (628) is his most famous work. It was brought to the middle east and translated into arabic.

The observations he made and published of the angles of some specific fixed stars, combined with measurements of the angles in modern times and the fact that the period of revolution of the earth around the fixed stars is 26,000 years, allow one to calculate his epoch.

For additional detail go to **Wikipedia** and look under *brahmagupta*.

## Mathematics

Brahmagupta made fundamental contributions to many branches of mathematics.

- **cyclic quadrilaterals (geometry)**
- **diophantine equations (arithmetic)**

Constructed the complete solutions of all first order diophantine equations and elucidated the structure of solutions of a remarkable class of second order diophantine equations; found Pythagorean triplets, and discovered solutions for the problem of finding triangles with integer sides whose areas are rational.

- **Number systems**

He knew about *zero* and its additive and multiplicative properties, although he was not quite sure of what happens when one divides by zero.

- **Trigonometry and applied mathematics**

He constructed tables of the sine function, knew of trigonometric identities (already discovered by Varahamihira), gave approximate values for  $\pi$ , and had ideas about gravitation.

His textbooks on astronomy were very influential and elevated him to the level of an *acharya*, a great teacher.

## Geometry

A *cyclic quadrilateral* is a quadrilateral whose 4 vertices lie on a circle. If  $a, b, c, d$  are the sides of such a quadrilateral, Brahmagupta's formula for its area is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where  $s$  is the *semiperimeter*, namely,

$$s = \frac{a + b + c + d}{2}.$$

If we take  $a = 0$  this specializes into the ancient formula for the area of a triangle, due to **Heron**.

He also derived the formulae for the lengths  $m, n$  of the two diagonals for such a quadrilateral, namely,

$$m^2 = \frac{(ab + cd)(ac + bd)}{ad + bc}, \quad n^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}.$$

## Diophantine equations

**Diophantus** of Alexandria lived around 250 A. D., and is chiefly remembered for his studies of polynomial equations in many unknowns with integer coefficients. These are called *Diophantine equations*. Some of the most famous problems in number theory ask for solutions of diophantine equations.

### *Fermat problem*

Show that when  $n \geq 3$  there are no positive integer solutions  $(x, y, z)$  to the equation

$$x^n + y^n = z^n.$$

For  $n = 2$  there are infinitely many solutions, the so-called *Pythagorean triples*.

### *Pell's equation*

Find all integer solutions  $(x, y)$  to

$$x^2 - Ny^2 = 1 \quad (N \text{ an integer } \geq 1).$$

Diophantine problems are still at the very center of modern number theory.



## Brahmagupta's work on Pell's equation

- **Structure of set of solutions**

Discovered that there are an *infinity* of solutions, and that the set of solutions forms a *commutative group* in the modern sense. The key to this is to define a suitable *multiplication* for pairs of solutions which will give a third solution.

$$(x, y) = (a, b) * (c, d) \quad x = ac + Nbd, \quad y = ad + bc.$$

This follows from *Brahmagupta's identity*:

$$(a^2 - Nb^2)(c^2 - Nd^2) = x^2 - Ny^2.$$

- **Construction of an infinity of solutions if one is known**

Thus, if we have one solution  $(p, q)$ , we can construct an *infinity* of solutions  $(p_k, q_k)$  ( $k = 0, 1, 2, \dots$ ,  $(p_0, q_0) = (p, q)$ ) by the following recursion formula

$$p_k = pp_{k-1} + Nqq_{k-1}, \quad q_k = qp_{k-1} + pq_{k-1}.$$

He also knew (in many cases at least) how to get a *minimal* solution in a finite number of steps, starting from some  $(p, q)$  such that  $p^2 - Nq^2 = m$  where  $m$  is a small integer.

## The Chakravala algorithm : its subsequent discovery and proof

Eventually, a precise algorithm known as the *Chakravala* or the *cycle method* was developed by **Bhaskara (1114–1185)**. In the west such problems were studied only from the 17<sup>th</sup> century onwards, by **Fermat (1601–1665)** to begin with. The solution to the Pell's equation and the proof of the validity of the Chakravala are consequences of the work of **Lagrange (1736–1813)** who developed a general theory of continued fractions of quadratic irrationalities from which these results followed easily.

For details see

Weil, A., *Number Theory : an approach through history from Hammurapi to Legendre*, Birkhäuser, 1984.

Varadarajan, V. S, *Algebra in Ancient and Modern Times*, AMS-Hindustan Book Agency, 1998.

## The Chakravala algorithm : details

The algorithm starts with  $(p_0, q_0)$  such that  $p_0^2 < N$  and is as close to  $N$  as possible; then  $q_0$  is defined as 1. The  $(p_i, q_i, m_i)$  ( $i \geq 1$ ) are calculated by the following formulae recursively, using intermediate quantities  $x_i$  ( $i \geq 1$ ). First  $x_{i+1}$  is found from

$$p_i + x_{i+1}q_i \equiv 0 \pmod{|m_i|}.$$

Then we use

$$\begin{aligned} p_{i+1} &= \frac{p_i x_{i+1} + N q_i}{|m_i|} \\ q_{i+1} &= \frac{p_i + x_{i+1} q_i}{|m_i|} \\ m_{i+1} &= \frac{x_{i+1}^2 - N}{m_i} \end{aligned}$$

In a finite number of steps ( $< 2\sqrt{N}$ ) we will arrive at  $m_k = 1$ ; the corresponding  $(p_k, q_k)$  is the minimal solution.

