Spherical transforms on semisimple Lie groups

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Table of Contents

1. Introduction

1.1. Summary of results
1.2. Notation

2. Asymptotic expansions

2.1. Preliminaries
2.2. Radial components and the rings $\mathfrak{g}^{[r]}$
2.3. The basic differential equations
2.4. Initial estimates and the function $\Theta$
2.5. The sequence $\{\Phi^0_n\}$
2.6. Expressions for $\Phi^0_n - \Phi^0_{n-1}$
2.7. An estimate for $\Phi - e^{\Gamma}\Phi^0_n$
2.8. The integrals $I_q$ and an expression for $e^{\Gamma}\Phi^0_n$
2.9. The matrices $\Omega_q$
2.10. The functions $\Theta_q$
2.11. Asymptotic expansions for $\psi\lambda$

3. The spherical transform on $\mathcal{S}^p(G)$

3.1. Summary
3.2. The tubes $T^s$ and $*_T^s$
3.3. The function $\xi$
3.4. The spaces $\mathcal{E}(T^s), \mathcal{E}(*_T^s)$
3.5. The space $\mathcal{S}^p(G)$
3.6. A formula for $\psi_q$
3.7. Analyticity and growth properties of $\psi_q$
3.8. Two lemmas
3.9. Formation of wave packets
3.10. Spherical transforms on $\mathcal{S}^p(G)$

Appendix

1. Introduction

1.1. Summary of results. Let $G$ be a connected semisimple Lie group with finite center. In a series of papers Harish-Chandra introduced the Schwartz space $\mathcal{C}(G)$ and carried out a Fourier analysis of its members. It is however clear from the definition of $\mathcal{C}(G)$ that it is one of a whole family of

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1 The results of the paper in the special case when $G/K$ has rank 1 formed the Ph.D. Thesis of this author submitted to the University of Illinois.

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spaces $C^p(G)$ ($p > 0$; $C^0(G) = C(G)$), and the question arises whether one can carry out a Fourier analysis of the members of these spaces also. In this paper we have restricted ourselves to the space $S^p(G)$ of all spherical functions in $C^p(G)$ and have characterized the spherical transforms of its elements completely.

Let $0 < p < 2$, $\varepsilon = 2/p - 1$. Then the space $S^p(G)$ is contained in the space $S^2(G)$ and so, for any $f \in S^p(G)$ we can associate its spherical transform $\hat{f}$. For a general element of $S^2(G)$, the corresponding transform lies only on $F_I$ (see §1.2 for notation); but if $f \in S^p(G)$, $\hat{f}$ extends analytically to the interior of a tubular region in $F$ containing $F_I$, which we denote by $F^\varepsilon$. Moreover $\hat{f}$ possesses certain natural growth and symmetry properties, and this leads us to introduce a space $\tilde{Z}(F^\varepsilon)$ of functions holomorphic in the interior of $F^\varepsilon$ and having these properties. The question then arises whether these conditions of symmetry, holomorphy and growth are all that are needed to specify elements of $S^p(G)$. The main theorem of this paper (cf. Ch. 3) answers this in the affirmative. We prove that $f \mapsto \hat{f}$ is an isomorphism of $S^p(G)$ with $\tilde{Z}(F^\varepsilon)$.

The main idea behind this paper can be described quite simply. We shall use notation to be introduced later. Let $\varphi(\lambda: \cdot)$ be the elementary spherical functions on $G$ and for $f \in S^p(G)$, let

$$\hat{f}(\lambda) = \int_G f(x) \varphi(-\lambda: x) dx.$$  

We begin by observing that $f \mapsto \hat{f}$ is an injective map of $S^p(G)$ into $\tilde{Z}(F^\varepsilon)$; this is not difficult provided we take into account the work of Harish-Chandra [4], [5] (cf. also Gangolli [12]). The question is then whether it is surjective. To this end, one starts with a function $a \in \tilde{Z}(F^\varepsilon)$ and forms the "wave packet"

$$(1.1.1) \quad \varphi_a(x) = \int_{F_I} a(\lambda) \varphi(\lambda: x) d\beta(\lambda) \quad (x \in G),$$

$d\beta$ being the Plancherel measure for $G/K$. If the Plancherel measure $d\beta(\lambda)$ is suitably normalized then it follows from Harish-Chandra's spherical transform theory [4] that $\varphi_a \in S^2(G)$ and $\hat{\varphi}_a = a$. The question of surjectivity of the map $f \mapsto \hat{f}$ is therefore reduced to proving that $\varphi_a \in S^p(G)$ for all $a \in \tilde{Z}(F^\varepsilon)$.

In order to prove this it is obvious that we need to know rather accurately how $\varphi(\lambda: x)$ behaves when $x$ goes to infinity on $G$, and $\lambda$ varies on $F_I$. For the case $p = 2$ Harish-Chandra developed some estimates which led to his transform theory [3], [4]. However, these estimates are not enough to handle the case for $0 < p < 2$. Roughly speaking, the differential equations satisfied by the $\varphi(\lambda: x)$ lead to a perturbation problem at infinity on $G$, and Harish-
Chandra’s work in [4] may be described as that of (essentially) replacing $\varphi(\lambda; x)$ by the solutions of the unperturbed equations. The case of general $p$ requires a knowledge of higher order terms corresponding to this perturbation problem. We use the classical Cauchy-Picard-Lipschitz iteration scheme to obtain the entire perturbation expansion, leading to the construction of an asymptotic series for $\varphi(\lambda; x)$ with error estimates which are good in $x$ and uniform in $\lambda \in \mathcal{F}_I$ (Ch. 2). We then replace $\varphi(\lambda; x)$ in (1.1.1) by its asymptotic series (terminated at a suitable stage), and estimate $\varphi_*$ by using the classical Paley-Wiener device of transferring integration into the complex domain. In order to be able to do this we need precise information concerning the analyticity and growth properties of the terms in the asymptotic series for $\varphi(\lambda; \cdot)$. This is done in §§ 3.6, 3.7. The main theorem is then proved by induction on $\dim(G)$. The inductive nature of the proof makes it necessary to relate the function spaces and tube domains associated with $G$ to their counterparts associated with reductive subgroups of $G$. This is done in §§ 3.2, 3.3, 3.4.

For the case $G = \text{SL}(2, \mathbb{R})$, the main theorem of this paper (Theorem 3.10.1) was obtained by Ehrenpreis and Mautner [10], in I and III. For the case when $G$ is either complex or of real rank 1, Helgason [8] (cf. also [7]) obtained the characterization of the space of spherical transforms of elements of $\mathcal{S}(G)$. Proceeding independently of Helgason and using entirely different methods, Trombi [11] determined the space of spherical transforms of the elements of $\mathcal{S}^p(G)$ for $0 < p < 2$, $G$ being of real rank 1. The methods of [11] form the point of departure for the present work.

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1.2. Notation. Let $G$ be a reductive Lie group with Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$) and complexification $\mathfrak{g}_c$. Our main concern in this paper is with connected semisimple Lie groups with finite center. However, the proofs of the main theorems go by an induction which makes it necessary to work with a slightly larger class of groups. We say that $G$ is of class $\mathcal{K}$ if

1. $G$ has only finitely many connected components
2. the centralizer of $\mathfrak{g}$ in $G$ is compact
3. $\text{Ad}(G) \subseteq G_c$ where $G_c$ is the connected complex adjoint group of $\mathfrak{g}_c$. 

Throughout this paper we shall make the assumption that \( G \) is of class \( \mathcal{K} \). We shall now set up our notation; the results which we shall assume are all known for semisimple groups (cf. [2], [3], [4], [6]); the modifications needed for the slightly more general case treated here are easy to supply.

Let \( G \) be a group of class \( \mathcal{K} \). We fix a maximal compact subgroup \( K \) of \( G \), and denote the corresponding Cartan involution by \( \theta \). The induced involution on \( g \) is also denoted by \( \theta \). \( \mathfrak{f} \subseteq g \) is the Lie algebra of \( K \), \( \mathfrak{s} = \{ X: X \in g, \theta X = -X \} \). \( \langle \cdot, \cdot \rangle \) is a non-singular symmetric bilinear form on \( g \times g \) such that (1) \( \langle \cdot, \cdot \rangle \) coincides with the Killing form on \( g \), \( g \) being the derived algebra of \( g \); (2) \( \mathfrak{f} \) and \( \mathfrak{s} \) are orthogonal under \( \langle \cdot, \cdot \rangle \); (3) \( X \mapsto -\langle X, \theta X \rangle \) is a positive definite quadratic form on \( g \). \( g \) becomes a real Hilbert space on defining \( \| X \|^2 = -\langle X, \theta X \rangle \) (\( X \in g \)). We treat \( g \) as a complex Hilbert space in the usual manner and write \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) for its scalar product and norm, respectively. The map \( k, X \mapsto k \exp X \) is an analytic diffeomorphism of \( K \times \mathfrak{s} \) onto \( G \). Put \( x^\gamma = \theta(x^{-\gamma}) \) (\( x \in G \)).

Let \( G = KAN \) be an Iwasawa decomposition of \( G \), and let \( a, n \) be the Lie algebras of \( A \) and \( N \) respectively. We assume \( a \subseteq \mathfrak{s} \). Let \( \Delta \) be the set of roots of \( (g, a) \). For \( \lambda \in \Delta \), let \( g_\lambda \) be the corresponding root subspace. Then \( n = \sum_{\lambda \in \Delta^+} g_\lambda \) where \( \Delta^+ \) is a positive system. Let \( \{ \alpha_1, \ldots, \alpha_d \} \) be the simple roots in \( \Delta^+ \) (\( d = \dim a \)). For \( x \in G \), \( \kappa(x) \in K \), \( H(x) \subseteq a \), and \( n(x) \subseteq N \) are defined by the equation \( x = \kappa(x) \exp H(x) n(x) \). We write log for the map of \( A \) onto \( a \) which inverts \( \exp: a \to A \). For \( \alpha \in \Delta \), \( s_\alpha \) is the corresponding Weyl reflection. \( n \) is the Weyl group of \( (g, a) \). We write \( a^\prime \) for the set of all \( H \in a \) where no root of \( (g, a) \) vanishes, and \( a^+ \) for the chamber where all the positive roots take positive values. Put \( A^+ = \exp a^\prime \). Then \( G = K \cdot \text{Cl}(A^+) \cdot K \), \( \text{Cl} \) denoting closure. The form \( \langle \cdot, \cdot \rangle \) is nonsingular on \( b \times b \) for any subspace \( b \) of \( a^+ \); so, for any real linear function \( \mu \) on \( b \) there is a unique element \( H_\mu \in b \) such that \( \langle H_\mu, H \rangle = \mu(H) \) (\( H \in b \)).

Let \( H_0 \neq 0 \) be in \( \text{Cl}(a^+) \). Let \( m_{10} \) be the centralizer of \( H_0 \) in \( g \). Write \( \Delta = \{ \alpha: \alpha \in \Delta, \alpha(H_0) = 0 \} \), \( \Delta^+ = \Delta \cap \Delta^+ \). Put \( \mathfrak{n}_0 = \sum_{\lambda \in \Delta^+} g_\lambda \), and \( N_0 = \exp \mathfrak{n}_0 \). \( m_{10} \) is \( \theta \)-stable. Let \( \mathfrak{f}_0 = m_{10} \cap \mathfrak{f} \), \( \mathfrak{s}_0 = m_{10} \cap \mathfrak{s} \). Then \( g = \mathfrak{f} \oplus \mathfrak{s}_0 \oplus \mathfrak{n}_0 = \mathfrak{f} \oplus \mathfrak{s}_0 \oplus \theta(\mathfrak{n}_0) \), both sums being direct. If \( a_0 \) is the subspace of \( a \) where all the elements of \( \Delta \) vanish, \( m_{10} \) is the centralizer of \( a_0 \) in \( g \). Let \( m_0 \) be the orthogonal complement of \( a_0 \) in \( m_{10} \), and \( \alpha = m_{10} \cap a_0 \). Put \( A_0 = \exp a_0 \), \( A = \exp a^\prime \).

Let \( M_0 \) be the centralizer of \( a_0 \) in \( G \), and define \( M_5 \) as the intersection of the kernels of the continuous homomorphisms of \( M_{10} \) into the multiplicative group \( \mathbb{R}^+ \) of positive reals. Then \( M_{10} \) splits as the direct product of \( M_0 \) and \( A_0 \), and \( M_0 \) is of class \( \mathcal{K} \). Write \( *n = n \cap m_{10} = n \cap m_0 \), \( *N = \exp *n \) and let
$K = K \cap M_1 = K \cap M_\alpha$. Then $K_0$ is a maximal compact subgroup of $M_\alpha$, and $M_0 = K_0 \cdot A_0 \cdot *N$ is the Iwasawa decomposition of $M_\alpha$. For $H \in \mathfrak{a}$, let $\rho(H)$, $\rho_\alpha(H)$, and $\rho(H)$ be the traces of $(1/2) \text{ad} H$ on $\mathfrak{n}_1$, $\mathfrak{n}_0$, and $\mathfrak{n}_\alpha$, respectively. Then $\rho = \rho_\alpha + *\rho$, $\rho_\alpha * \alpha = 0$, and $*\rho |_{\mathfrak{n}_\alpha} = 0$. We define the homomorphism $d_0$ of $M_1$ into $\mathbb{R}^+$ by $d_0(m) = |\det(\text{Ad}(m)|_{\mathfrak{n}_1})|^{1/2}$. Then $d_0(m \alpha) = e^{\rho_\alpha(\log \alpha)} (m \in M_\alpha, \alpha \in A_0)$. $M_1$ normalizes $\mathfrak{n}_1$.

$F$ is the dual of $\mathfrak{a}_0$. $F_R$ (resp. $F_I$) is the real linear subspace of all elements of $F$ which take real (resp. pure imaginary) values on $\mathfrak{a}$. The form $\langle \cdot, \cdot \rangle$ induces an isomorphism of $\mathfrak{a}$ with $F_R$ and converts $F_R$ into a real Hilbert space. $F$ therefore becomes a complex Hilbert space. We write $(\cdot, \cdot)$ and $|| \cdot ||$ for the scalar product and norm in $F$. $F_R$ induces a conjugation in $F$.

$G$ is the enveloping algebra of $g_0$. We denote by $\mathfrak{f}_1, \mathfrak{h}, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{a}_0$ etc. the subalgebras of $G$ containing $1$ and generated by $\mathfrak{f}, \mathfrak{a}, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{a}_0$ etc., respectively. We use Harish-Chandra's notation for denoting the action of differential operators; thus, if $M$ is a $C^\infty$-manifold, $f$ a smooth function (scalar or vector valued) and $D$ a differential operator (scalar or matrix), we write $f(x; D)$ for $(Df)(x)$ ($x \in M$).

A function $f$ on $G$ is called spherical if $f(k_1 x k_2) = f(x)$ for $k_1, k_2 \in K$, $x \in G$. Let $D$ be the centralizer of $K$ in $G$. A spherical function $\varphi$ on $G$ is called elementary if it is analytic, if $\varphi(1) = 1$, and if it is an eigenfunction for each $q \in D$. For any $\lambda \in F$ let

$$\varphi_\lambda(x) = \varphi(\lambda \cdot x) = \int_K e^{(1-\rho)(H(\lambda x)k)}dk \quad (x \in G),$$

$dk$ being the Haar measure on $K$ with $\int_K dk = 1$. For fixed $x \in G$, $\varphi(\cdot \cdot x)$ is holomorphic on $F$, and one has $\varphi_\lambda = \varphi_\lambda^{\text{conj}} = \varphi_\lambda (\lambda \in F, s \in \mathfrak{m})$. The $\varphi_\lambda (\lambda \in F)$ are precisely all the elementary spherical functions on $G$. For any $\mathfrak{a} \in G$, we can find a unique element $u_\mathfrak{a} \in \mathfrak{a}$ such that $a - u_\mathfrak{a} \in f\mathfrak{f} + \mathfrak{m}_\mathfrak{a}$. For any $q \in D$, define $\nu(q) \in \mathfrak{h}$ by $\nu(q) = e^\rho \circ u_\mathfrak{a} \circ e^{-\rho}$. Then $\nu(q \mapsto \nu(q))$ is known to be a homomorphism of $D$ with kernel $D \cap \mathfrak{h} = D \cap f\mathfrak{f}$, and range $I(m)$, the subalgebra of $\mathfrak{a}$ of all $m$-invariant elements. Moreover,

$$\varphi(\lambda \cdot x; q) = \nu(q)(\lambda) \varphi(\lambda \cdot x) \quad (\lambda \in F, x \in G).$$

Note that we are regarding elements of $\mathfrak{a}$ as polynomials on $F$ in the natural fashion.

2. Asymptotic expansions

2.1. Preliminaries. We use the notation of Chapter 1. The aim of this chapter is to study the behavior at infinity on $G$ of the $\varphi_\lambda$, $\lambda \in F$. As $G = K \text{Cl}(A^+)K$, it is enough to examine $\varphi_\lambda(h)$ as $h \in A^+$ goes to infinity. It is
clearly necessary to take into account the fact that as \( h \to \infty \) in \( A^+ \),
\( \alpha(\log h) \) may not \( \to \infty \) for some positive roots \( \alpha \). This will be handled as follows. We fix \( H_0 \neq 0 \) in \( \text{Cl}(a^+) \), and (cf. § 1.2) obtain the asymptotic expansion of the function \( \varphi_\lambda(m \exp H) \) \( (m \in M_\lambda) \) as \( H \in a_\lambda \) tends to infinity in such a way that \( \alpha(H) \to \infty \) for all \( \alpha \in \Delta^+ \setminus \Delta^+ \), with error estimates that are uniform for \( \lambda \in \mathcal{F}_\lambda \).

The method for obtaining these expansions is quite simple. We transcribe the differential equations for \( \varphi_\lambda \) from \( G \) to \( M_{\lambda_0} \) (§ 2.3). It turns out that these equations are perturbations of the differential equations which are satisfied by the elementary spherical functions of \( M_{\lambda_0} \). We use the Cauchy-Picard-Lipschitz iterative scheme to obtain the asymptotic expansions corresponding to this problem (§ 2.5–§ 2.11).

Fix \( H_0 \neq 0 \) in \( \text{Cl}(a^+) \) and use notation of § 1.2. We number the \( \alpha_i \)'s so that \( \alpha_i, \ldots, \alpha_v \) are the simple roots that are \( > 0 \) at \( H_0 \). For any endomorphism \( T \) of \( g_\lambda \), let \( ||T|| = ||Q T|| \), \( Q \) being the orthogonal projection on \( p \). Let \( \gamma(m) = ||\text{Ad}(m^{-1})||_{\nu_\lambda} (m \in M_{\lambda_0}) \) and let \( M_{\lambda_0}^{+} = \{m : m \in M_{\lambda_0}, \gamma(m) < 1\} \). \( M_{\lambda_0}^{+} \) is open in \( M_{\lambda_0} \). It is easy to see that \( A^+ \subseteq M_{\lambda_0}^{+} \) and that \( M_{\lambda_0}^{+} = K_\lambda \cdot A \cdot K_\lambda \) where \( A \) is the set of all \( h \in \text{Cl}(A^+) \) with \( \lambda(\log h) > 0 \) for \( \lambda \in \Delta^+ \setminus \Delta^+ \). If we put \( \beta(H) = \min_{\gamma_{\lambda_0} \leq \gamma_{\lambda_0} + \lambda(H)} (H \in a) \), we see that \( \gamma(h) = e^{-\beta(\log h)} (h \in A) \). It is clear from these observations that \( M_{\lambda_0}^{+} \) meets each connected component of \( M_{\lambda_0} \).