## Examples of Brahmagupta multiplication

Here are some examples of solutions of some Pell's equations.

$$x^2 - 3y^2 = 1, (x, y) = (2, 1).$$

Application of the Brahmagupta multiplication gives, the following sequence of solutions:

$$(2, 1), (7, 4), (26, 15), (97, 56), (362, 209), (1351, 780), \dots$$

Starting from the solution  $(1, 1)$  to  $x^2 - 3y^2 = -2$  we get solutions to this equation

$$(1, 1), (5, 3), (19, 11), (71, 41), (265, 153), \dots$$

In the general case (this is an exercise)  $(1, 0)$  is the identity element and  $(a, -b)$  is the inverse of  $(a, b)$ .

The solution  $(265, 153)$  leads to

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

used by **Archimedes** in his famous approximation for  $\pi$ :

$$3\frac{10}{71} < \pi < \frac{22}{7}.$$

## Examples of solutions of Pell's equation

Here are some solutions where the numbers are large and so it is clear that one cannot find these solutions by trial and error.

$$x^2 - 13y^2 = 1, \quad (x, y) = (649, 180)$$

$$x^2 - 61y^2 = 1, \quad (x, y) = (1766319049, 226153980)$$

$$x^2 - 67y^2 = 1, \quad (x, y) = (48842, 5967)$$

These are the *smallest positive solutions*!!

For the problem in the quotation at the beginning,

$$x^2 - 92y^2 = 1, \quad (x, y) = (1151, 120).$$

The *Chakravala* algorithm produces this in the 8<sup>th</sup> step:

$$(9, 1), (10, 1), (19, 2), (48, 5), (211, 22) \\ (470, 49), (681, 71), (1151, 120).$$

For a table of the minimal solutions up to  $N = 102$  see

<http://mathworld.wolfram.com/PellEquation.html>

For more details see

V. S. Varadarajan, *Algebra in Ancient and Modern Times*, AMS-Hindustan Book Agency, 1998.

## RAMANUJAN (1887–1920)

### Early life

Srinivasa Ramanujan was born in a lower middle class family in 1887 in a small town in South India and grew up in *Kumbakonam*, a town about 160 miles southwest of Chennai, the major southern city in India. His brilliance in mathematics was recognized early by friends and others and he received help in securing a very modest clerical job in the Port Trust of Chennai so that he could pursue his mathematical ideas.

### Letters to Hardy

On the 16<sup>th</sup> of January, 1913 he wrote to G. H. Hardy, a famous English mathematician at Trinity College, Cambridge, detailing some of his results and asking Hardy to look through them. This is one of the most famous letters in the entire history of mathematics, and is the first of two letters to Hardy; the second one was written on the 27<sup>th</sup> of February, 1913, after a reply from Hardy to the first letter.

## **Invitation to Cambridge**

Impressed by Ramanujan's results as described in the letters, and realizing that the results that Ramanujan had described were only the tip of an iceberg, Hardy invited him to visit Cambridge, and helped Ramanujan secure a fellowship for this purpose. Many obstacles had to be overcome since Ramanujan's religious practices prohibited travel across the seas. Eventually Ramanujan came to Cambridge in 1914 and stayed on till 1919. He wrote many famous papers, by himself and with Hardy.

## **Return to India and last illness**

Ramanujan became ill in the last year of his stay in Cambridge. The source of his illness was not diagnosed and he returned to India in 1919. But after a year of continued health problems he died in 1920. Opinions vary as to what his illness was due to but there has been no conclusive determination.

## Some theorems from his letters to Hardy



## Hardy's reaction

The first question was whether I could recognize anything. I had proved things rather like (1.7) myself, and seemed vaguely familiar with (1.8). Actually (1.8) is classical; it is a formula of Laplace, first proved properly by Jacobi; and (1.9) occurs in a paper published by Rogers in 1907.....

The formulae (1.10)–(1.13) were on a different level and obviously both difficult and deep. .... A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.....

## Hardy's assessment

“It was his insight into algebraical formulae, transformation of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples.....But with his memory, his patience, and his power of calculation he combined a feeling for form, and a capacity for rapid modification of his hypotheses, that were often truly startling, and made him, in his own peculiar field, without a rival in his day.....”

“There have been a good many more important, and I suppose one must say greater, mathematicians than Ramanujan in the last fifty years, but no one who could stand up to him on his own ground. Playing the game of which he knew the rules, he could give any mathematician in the world fifteen...”

## Work

The following is a partial list of areas in which Ramanujan made his remarkable contributions.

- Summation of convergent and divergent series
- Prime numbers and their distribution
- Hypergeometric series
- Continued fractions
- Number theory
- Modular forms
- Modular equations
- Definite integrals

Interest in his work is growing rapidly even today. His formulae are finding applications in diverse fields.

## The Ramanujan Notebooks

Before he came to Cambridge Ramanujan had written down his results (without any proofs mostly) in Notebooks. These were published and have made many people busy for years trying to prove the theorems announced in them. **Bruce Berndt** has edited all of them and has given proofs to most, if not all, of the theorems stated in the Notebooks. The Berndt volumes have been published by Springer.

## Examples

### Summation of convergent series

$$1 - 5 \left(\frac{1}{2}\right)^3 + 9 \left(\frac{1.3}{2.4}\right)^3 - \dots = \frac{2}{\pi}$$
$$\frac{1^{13}}{e^{2\pi} - 1} + \frac{2^{13}}{e^{4\pi} - 1} + \frac{3^{13}}{e^{6\pi} - 1} + \dots = \frac{1}{24}$$

### Summation of divergent series

$$1 + 2 + 3 + 4 + \dots = -\frac{1}{12}$$
$$1 - 1! + 2! - 3! + 4! - \dots = 0.596 \dots$$

### The Ramanujan function $\tau(n)$

Let

$$\Delta(x) = x \left( \prod_{n \geq 1} (1 - x^n) \right)^{24} = \sum_{n=1}^{\infty} \tau(n) x^n$$

Then

- (a)  $\tau(mn) = \tau(m)\tau(n)$  if  $(m, n) = 1$
- (b)  $\tau(p^r) = \tau(p)\tau(p^{r-1}) - p^{11}\tau(p^{r-2})$  ( $p$  prime,  $r \geq 1$ )
- (c)  $|\tau(p)| \leq 2p^{\frac{11}{2}}$  ( $p$  a prime)

**The summation formula**  $1 + 2 + 3 + 4 + \dots = -\frac{1}{12}$

The zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

is defined for  $s > 1$  as the series is convergent. It can be continued as a complex analytic function to the entire complex  $s$ -plane except for a pole at  $s = 1$  where it behaves like

$$\frac{1}{s-1}.$$

**Riemann (1826-1866)** proved these facts and also discovered its *functional equation*:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos \frac{1}{2}s\pi \Gamma(s) \zeta(s)$$

This can be used to sum the divergent series

$$1 + 2 + 3 + 4 + \dots = \zeta(-1)$$

A simple calculation gives

$$\zeta(-1) = -\frac{1}{12}.$$

## Remarks

Ramanujan had another method presumably although he knew of the functional equation which he had derived himself (**L. Euler (1707–1783)**) had obtained the functional equation 100 years before Riemann!)

This method of using zeta function is very much prevalent in modern mathematics, and modern physics, as a technique to give meaning to certain infinite determinants. It is called the *zeta function regularisation*.

## The $\tau$ -function

Ramanujan defined the  $\tau$ -function in his famous paper *On certain arithmetical functions*, Trans. Cambridge Phil. Soc., XXII, No., 9 (1916), 159–184; *Collected Papers*, 136–162). He tabulated its values up to  $n = 30$ . He conjectured, on the basis of this scanty numerical evidence, the properties (a)-(c) described earlier! The relations (a) and (b) allowed Ramanujan to prove that

$$\sum \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$$

where the product is over the primes.

**L. J. Mordell (1888–1972)** proved (a) and (b) and these results were extended by **E. Hecke (1887–1947)** to a huge class of modular forms. The conjecture (c) defeated everybody till **Pierre Deligne (1944–)** proved that they would follow from the famous conjectures of **A. Weil (1906–1998)**. When he later proved the Weil conjectures, his proof of the Ramanujan conjecture was complete. Deligne's proof uses the full resources of modern algebraic geometry; no elementary proof of Ramanujan's conjecture is known.



## Discussion

Products taken over the primes like

$$\prod_p \frac{1}{P_p(p^{-s})}$$

where  $P_p = 1 + \dots$  is a polynomial of fixed degree, are called *Euler products* because Euler was the first to discover and study examples of them. The most famous Euler product is for the Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

which Euler used to prove analytically that there are an infinity of primes. **Lejune Dirichlet (1805–1859)** defined more general products where  $P_p$  is of degree 1 to show that every arithmetic progression  $an + b$  where  $b$  is prime to  $a$  contains an infinity of primes. Before Ramanujan no one had considered Euler products of degree 2. It was a great insight of Ramanujan to realize that modular forms are a source of such higher degree Euler products.

Euler products arise naturally in algebraic geometry, partly explaining the use of algebraic geometric methods in Deligne's solution. That the Weil conjectures should imply the Ramanujan conjecture was *in principle* known to many, but no one had the technical power to carry through the idea rigorously except Deligne.

## References

- [1] G. H. Hardy, *Ramanujan*, Chelsea (this is available in AMS now).
- [2] B. Berndt, and R. A. Rankin, *Ramanujan : Letters and Commentary*, AMS, 1995.