Let \( D \) be the set of all elements of the form \( m_0 \alpha_1 + \cdots + m_v \alpha_v \) where the \( m_j \) are all integers \( \geq 0 \), and \( L \), the subset of \( D \) of those elements which are linear combinations of \( \alpha_1, \ldots, \alpha_v \) only. Let \( D^+ = D \setminus \{0\}, L^+ = L \setminus \{0\} \). For \( q = m_0 \alpha_1 + \cdots + m_v \alpha_v \), let \( o(q) = m_1 + \cdots + m_v \). If \( q \in L \) and \( H \in \text{Cl}(a^+) \), it is obvious that \( q(H) \geq o(q) \beta(H) \) and \( \beta(H) = \min_{\gamma_{\lambda_0} \leq \gamma_{\lambda_0} + \lambda(H)} (H \in a) \). We write \( \{H_1, \ldots, H_v\} \) for the basis of \( a_\lambda \) dual to \( \{\alpha_1, \ldots, \alpha_v\} \).

2.2. Radial components and the rings \( \mathcal{R}^{(r)} \). We define \( M_{1_0} \) to be the set of all \( m \in M_{1_0} \) for which \( \text{Ad}(m^{-1}) - \text{Ad}(\theta(m^{-1})) \) is invertible on \( n_0 + \theta(n_0) \). \( M_{1_0} \cap A = A^+ \) is the set of all \( h \in A \) such that \( \lambda(\log h) \neq 0 \) for \( \lambda \in \Delta^+ \setminus \Delta^+ \), and it is clear that \( M_{1_0} = K_\lambda \cdot A \cdot K_\lambda \) while \( M_{1_0}^{+} \subseteq M_{1_0} \). A standard differential calculation reveals that the map \( k_1, m, k_2 \mapsto k_1 mk_2 \) of \( K \times M_{1_0} \times K \) into \( G \) is submersive everywhere. So, \( K M_{1_0} K \) is open in \( G \), and one can associate, with any differential operator \( D \) on this open set, a differential operator \( \bar{D} \) on \( M_{1_0} \), called a radial component of \( D \), with the following property: \( f(m; D) = f(m; \bar{D}) \) for all \( C^\infty \) spherical functions \( f \) on \( G \) and all \( m \in M_{1_0} \). When \( D \in \mathcal{S} \), the coefficients of \( \bar{D} \) belong to a certain algebra of functions on \( M_{1_0} \). This
algebra was first introduced by Harish-Chandra who obtained certain estimates for its elements [4]. We shall describe in this section the refinements of these estimates that are necessary for our work.

For $m \in M_0^+$ we define $c(m)$ to be the endomorphism $\text{Ad}(m^{-1})_{\mu_0}(\text{Ad}(m^{-1}) - \text{Ad}(\theta(m^{-1})))_{\mu_0}^{-1}$ of $\mathfrak{g}_0$, where the suffix $\mu_0$ indicates restriction to $\mathfrak{g}_0$. $c(m \mapsto c(m))$ is an analytic map of $M_0^+$ into the vector space of endomorphisms of $\mathfrak{g}_0$. Let $\mathcal{R}_0$ be the algebra of functions on $M_0^+$ generated by the matrix entries of $c$ relative to some basis of $\mathfrak{g}_0$. Note that 1 is not included in $\mathcal{R}_0$.

Let $\lambda$ be the usual symmetrizer map of the symmetric algebra $S(\mathfrak{g}_0)$ (over $\mathfrak{g}_0$) onto $\mathfrak{g}_0$, and let $\mathfrak{g}_0 = \lambda(S(\mathfrak{g}_0))$. As $\mathfrak{g} = \mathfrak{f} + \mathfrak{s}_0 + \theta(\mathfrak{n}_0)$ is a direct sum, it is clear (cf. [4]) that $\mathfrak{g} = \theta(\mathfrak{n}_0) \mathfrak{g}_0 + \mathfrak{g}_f + \mathfrak{g}_0$ (direct sum), and so, for any $b \in \mathfrak{g}_0$, one can find a unique $\mu_\alpha(b) \in \mathfrak{g}_0$ such that

$$b \equiv \mu_\alpha(b) \mod (\mathfrak{g}_f + \theta(\mathfrak{n}_0)\mathfrak{g}_0).$$

(2.2.1)

The significance of the algebra $\mathcal{R}_0$ is clear from the following lemma.

**Lemma 2.2.1.** Let $b \in \mathfrak{g}_0$. Then we can find functions $\psi_i \in \mathcal{R}_0$ and elements $\mu_i \in \mathfrak{W}_i$ ($1 \leq i \leq b$), with $\deg \mu_i < \deg b$ for all $i$, such that the analytic differential operator on $M_0^+$ defined by $\delta_i(b) = \mu_i(b) + \sum_{1 \leq i \leq b} \psi_i \circ \mu_i$ is a radial component of $b$ on $M_0^+$.

**Proof.** This is a slight strengthening of Lemma 9 of [4]; a small variation of that proof is adequate and we omit it.

Let $\mathcal{R}$ be the algebra generated by the functions $\mu \mathfrak{f}$ ($\mu \in \mathfrak{W}_0$, $\mathfrak{f} \in \mathcal{R}_0$). For any function $f$ on $M_0^+$ let us define $f^\circ$ and $f^\circ$ by $f^\circ(m) = f(m^{-1})$ and $f^\circ(m) = f(\theta(m))$, for $m \in M_0^+$. We denote by $\mathcal{A}$ the algebra generated by the $a_{a,b}$ and $a_{a,b}$, where $a_{a,b}$ are the matrix entries of the representation $m \mapsto \text{Ad}(m)^{\mu_0}$ of $M_0^+$ in $\mathfrak{g}_0$. In both cases 1 is not included as a generator. $\mathcal{A}$ is closed under differentiations by elements of $\mathfrak{W}_0$. Since $\gamma(\theta(m^{-1})) = \gamma(m) < 1$ for $m \in M_0^+$, it follows that for each $f \in \mathcal{A}$, $\exists$ a constant $c_f > 0$ such that

$$|f(m)| \leq c_f \gamma(m) \quad (m \in M_0^+).$$

(2.2.2)

Fix an integer $r \geq 1$. For any ring $\mathcal{D}$, let $\mathcal{D}^{[r]}$ be the ring of $r \times r$ matrices with entries from $\mathcal{D}$. $\mathcal{D}^{[1]}$ is identified with $\mathcal{D}$. Let $\mathcal{L}_r$ be the set of all $r \times r$ matrix functions $L$ on $M_0^+$ with the following property:

$\exists$ a constant $c_L > 0$, an integer $s_L \geq 0$, and a map $q \mapsto L_q$ of $L^+$ into $\mathcal{A}^{[r]}$ such that

$$\|L_q(m)\| \leq c_L o(q)^{s_L} \gamma(m)^s \quad (m \in M_0^+, q \in L^+)$$

$$L(m \exp H) = \sum_{q \in L^+} e^{-q(H)} L_q(m) \quad (m \in M_0^+, H \in \alpha_0^+).$$

Here $L^+$ is as in § 2.1, $\| \cdot \|$ is the matrix norm, and $\alpha_0^+$ is the set of all $H \in \alpha_0$ such that $\alpha_j(H) > 0$ for $1 \leq j \leq \nu$. The first condition in (2.2.3) ensures the
absolute convergence of the series for $L(m \exp H)$. By Lemma 2 of the Appendix, the map $q \mapsto L_q$ satisfying (2.2.3) is uniquely determined by $L$. It follows at once from (2.2.3) and this uniqueness that for all $q \in L^+$

$$(2.2.4) \quad L_q(m \exp H) = e^{-q(m)} L_q(m) \quad (m \in M_{10}, H \in a_n).$$

Cauchy multiplication of the series of the type (2.2.3) shows that $\mathcal{L}_r$ is an algebra.

**Lemma 2.2.2.** Let $r$ be any integer $\geq 1$. Then $\mathcal{R}^{[r]} \subseteq \mathcal{L}_r$. Suppose $B \in \mathcal{R}^{[r]}$, $\mu \in M_{10}$. Then $(\mu B)_q = \mu B_q$ on $M_{10}$ for all $q \in L^+$. Moreover, $\exists C = C_{\beta, \gamma} > 0$ and $k = k_{\beta, \gamma} \geq 0$ such that for all $m \in M_{10}^+$, $q \in L^+$,

$$(2.2.5) \quad \| B(m; \mu) \| \leq C \gamma(m)^2 (1 - \gamma(m))^2 \gamma^k,$$

$$\| B_q(m; \mu) \| \leq C \alpha(k)^2 \gamma(m)^2.$$

**Proof.** We follow closely the proof of Lemma 7 of [4]. Let $p = \dim u_0$. We identify, for purposes of this proof, endomorphisms of $u_0$ with their matrices in some orthonormal basis. We write $B(m) = \text{Ad}(\theta(m^{-1}))m_{10}^{-1}$, $m \in M_{10}$. Then $B \in \mathcal{R}^{[p]}$, $(1 - B)^{-1} = 1 - C$, and $\| B(m) \| \leq \gamma(m)^2 < 1$ for $m \in M_{10}^+$. For any $\alpha \in L^+$ let $u_{0\alpha} = \sum_{1 \leq a \leq \alpha} g_a$. Then $u_0$ is the orthogonal direct sum of the $u_{0\alpha}$. Write $E_\alpha$ for the orthogonal projection of $u_0$ onto $u_{0\alpha}$. Then it follows from the definition of $c$ that $c(m) = -\sum_{r \geq 1} B(m)^r$ for $m \in M_{10}^+$. Consequently, for $m \in M_{10}^+$, $H \in a_n^+$,

$$c(m \exp H) = \sum_{q \in L^+} e^{-q(m)} c_q(m),$$

where $c_q(m) = -\sum B(m)^q E_\alpha$, the sum extending over all pairs $(s, \alpha)$ such that $s$ is an integer $\geq 1$, $\alpha \in L^+$ and $2s\alpha = q$. This implies at once that $c \in \mathcal{L}_p$.

We shall now prove by induction on $\deg \mu$ that $\mu c \in \mathcal{L}_p$ for any $\mu \in M_{10}$. Let $\mu \in M_{10}$ and $\deg \mu = \alpha \geq 1$. By Lemma 6 of [4] we can find constant $p \times p$ matrices $s_{ij}$ and elements $\mu_{ij} \in M_{10}$ of degree $\alpha \leq (1 \leq i \leq 4, 1 \leq j \leq t)$ such that

$$\mu c = \sum_{1 \leq j \leq t} (1 - c)[s_{ij} B s_{i2} (\mu_{ij} c) + s_{ij} B s_{i3}]$$

on $M_{10}^+$. The induction hypothesis then leads to $\mu c \in \mathcal{L}_p$.

Since $\mathcal{L}_p$ is an algebra we conclude that $\mathcal{R}^{[p]} \subseteq \mathcal{L}_p$. So $\mathcal{R} \subseteq \mathcal{L}_r$, and hence $\mathcal{R}^{[r]} \subseteq \mathcal{L}_r$, for any integer $r \geq 1$.

Suppose $B \in \mathcal{R}^{[r]}$. Then $\nu B \in \mathcal{R}^{[r]} \subseteq \mathcal{L}_r$, for all $\nu \in M_{10}$, so that $(\nu B)_q$ is well-defined for $q \in L^+$ (cf. remarks following (2.2.3)). We shall now prove by induction on $\deg \mu$ that $(\mu B)_q = \mu B_q$ for all $q \in L^+$, $\mu \in M_{10}$. The case $\deg \mu = 0$ is trivial. Let $s \geq 1$ and let us assume that the result has been proved for elements of degree $\leq s - 1$. It is enough to consider the case $\mu = X \nu$ where
\( X \in \mathfrak{m}_{10} \) and \( \nu \in \mathfrak{M}_{10} \) is of degree \( \leq s - 1 \). Let \( m \in M_{10}^+ \). Choose \( t_0 > 0 \) such that \( m \exp tX \in M_{10}^+ \) for \( 0 \leq t \leq t_0 \). Now, for \( H \in a_5^+ \),

\[
(\mu B)(m \exp H \exp tX) = \sum_{q \in L^+} e^{-\nu(H)}(\mu B)_q(m \exp tX) \quad (0 \leq t \leq t_0).
\]

As \( \mu B \in \mathfrak{R}_r \), the \( (\mu B)_q \) satisfy estimates of the type \( (2.2.3) \). So we may integrate the sum on the right term by term. The function on the left is the derivative with respect to \( t \) of the function \( t \mapsto (\nu B)(m \exp H \exp tX) \). If we identify, by virtue of Lemma 2 of the Appendix, the coefficients of \( e^{-\nu(H)} \) on both sides, we get, for \( 0 \leq v \leq t_0 \) and \( q \in L^+ \),

\[
\int_0^t (\mu B)_q(m \exp tX) dt = (\nu B)_q(m \exp vX) - (\nu B)_q(m).
\]

Differentiating at \( v = 0 \) we get \( (\mu B)_q(m) = (\nu B)_q(m; X) = B_q(m; \mu) \) by the induction hypothesis.

We finally come to the estimates in \( (2.2.5) \). The first of these is just Lemma 7 of [4]. Since \( \mu B \in \mathfrak{R}_r \) and \( (\mu B)_q = \mu B_q \), the second follows from \( (2.2.3) \). This finishes the proof of the Lemma.

2.3. The basic differential equations. We shall now transcribe the differential equations of the \( \varphi_j \) from \( G \) to \( M_{10}^+ \). We shall often write \( \nu_g \) for the homomorphism \( \nu \) of \( \mathcal{O} \) onto \( I(\nu) \). One knows [3] that \( \nu_{g(a)} \) maps \( \mathcal{O} \cap \lambda(S(\mathfrak{h})) \) bijectively onto \( I(\nu) \). Let \( \mathfrak{w}_0 \) be the Weyl group of \( (\mathfrak{n}_{10}, a) \), and \( I(\nu) \) the subalgebra of \( \mathfrak{w}_0 \)-invariant elements of \( \mathfrak{A} \). Let \( \mathcal{O}_0 \) be the centralizer of \( K_0 \) in \( \mathfrak{M}_{10} \). As \( I(\nu) \subseteq I(\nu_\mathfrak{w}) \), for each \( q \in I(\nu) \), there is a unique \( q' = \nu_q(q) \in \mathcal{O}_0 \cap \mathfrak{S}_\mathfrak{r} \) such that \( \nu_{q_0}(q_0(q)) = \nu_{q_0}(q) \). It is easily verified (cf. [4]) that \( \ker(\nu_q) = \mathcal{O} \cap \mathfrak{S}_\mathfrak{r} \) and that \( \nu_q(q q_2) = \nu_q(q_2 \nu_q(q_2) = \nu_q(q_2) \nu_q(q_1) \nu_q(q_1) \) for \( q_1, q_2 \in I(\nu) \), both congruences modulo \( \mathfrak{m}_{10}^+ \).

Let \( w = [w], w_0 = [w_0], r = w/w_0 \). We choose elements \( s_1, s_2, \ldots, s_r \) in \( \mathfrak{w} \) such that \( \mathfrak{w} = \bigcup_{i \leq i \leq r} s_i \mathfrak{w}_0 \). Let \( H(\nu) \) be the space of \( w \)-harmonic elements in \( \mathfrak{A} \). Then \( \dim H(\nu) = w \), and we choose a basis \( u_1, u_2, \ldots, u_w \) for \( H(\nu) \) such that (a) \( u_i \) are homogeneous (b) \( u_1, \ldots, u_r \) span \( H(\nu) \cap I(\nu) \). It is known that \( u_1, \ldots, u_w \) form a free module basis for \( \mathfrak{A} \) considered as an \( I(\nu) \)-module. So \( u_1, \ldots, u_r \) form a free module basis for \( I(\nu) \) considered as an \( I(\nu) \)-module. We now define \( v_p \in \mathcal{O}_0 \cap \mathcal{O}_0 \) by \( \nu_{m_{10}/a}(v_p) = u_p \), \( 1 \leq p \leq r \). It is then clear that for any \( v \in \mathcal{O}_0 \), there are \( q_p^* \in \mathcal{O}_0 \cap \lambda(S(\mathfrak{h})) \) (unique) such that \( v = \sum_{1 \leq p \leq r} v_p q_p^* \) modulo \( \mathfrak{m}_{10}^+ \). We then have the following basic result on differential equations. It is essentially Lemma 19 of [4] and we omit its proof.

**Lemma 2.3.1.** Let \( d_0 \) be as in § 1.2. Let \( v \in \mathcal{O}_0 \) and let \( q_p^* \) be as above. Then we can find functions \( g_j \in \mathfrak{R} \) and elements \( \mu_j \in \mathcal{O}_0 \) \( (1 \leq j \leq s) \) such that for all spherical \( f \in C^\infty(\mathcal{G}) \) and \( m \in M_{10}^+ \).
(2.3.1) \[ f(m; v \circ d_0) = \sum_{i \leq p \in \mathbb{Z}} f(m; v_p \circ d_0 \circ q_p^*) + \sum_{1 \leq j \leq l} g_j(m) f(m; \mu_j \circ d_0). \]

Let \( F \) be as in § 1.2. We define \( \Phi \) as the \( r \times 1 \) matrix function whose \( p^{th} \) component \( \Phi(\cdot; \cdot)_p \) is given by \( \Phi(\lambda; m)_p = \varphi(\lambda; m; v_p \circ d_0) \) for \( 1 \leq p \leq r \), \( \lambda \in \mathfrak{F}, \ m \in M_0 \). Suppose now \( v \in \mathcal{S}_0 \). Write \( q_{p'p} = q_{p'p}^*, \ 1 \leq p, \ p' \leq r \), and define \( \Gamma(\lambda; v) \) as the \( r \times r \) matrix whose \( pp' \)th element \( \Gamma(\lambda; v)_{pp'} \) is \( \nu(q_{pp'})^*(\lambda) \) \((\lambda \in \mathfrak{F})\). By Lemma 2.3.1 we can select \( g_{pp'} \in \mathcal{S}, \mu_{pp'} \in \mathcal{S}_0 \) \((1 \leq p \leq r, 1 \leq j \leq l)\) such that for all spherical functions \( f \in C^\infty(G) \) and \( m \in M_0 \),

\[ f(m; v v_p \circ d_0) = \sum_{1 \leq p' \leq r} f(m; v_{p'} \circ d_0 \circ q_{p'p}^*) + \sum_{1 \leq j \leq l} g_{pp'}(m) f(m; \mu_{pp'} \circ d_0) \]

\((1 \leq p \leq r)\). Let \( B_v \) be the \( r \times r \) matrix differential operator on \( M_0' \) whose \( pp' \)th entry \( (B_v)_{pp'} \) is the differential operator \( \partial_{p'} \sum_{1 \leq j \leq l} g_{pp'} \circ \mu_{pp'} \). We then get the following lemma.

**Lemma 2.3.2.** Let \( v \in \mathcal{S}_0 \) and \( B_v \) as above. Then, for \( \lambda \in \mathfrak{F}, \)

\[ \Phi(\lambda; m; v) = \Gamma(\lambda; v) \Phi(\lambda; m) + \Phi(\lambda; m; B_v) \quad (m \in M_0'). \]

Consider the basis \( \{ H_i, \cdots, H_j \} \) of \( a_0 \) (cf. § 1.2); \( H_j \in \mathcal{S}_0 \). Fix \( \eta \in \mathfrak{M}_{10} \). For any \( H \in a_0 \) we define \( B_{H, \eta} \) so that \( B_{H, \eta} \) is linear in \( H \) and \( B_{H, \eta} = \eta \cdot B_{H, \eta} \), \( 1 \leq j \leq \nu, \) \( B_{H, \eta} \) being as in Lemma 2.3.2 (with \( v = H_j \)). It is clear that there is an integer \( L_\eta \geq 1, a_{\omega, \eta} \in a_0^* (= \text{dual of } a_0), \mu_{\omega, \eta} \in \mathfrak{M}_{10}, \) \( B_{\omega, \eta} \in \mathcal{S}[r^1] \) \((1 \leq \omega \leq L_\eta)\) such that

\[ B_{H, \eta} = \sum_{1 \leq \omega \leq L_\eta} a_{\omega, \eta}(H) B_{\omega, \eta} \ast \mu_{\omega, \eta} \quad (H \in a_0). \]

We may now put \( v = H_j \) in (2.3.2) and differentiate with respect to \( m \). We then obtain the following lemma.

**Lemma 2.3.3.** For \( \lambda \in \mathfrak{F}, H \in a_0, \eta \in \mathfrak{M}_{10}, m \in M_0', \) we have

\[ \Phi(\lambda; m; H \eta) = \Gamma(\lambda; H) \Phi(\lambda; m; \eta) + \Phi(\lambda; m; B_{H, \eta}). \]

We now introduce the vector functions \( \Phi, \Phi^0 \) and the matrix differential operator \( B_{\lambda; \eta}(\lambda; t) \). For \( \lambda \in \mathfrak{F}, m \in M_0, t \in \mathbb{R}, H \in a_0 \) we define

\[ \Phi(\lambda; m; t; H) = \Phi(\lambda; m \exp t H) \]

and \( \Phi^0(\lambda; m; t; H) = e^{-t \Gamma(\lambda; H)} \Phi(\lambda; m); \) and put, for \( \eta \in \mathfrak{M}_{10}, \)

\[ B_{H, \eta}(\lambda; t) = e^{-t \Gamma(\lambda; H) \circ B_{H, \eta} \circ e^{t \Gamma(\lambda; H)}}. \]

Lemma 2.3.3 leads at once to

**Lemma 2.3.4.** Let \( \lambda \in \mathfrak{F}, \) and let \( m \in M_0, t \in \mathbb{R}, H \in a_0 \) be such that \( m \exp t H \in M_0' \). Then, for \( \eta \in \mathfrak{M}_{10}, \)

\[ \left( \frac{d}{ds} \Phi(\lambda; m \exp sH; \eta) \right)_{s=t} = \Gamma(\lambda; H) \Phi(\lambda; m \exp t H; \eta) + \Phi(\lambda; m \exp t H; B_{H, \eta}) \]

\[ \left( \frac{d}{ds} \Phi^0(\lambda; m \exp sH; \eta; s; H) \right)_{s=t} = \Phi^0(\lambda; m \exp t H; B_{H, \eta}(\lambda; t); t; H). \]
2.4. Initial estimates and the function $\Theta$. Our object is to study the behavior of $\Phi(\lambda; m \exp sH; \eta)$ when $s \to +\infty$, using the differential equations of Lemma 2.3.4. Notice that the equation for $\Phi$ is, except for the perturbing second term, a linear system. Further, as we shall see presently, all eigenvalues of $\Gamma(\lambda; H)$ are pure imaginary when $\lambda \in \mathcal{F}$, and $H \in a_\nu$. So there is no difficulty in studying these equations. We shall now introduce the initial estimates that are necessary for our work.

Let notation be as in § 2.3. Then, for any $u \in I(w_0)$, $\exists$ unique $z_{u\nu} \in I(w)$ such that $uu_p = \sum_{1 \leq \nu \leq r} z_{u\nu}u_{\nu}^\nu$, $1 \leq p \leq r$. Let $\bar{\Gamma}(\lambda; u)$ be the $r \times r$ matrix whose $pp^{\text{th}}$ entry is $z_{u\nu}^\nu(\lambda)$ ($\lambda \in \mathcal{F}$). For fixed $u \in I(w_0)$, the entries of $\bar{\Gamma}(\cdot; u)$ are $w$-invariant polynomials on $\mathcal{F}$, while, for fixed $\lambda \in \mathcal{F}$, $u \mapsto \bar{\Gamma}(\lambda; u)$ is a representation of the algebra $I(w_0)$ by $r \times r$ matrices. It is clear that $\Gamma(\lambda; v) = \bar{\Gamma}(\lambda; v_{w_0}(v))$ for $\lambda \in \mathcal{F}$, $v \in \mathcal{O}_\nu$. Let $\pi$ (resp. $\pi_\nu$) be a non zero $w$-skew (resp. $w_\nu$-skew) polynomial on $\mathcal{F}$ of minimal degree (cf. [3]). Let $\mathcal{F}'$ be the set of all $\lambda \in \mathcal{F}$ such that $\pi(\lambda) \neq 0$. The following two lemmas summarize the properties of $\Gamma(\lambda; \cdot)$ (cf. [3, §3]; [4, Lemma 19]).

**Lemma 2.4.1.** Let $e_p(\lambda)$ be the vector in $C'$ whose $j^{\text{th}}$ component is $u_j(s_p^{-1}\lambda)$ ($1 \leq p, j \leq r, \lambda \in \mathcal{F}$). Then, for $\lambda \in \mathcal{F}'$, the $e_p(\lambda)$ form a basis for $C'$; moreover, for any $u \in I(w_0)$, $\lambda \in \mathcal{F}$

$$\bar{\Gamma}(\lambda; u)e_p(\lambda) = u(s_p^{-1}\lambda)e_p(\lambda)$$

for $1 \leq p \leq r$.

In particular, the eigenvalues of $\bar{\Gamma}(\lambda; H)$ are the numbers $(s_p^{-1}\lambda)H)$ for $\lambda \in \mathcal{F}$, $H \in a_\nu$.

**Lemma 2.4.2.** $\exists$ rational functions $w^1, \ldots, w^r$ on $\mathcal{F}$, invariant under $w_0$ and well defined on $\mathcal{F}'$, such that

$$\sum_{1 \leq p \leq r} u^k(s_p^{-1}\lambda)u_j(s_p^{-1}\lambda) = \delta_{jk}$$

for $\lambda \in \mathcal{F}'$, $1 \leq j, k \leq r$. Let $F$ be the $r \times r$ matrix with entries $u^k u_j$ ($1 \leq j, k \leq r$). Then $(\pi/\pi_\nu)u^k u_j$ are polynomials on $\mathcal{F}$ for $1 \leq j, k \leq r$, and, for any $\lambda \in \mathcal{F}'$, the matrices $F(s_p^{-1}\lambda)$ ($1 \leq p \leq r$) are the projections $C' \to C \cdot e_p(\lambda)$ corresponding to the direct sum $C' = \sum_{1 \leq p \leq r} C \cdot e_p(\lambda)$.

In particular, the matrices $F(s_p^{-1}\lambda)$ ($1 \leq p \leq r$) mutually commute. They are clearly the spectral projections for all the $\bar{\Gamma}(\lambda; u)$.

The initial estimates for our differential equations are obtained in terms of the spherical functions $\Xi$ and $\sigma$ ([5]). $\Xi_\nu = \Xi = \varphi_\nu$ (cf. (1.2.1)); the function $\sigma$ is the unique continuous spherical function on $G$ such that $\sigma(k \exp X) = \|X\|$ ($k \in K, X \in S$). We assume the reader to be familiar with the basic properties of $\Xi$ and $\sigma$ (cf. [5, §7]); especially useful is the following estimate
(2.4.1) \[ e^{-\rho(\log h)} \leq \Xi(h) \leq c_0 e^{-\rho(\log h)(1 + \sigma(h))^{d_\pi}} \quad (h \in \text{Cl}(A^+)) , \]
c_0 > 0 being a constant and \( d_\pi = \deg(\pi) \). We now have the following lemma; it contains the initial estimates we need. We write \( \Xi_\circ \) for \( \Xi_{10} \).

**Lemma 2.4.3.** \( \exists \) constants \( c > 0, c_0 > 0, b \geq 0, b_\pi \geq 0 \) such that for all \( \lambda \in \mathcal{F}_I, u \in \mathbb{R}, t \geq 0, H \in a_\circ, \eta \in \mathcal{W}_{10}, m \in M^+_I \),

\[
\begin{align*}
|| e^{u\Gamma(\lambda; H)} || & \leq c(1 + || H || b)(1 + || u || b)(1 + || \lambda || b) \\
\| \Phi(\lambda; m; \eta) \| & \leq c_\pi(1 + || \lambda || b)(1 + \sigma(m)) \Xi_\circ(m) \\
\| \Phi^0(\lambda; m; \eta; t; H) \| & \leq c_\pi(1 + || \lambda || b)(1 + || H || b)(1 + \sigma(m)) \Xi_\circ(m) .
\end{align*}
\]

**Proof.** The first estimate is essentially corollary to Lemma 19 of [4]; together with the estimate for \( \Phi \), this leads to the estimate for \( \Phi^0 \). So, it remains to prove the estimate for \( \Phi \). As \( d_0 \) is a continuous homomorphism of \( M_{10} \) into \( \mathbb{R}^+ \), it is clear that \( \zeta \mapsto d_0^{-1} \circ \zeta \circ d_0 \) is an algebra automorphism of \( \mathcal{W}_{10} \). So, by Lemma 46 of [3] we can find, given \( \eta \in \mathcal{W}_{10}, 1 \leq p \leq r \), constants \( c' = c'_{p,r} > 0 \) and \( r' = r'_{p,r} > 0 \) such that

\[
| \varphi(\lambda; m; \eta \nu_{p} \circ d_0) | \leq c'(1 + || \lambda ||)^r \Xi_\circ(m) d_0(m)
\]
for all \( \lambda \in \mathcal{F}_I \) and \( m \in M_{10} \). Now, by (2.4.1), \( \Xi(h)d_0(h) \leq c_0 \Xi_\circ(h)(1 + \sigma(h))^{d_\pi} \) for all \( h \in \text{Cl}(A^+) \), from which, as \( M^+_I \subseteq K_0 \text{Cl}(A^+) K_0 \), we have \( d_0(m) \Xi_\circ(m) \leq c_\pi \Xi_\circ(m)(1 + \sigma(m))^{d_\pi} \) for \( m \in M^+_I \). So

\[
| \varphi(\lambda; m; \eta \nu_{p} \circ d_0) | \leq c_\pi c'(1 + || \lambda ||)^r (1 + \sigma(m))^{d_\pi} \Xi_\circ(m)
\]
for \( \lambda \in \mathcal{F}_I, m \in M^+_I, 1 \leq p \leq r \). This leads to the estimate for \( \Phi \).

Let \( H \in a_\circ^+ \). Then \( \gamma(m \exp tH) \leq \gamma(m)e^{-t\beta(H)} \) for \( t \geq 0, m \in M_0 \). This shows that for fixed \( m \), \( m \exp tH \in M^+_I \) for all large \( t \) and (cf. (2.2.5)) that \( \| B(m \exp tH) \| = O(e^{-2t\beta(H)}) \) for any \( B \in \mathcal{R}^{1,1} \). Now,

\[
\Phi^0(\lambda; m \exp tH; B_{H,\eta}(\lambda; t): t; H)
\]
becomes, after (2.3.3),

\[
\sum_{r \in \mathbb{R}^+} a_{w,\tau}(H) e^{-t\Gamma(\lambda; H)} \bar{B}_{w,\tau}(m \exp tH) \Phi(\lambda; m \exp tH; \mu_{w,\tau}) .
\]

So, using Lemma 2.4.3 and the above observations we find that

\[
\int_{t_0(m)}^\infty \| \Phi^0(\lambda; m \exp tH; B_{H,\eta}(\lambda; t): t; H) \| \, dt < \infty ;
\]

here \( t_0(m) > 0 \) is such that \( m \exp tH \in M^+_I \) for \( t \geq t_0(m) \). Lemma 2.3.4 now implies that

\[
\lim_{t \to \infty} e^{-t\Gamma(\lambda; H)} \Phi(\lambda; m \exp tH; \eta)
\]
exists for \( \lambda \in \mathcal{F}_I, m \in M_{10}, \eta \in \mathcal{W}_{10} \). It follows from the work of Harish-Chandra
(cf. [4, §§ 7, 8]) that this limit is independent of \( H \); and that if \( \Theta(\lambda: m) \) denotes this limit when \( \eta = 1, \Theta(\lambda: \cdot) \in C^\omega(M_{10}) \) and the limit for arbitrary \( \eta \) is \( \Theta(\lambda: m; \eta) \) \( (\lambda \in \mathcal{F}_I) \). The next lemma summarizes the main properties of \( \Theta \).

**Lemma 2.4.4.** For \( \lambda \in \mathcal{F}_I, m \in M_{10}, v \in \mathcal{O}_0, H \in a_0^+ \),

\[
\Theta(\lambda: m; v) = \Gamma(\lambda: v)\Theta(\lambda: m), \quad \Theta(\lambda: m \exp H) = e^{\Gamma(1; H)}\Theta(\lambda: m).
\]

For each \( \lambda \in \mathcal{F}_I, \Theta(\lambda: \cdot) \) is an analytic spherical function on \( M_{10} \) while, for any \( \eta \in \mathcal{M}_{10}, \Theta(\cdot: \cdot; \eta) \) is continuous on \( \mathcal{F}_I \times M_{10} \), and we can find constants \( c_\eta > 0, b_\eta \geq 0 \) such that for all \( (\lambda, m) \in \mathcal{F}_I \times M_{10}^\circ \),

\[
||\Theta(\lambda: m; \eta)|| \leq c_\eta(1 + ||\lambda||)^{b_\eta}(1 + \sigma(m))^{\delta}\Xi_\eta(m).
\]

(2.4.3)

**Proof.** Everythig except (2.4.3) is proved in [4, §§ 7, 8]. Fix \( \bar{H} \in a_0^+ \) and write \( B_{\bar{H}, \nu} = \sum_{l \leq \nu} B_l \circ \mu_l \) for suitable \( B_l \in \mathcal{R}[r] \) and \( \mu_l \in \mathcal{M}_{10}, 1 \leq l \leq a \). Then, for \( \lambda \in \mathcal{F}_I \) and \( m \in M_{10}^\circ \), \( \Theta(\lambda: m; \gamma) - \Phi(\lambda: m \exp \bar{H}; \gamma: 1; \bar{H}) \) is found to be

\[
\sum_{l \leq \nu} \int_0^\infty e^{-\Theta(1; \bar{H})}B_l(m \exp tH)\Phi(\lambda: m \exp t\bar{H}; \mu_l)dt.
\]

The estimate (2.4.3) now follows from Lemma 2.4.3. We omit the details.

It is possible to determine \( \Theta \) explicitly. Let \( \theta(\lambda: \cdot) \) be the elementary spherical functions on \( M_{10} \), so that, for \( \lambda \in \mathcal{F}_I, m \in M_{10} \)

\[
\theta(\lambda: m) = \int_{\mathbb{K}_0} e^{(2 - r)(H(m; k))} dk'.
\]

It is known [3] that there is a meromorphic function \( \varphi \) on \( \mathcal{F} \) with the property that for each \( \lambda \in \mathcal{F}_I \) \( (= \mathcal{F} \cap \mathcal{F}_I) \), the difference between \( e^{\varphi(\log h)}\varphi(\lambda: \log h) \) and \( \sum_{\alpha \in \Delta} \mu(\alpha) \epsilon(\alpha) e^{\varphi(\log h)} \) tends to 0 as \( h \to \infty \) in such a manner that \( \alpha(\log h) \to +\infty \) for all \( \alpha \in \Delta^+ \). We denote by \( c_0 \) the corresponding function for the group \( M_{10} \). Let \( \omega = c/c_0 \). The following lemma now follows immediately from [3], [4], and [1]:

**Lemma 2.4.5.** There is an \( \varepsilon > 0 \) such that \( \pi c_0, \pi_\omega c_0 \) and \( (\pi_\omega/\pi_0)\omega \) are holomorphic in the open tube \( \{\lambda \in \mathcal{F}, ||\lambda_R|| < \varepsilon\}; \omega \) and \( (\pi_\omega/\pi_0)\omega \) are \( \omega_\nu \)-invariant. The components \( \Theta(\cdot: \cdot; \cdot) \) are given by the following formulae, valid for \( \lambda \in \mathcal{F}_I, m \in M_{10}, 1 \leq j \leq r \):

\[
\Theta(\lambda: m) = \sum_{l \leq \nu} \omega(s^{-1}_p\lambda)u_j(s^{-1}_p\lambda)\theta(s^{-1}_p\lambda: m).
\]

(2.4.4)

We omit the proof. We note that for \( \lambda \in \mathcal{F}_I, \lambda_R, \lambda_I \in \mathcal{F}_I \) are uniquely defined by the equation \( \lambda = \lambda_R + (-1)^{i/2}\lambda_I \).

2.5. The sequence \( \{\Phi_\nu\} \). We shall now use the classical Cauchy-Picard scheme to set up a sequence of approximations to \( \Phi \). It follows quickly from the definition of \( \Theta \) and Lemma 2.3.4 that the following integral equation is valid, for all \( \lambda \in \mathcal{F}_I, m \in M_{10}^+, H \in a_0^+, t \geq 0 \):
\[ \Phi^0(\lambda; m; t; H) = e^{-t \Gamma(1; H)} \Theta(\lambda; m) - \int_0^\infty \Phi^0(\lambda; m \exp \tau H; B_{H,t}(\lambda; \tau + t): \tau + t; H) d\tau. \]

We now put \( \Phi^0(\lambda; m; t; H) = e^{-t \Gamma(1; H)} \Theta(\lambda; m) \), and define, for \( n \geq 1 \), \( \lambda \in \mathcal{F}_I \), \( m \in M_{10}^+ \), \( H \in a^+_o \), \( t \geq 0 \),

\[
(2.5.1) \quad \Phi^n(\lambda; m; t; H) = e^{-t \Gamma(1; H)} \Theta(\lambda; m) - \int_0^\infty \Phi^{n-1}(\lambda; m \exp \tau H; B_{H,t}(\lambda; \tau + t): \tau + t; H) d\tau.
\]

**Lemma 2.5.1.** The functions \( \Phi^n(\lambda; m; t; H) \) are well defined. For fixed \( \lambda, t, H \), they are of class \( C^\omega \) on \( M_{10}^+ \), while for fixed \( \lambda, H \), and \( \eta \in \mathcal{W}_{10}, m, t \mapsto \Phi^n(\lambda; m; \eta; t; H) \) is continuous on \( M_{10}^+ \times [0, \infty) \). Given \( n \geq 0 \) and \( \eta \in \mathcal{W}_{10} \), we can select \( a_{n,\eta} \geq 0 \) and a positive locally bounded function \( A_{n,\eta} \) on \( \mathcal{F}_I \times M_{10}^+ \times a^+_o \) such that for all \( \lambda \in \mathcal{F}_I, m \in M_{10}^+, H \in a^+_o, t, t' \geq 0 \),

\[
(2.5.2) \quad ||\Phi^n(\lambda; m \exp t' H; \eta; t; H)|| \leq A_{n,\eta}(\lambda; m; H)[(1 + t)(1 + t')]^{a_{n,\eta}}.
\]

Finally, for \( n \geq 1, \eta \in \mathcal{W}_{10} \), and \( \lambda, m, t, H \) as above,

\[
(2.5.3) \quad \Phi^n(\lambda; m; \eta; t; H) = e^{-t \Gamma(1; H)} \Theta(\lambda; m; \eta) - \int_0^\infty \Phi^{n-1}(\lambda; m \exp \tau H; B_{H,t}(\lambda; \tau + t): \tau + t; H) d\tau.
\]

**Proof.** We use induction on \( n \). For \( n = 0 \), the assertions follow from Lemmas 2.4.3 and 2.4.4. Let \( n \geq 1 \). Fix \( \eta \in \mathcal{W}_{10} \). Then, given any \( B \in \mathcal{R}^{[r]} \), one can select a positive locally bounded function \( c_n \) on \( M_{10}^+ \) such that

\[
||B(m \exp t' H \exp \tau H)|| \leq c_n(m)e^{-2\tau \beta(H)}
\]

for all \( m \in M_{10}^+, H \in a^+_o \), and \( \tau, t' \geq 0 \). Moreover, by the inductive assumption, (2.5.2) is satisfied by \( \Phi^{n-1} \). From these we obtain, in view of (2.3.3) and the first estimate in (2.4.2)

\[
(2.5.4) \quad ||\Phi^{n-1}(\lambda; m \exp t' H \exp \tau H; B_{H,t}(\lambda; \tau + t): \tau + t; H)|| \leq A'_{n,\eta}(\lambda; m; H)[(1 + t)(1 + t')]^{a_{n,\eta} e^{-2\tau \beta(H)}}
\]

for all \( \lambda \in \mathcal{F}_I, m \in M_{10}^+, H \in a^+_o, \tau, t, t' \geq 0 \), where \( A_{n,\eta}' \) is a positive locally bounded function and \( a_{n,\eta}' \) is a constant \( \geq 0 \). From (2.5.4) we may now conclude, using standard arguments in analysis that \( \Phi^n \) is well-defined by (2.5.1), has the required smoothness properties, and that its derivatives are given by (2.5.3). The estimate (2.5.2) for \( \Phi^n \) follows from (2.5.3), (2.5.4), and the estimates (2.4.2) and (2.4.3). This proves the lemma by induction.

### 2.6. Expressions for \( \Phi^0 - \Phi^n \) and \( \Phi^n - \Phi^{n-1} \)

For any integer \( l \geq 1 \) let \( \mathcal{D}_I \) be the vector space spanned by the homogeneous polynomials of degree \( l \) on \( a_{o^l} \), and let \( \mathcal{R}^{[r(l)]} = \mathcal{R}^{[r]} \times \cdots \times \mathcal{R}^{[r]} \) (\( l \) factors); elements of \( \mathcal{R}^{[r(l)]} \) will be
denoted by \( B = (B_i, \ldots, B_n) \). We shall now define certain maps \( P_{n,\eta}(M_{10} \times R^{[r]}(n) \rightarrow \mathcal{P}_n) \) for integers \( n \geq 1 \) and elements \( \eta \in M_{10} \), inductively on \( n \), as follows. Let \( a_{\omega,\eta}, \tilde{B}_{\omega,\eta}, \mu_{\omega,\eta} \) be as in (2.3.3). We define, for \( \mu \in M_{10} \) and \( B \in R^{[r]} \),

\[
P_{1,\eta}(\mu; B) = -\sum_{1 \leq \omega \leq \eta, \rho_{\omega,\eta} = \mu, \tilde{B}_{\omega,\eta} = B} a_{\omega,\eta}
\]

(the right side is taken to be 0 when the sum is vacuous). Clearly \( P_{1,\eta} \) has finite support for each \( \eta \). If \( n > 1 \), \( \eta \in M_{10} \) we define, for \( \mu \in M_{10} \) and \( B = (B_i, \ldots, B_n) \in R^{[r]}(n) \),

\[
P_{n,\eta}(\mu; B) = -\sum_{1 \leq \omega \leq \eta, \rho_{\omega,\eta} = B_1} a_{\omega,\eta} \cdot P_{n-1,\rho_{\omega,\eta}}(\mu; B_2, \ldots, B_n);
\]

once again, the right side is taken to be 0 when the sum is vacuous. It is clear that \( supp P_{n,\eta} \) is finite for each \( \eta \) and that the \( P_{n,\eta} \) are well defined.

Let \( R^{[n]} = \{ \tau; \tau = (\tau_1, \ldots, \tau_n), \tau_i \geq 0 \text{ for } 1 \leq i \leq n \} \). For any integer \( n \geq 1, \lambda \in F, H \in a^+_0, \tau = (\tau_1, \ldots, \tau_n) \in R^{[n]}_+, t \geq 0 \), and \( B = (B_t, \ldots, B_n) \in R^{[r]}(n) \), we define \( P(B; \tau; \lambda; m; t; H) \) to be the product

\[
\prod_{1 \leq j \leq n} e^{-r_j \Gamma_j(t; H) B_j(m \exp(\tau_1 + \cdots + \tau_j + t) + H)};
\]

here, and in the sequel, we use the symbol \( \prod_{1 \leq j \leq n} a_j \) to denote the ordered product \( a_1 \cdots a_n \) when the \( a_j \) are elements of a possibly non-commutative algebra. Finally, let \( d\tau = d\tau_1 \cdots d\tau_n \).

**Lemma 2.6.1.** Let \( n \geq 1, \eta \in M_{10} \). Then, for all \( \lambda \in F, m \in M_{10}^+, H \in a^+_0, t \geq 0 \), we have the formulae:

\[
e^{\Gamma_0(t; H)(\Phi^0 - \Phi^0_{-\lambda})(\lambda; m \exp t H; \eta; t; H)}
= \sum_{\mu, B} P_{n,\eta}(\mu; B; H) \times \int_{R^{\eta}} P(B; \tau; \lambda; m; t; H) \Phi(\lambda; m \exp(\tau_1 + \cdots + \tau_n + t) + H; \mu) d\tau;
\]

\[
e^{\Gamma_0(t; H)(\Phi^0 - \Phi^0_{-\lambda})(\lambda; m \exp t H; \eta; t; H)}
= \sum_{\mu, B} P_{n,\eta}(\mu; B; H) \times \int_{R^{\eta}} P(B; \tau; \lambda; m; t; H) e^{(r_1 + \cdots + r_n + t) \Gamma_0(t; H)} \Theta(\lambda; m; \mu) d\tau.
\]

**Proof.** It follows from (2.2.5) and (2.4.2) that if we fix \( \lambda \in F, m \in M_{10}^+, H \in a^+_0, t \geq 0 \) and \( B \in R^{[r]}(n) \), then, for all \( h \geq 0 \),

\[
\int_{R^{[n]}} \| P(B; \tau; \lambda; m; t; H) \| (1 + \tau_1)^h \cdots (1 + \tau_n)^h d\tau < \infty.
\]

Together with the estimates (2.4.2) and (2.4.3) for \( \Phi \) and \( \Theta \) this shows that the integrals in the formulae above are all absolutely convergent. We shall
prove these formulae by induction on \(n\). Fix \(\lambda, m, H, \text{ and } t\) as indicated. From the integral equation set up for \(\Phi^0\) at the beginning of §2.5 and the definition of \(\Phi^n\) we find easily that

\[
e^{t \Gamma^{(1; H)}(\Phi^0 - \Phi^n)}(\lambda : m \exp tH; \gamma : t : H)
\]

becomes equal to

\[
-\int_0^\infty e^{-t \Gamma^{(1; H)}(\Phi)}(\lambda : m \exp(\tau + t)H; B_{H, \eta}) d\tau
\]
on using (2.3.3) and (2.6.1) this reduces to the right side of the first formula when \(n = 1\). If \(n \geq 2\) we find in the same way that

\[
e^{t \Gamma^{(1; H)}(\Phi^0 - \Phi^{n-1})}(\lambda : m \exp tH; \gamma : t : H)
\]
is equal to

\[
-\int_0^\infty e^{-t \Gamma^{(1; H)}(\Phi^0 - \Phi^{n-2})}(\lambda : m \exp(\tau_1 + t)H; B_{H, \eta} \circ e^{(\tau_1 + t) \Gamma^{(1; H)}} \tau_1 + t : H) d\tau_1.
\]

On using (2.3.3) and the formula for \(\Phi^0 - \Phi^{n-2}\) which is available from the induction hypothesis, and using (2.6.2), this reduces to the right side of the first formula. The second formula is proved in a similar fashion.

2.7. An estimate for \(\Phi - e^T \Phi^n\). We now have

**Lemma 2.7.1.** Let \(n\) be an integer \(\geq 0\) and \(\eta \in \mathcal{M}_{10}\). Then there are constants \(C_n, \gamma > 0\) and \(a_{n, \eta} \geq 0\) such that for all \(\lambda \in \mathcal{F}_t\), \(m \in M_{10}^+\) and \(H \in \alpha_0^+\) with \(\beta(H) \geq 1\),

\[
(2.7.1) \quad \| \Phi(\lambda : m \exp H; \eta) - e^{\Gamma^{(1; H)}(\Phi^n)}(\lambda : m \exp H; \gamma : 1 : H) \| \\
\leq C_{n, \gamma}((1 + \| H \|)(1 + \| \lambda \|))^{a_{n, \eta}}(\gamma(m) e^{-\beta(H)})^{2(n+1)}(1 + \sigma(m))^b \Xi_\delta(m).
\]

**Proof.** From Lemma 2.6.1 and the simple relation between \(\Phi\) and \(\Phi^0\) it follows that the left side in (2.7.1) is majorized by

\[
(\ast) \quad \sum_{n, H} |P_{n+1, \gamma}(\mu : B : H)| \int_{R^{(n+1)}} |P(B : \tau : \lambda : m : 1 : H)| \\
\times \| \Phi(\lambda : m \exp(\tau_1 + \cdots + \tau_{n+1} + 1)H; \mu) \| d\tau.
\]

Since \(\text{supp} P_{n+1, \gamma}\) is finite, it is obvious that there is a constant \(c' = c'_{n, \eta} > 0\) such that \(|P_{n+1, \gamma}(\mu : B : H)| \leq c'(1 + \| H \|)^{n+1}\) for all \(\mu, B, \text{ and } H\). On the other hand, for fixed \(\mu \in \mathcal{M}_0\) and \(B = (B_1, \cdots, B_{n+1}) \in \mathcal{R}^{(n+1)}\), it follows from (2.4.2) and the definition of \(P\) in §2.6 that we can find constants \(C' = C'_{\mu : B} > 0\) and \(s = s(\mu : B) \geq 0\) with the following property: for all \(\lambda \in \mathcal{F}_t\), \(m \in M_{10}^+\), \(\tau \in R^{(n+1)}, H \in \alpha_0^+\),

\[
|P(B : \tau : \lambda : m : 1 : H)| \| \Phi(\lambda : m \exp(\tau_1 + \cdots + \tau_{n+1} + 1)H; \mu) \|
\]
is majorized by
\[ C' \prod_{i \leq j \leq n+1} (1 + \tau_i)^{2b} [(1 + \|H\|)(1 + \|\lambda\|)](1 + \sigma(m))^c \beta(m) \times \prod_{i \leq j \leq n+1} \|B_i(m \exp(\tau_1 + \cdots + \tau_j + 1)H)\| \cdot \]

Now, it follows from (2.2.5) that for any \( B \in \mathcal{R}^{(r)} \) we can find a constant \( a = a_B > 0 \) such that \( \|B(m \exp tH)\| \leq c\gamma(m)^b e^{-2t\beta(H)} \) for all \( m \in M_{+1}^{(r)} \), \( t \geq 0 \) and \( H \in \mathcal{A}_m^{(r)} \) with \( \beta(H) \geq 1 \). So, for fixed \( \mu, B \), there are constants \( C = C(\mu, B) > 0 \) and \( s = s(\mu; B) \geq 0 \) such that the integrand in the sum (\( \ast \)) corresponding to \( \mu, B \) is majorized by
\[
C[(1 + \|H\|)(1 + \|\lambda\|)](1 + \sigma(m))^c \beta(m) e^{-\beta(H)^2(n+1)} \times \prod_{i \leq j \leq n+1} (1 + \tau_j)^{2b} e^{-2\tau_j}
\]
for all \( \lambda \in \mathcal{F}_I, m \in M_{+1}^{(r)} \tau \in \mathbb{R}^{(r)(n)}_+ \) and \( H \in \mathcal{A}_m^{(r)} \) with \( \beta(H) \geq 1 \). Since (\( \ast \)) is essentially a finite sum, these estimates lead at once to (2.7.1).

2.8. The integrals \( I_q \) and an expression for \( e^\tau \Phi_n^\tau \). Let \( B \in \mathcal{R}^{(r)} \). Then, for \( m \in M_{+1}^{(r)} \), \( H \in \mathcal{A}_m^{(r)} \) we have the expansion
\[
\begin{align*}
B(m \exp H) &= \sum_{q \in \mathbb{L}^+} e^{-\|H\|} B_q(m); \\
B_q &\in \mathbb{Q}^{(r)},
\end{align*}
\]

\( B \) and \( B_q \) satisfy the estimates (2.2.5), and in addition we have \( B_q(m \exp H) = e^{-\|H\|} B_q(m) \) (cf. (2.2.4)). Let \( B = (B_1, \cdots, B_n) \in \mathcal{R}^{(r)(n)} \). We then define, for \( \lambda \in \mathcal{F}_I, H \in \mathcal{A}_m^{(r)} \), \( m \in M_{+1}^{(r)} \tau = (\tau_1, \cdots, \tau_n) \in \mathbb{R}^{(r)(n)}_+ \), and \( q = (q_1, \cdots, q_n) \in \mathbb{L}^{(r)(n)}_+ \),
\[
(2.8.1) \quad P_q(B; \tau; \lambda; m; H) = \prod_{1 \leq k \leq n} e^{-H_k} B_{k_q}(m),
\]
where \( B_{k_q} \) is the coefficient of \( e^{-\|H\|} \) in the expansion of \( B_k(m \exp H) \). For \( H' \in \mathcal{A}_m^{(r)} \), the other variables being fixed as above,
\[
(2.8.2) \quad P_q(B; \tau; \lambda; m \exp H'; H) = e^{-H_1 - \cdots - H_n} H^{(r)(n)} P_q(B; \tau; \lambda; m; H).
\]

We now put, for \( \lambda \in \mathcal{F}_I, H \in \mathcal{A}_m^{(r)} \), \( B \in \mathcal{R}^{(r)(n)} \), \( q \in \mathbb{L}^{(r)(n)}_+ \), and \( m \in M_{+1}^{(r)} \),
\[
(2.8.3) \quad I_q(B; \lambda; m; H) = \int_{R^{(r)(n)}_+} P_q(B; \tau; \lambda; m; H) e^{\tau_1 + \cdots + \tau_n} H^{(r)(n)} d\tau.
\]

The integrand is majorized by
\[
\prod_{1 \leq k \leq n} \left( \|B_{k_q}(m)\| \|e^{-\tau_k H^{(r)(n)}}\| \right) \|e^{\tau_1 + \cdots + \tau_n} H^{(r)(n)}\| e^{-\beta(H)(\tau_1 + \cdots + \tau_n)};
\]
in view of (2.4.2) we may conclude that the integral (2.8.3) is absolutely convergent. From (2.4.2) and (2.2.5) it is clear that for fixed \( \mu \in \mathbb{M}_{+1}^{(r)} \), \( B \in \mathcal{R}^{(r)(n)} \) one can find constants \( C = C(\mu, n, B) > 0 \) and \( s = s(\mu, n, B) \geq 0 \) such that \( \|P_q(B; \tau; \lambda; m; \mu; H)\| \) is majorized by
\[
C[(1 + \|H\|)(1 + \|\lambda\|)]^{2n} \gamma(m)^{2n} \cdot \cdots \cdot o(q_n) \cdot \cdots \cdot o(q_n) \cdot [(1 + \tau_1) \cdots (1 + \tau_n)]^b \times \exp \left( -\sum_{k=1}^n \tau_k \sum_{j=k}^n q_j \right)
\]
for all \( \lambda \in \mathcal{F}_I, H \in \mathcal{A}_m^{(r)} \), \( m \in M_{+1}^{(r)} \), \( \tau \in \mathbb{R}^{(r)(n)}_+ \), \( q \in \mathbb{L}^{(r)(n)}_+ \).
LEMMA 2.8.1. Let $\lambda \in \mathcal{F}_I$, $m \in M_{10}^+$, $H \in \mathfrak{a}^+_\alpha$, $B \in \mathcal{R}^{[r]}[m]$. Then
\[
\int_{\mathbb{R}^+[\mathfrak{a}]} P(B : \tau : \lambda : m : 1 : H) e^{\tau_1 + \cdots + \tau_n} \Gamma(\lambda ; H) d\tau = \sum_{q \in \mathbb{L}^+[\mathfrak{a}]} e^{-\langle q_1 + \cdots + q_n \rangle} I_q(B : \lambda : m : H).
\]

Proof. Let $\lambda$, $m$, $H$, and the $B_i$ be as above. For $1 \leq p \leq n$, we have
\[
B_p(m \exp(\tau_1 + \cdots + \tau_p + 1))H = \sum_{q \in \mathbb{L}^+[\mathfrak{a}]} e^{-\langle q_1 + \cdots + q_p \rangle} B_{p,q}(m)
\]
for all $\tau = (\tau_1, \cdots, \tau_n) \in \mathbb{R}_+^n$. Since these series converge absolutely, we have
\[
P(B : \tau : \lambda : m : 1 : H) = \sum_{q \in \mathbb{R}^+[\mathfrak{a}]} e^{-\langle q_1 + \cdots + q_n \rangle} P_q(B : \tau : \lambda : m : H)
\]
for all $\tau \in \mathbb{R}_+^n$. Multiplying by $e^{\langle q_1 + \cdots + q_n \rangle} \Gamma(\lambda ; H)$ and integrating term by term we obtain the required formula; the estimate for $\| P_q(B : \tau : \lambda : m : H) \|$ obtained above implies that
\[
\sum_{q \in \mathbb{L}^+[\mathfrak{a}]} e^{-\langle q_1 + \cdots + q_n \rangle} \int_{\mathbb{R}_+^n} \| P_q(B : \tau : \lambda : m : H) \| e^{\langle q_1 + \cdots + q_n \rangle} \Gamma(\lambda ; H) \| d\tau < \infty
\]
and shows that this operation is justified.

LEMMA 2.8.2. Let $n$ be an integer $\geq 0$, $\eta \in \mathbb{M}_{10}$. Then for $\lambda \in \mathcal{F}_I$, $m \in M_{10}^+$, and $H \in \mathfrak{a}^+_\alpha$, we have
\[
e^{\Gamma(\lambda ; H)} \Phi_q(\lambda : m \exp H ; \eta : 1) = \Theta(\lambda : m \exp H ; \eta) + \sum_{s \leq s \leq m} \sum_{q \in \mathbb{R}_{10}^{[q]}(s)} P_{s,s}(\mu : B : H) \times \sum_{q \in \mathbb{L}^+[\mathfrak{a}]} e^{-\langle q_1 + \cdots + q_n \rangle} I_q(B : \lambda : m : H) \Theta(\lambda : m \exp H ; \mu).
\]

Proof. For $n = 0$ this is obvious. Let $n \geq 1$. Then $e^{\Gamma(\lambda ; H)} \Phi_q(\lambda : m \exp H ; \eta : 1) = 1$ is equal to
\[
\Theta(\lambda : m \exp H ; \eta) + \sum_{s \leq s \leq m} e^{\Gamma(\lambda ; H)} (\Phi^0_s - \Phi^0_{s-1}) \langle \lambda : m \exp H ; \eta : 1 \rangle
\]
for $\lambda$, $m$, $H$ as above. Lemmas 2.6.1 and 2.8.1 lead to the required result.

From (2.8.2) it is clear that for all $H' \in \mathfrak{a}_0$, $m \in M_{10}$, $\lambda \in \mathcal{F}_I$, $H \in \mathfrak{a}^+_\alpha$, (2.8.4)
\[
I_q(B : \lambda : m \exp H' : H) = e^{-\langle q_1 + \cdots + q_n \rangle} I_q(B : \lambda : m : H).
\]
Also, from (2.4.2) and the majorant obtained earlier for $\| P_q(B : \tau : \lambda : m : H) \|$, it is clear that for fixed $n \geq 1$ and $B \in \mathcal{R}^{[r]}[n]$, (2.8.5)
\[
\| I_q(B : \lambda : m : H) \| \leq C(1 + \| H \|(1 + \| \lambda \|)^{-1} + \gamma(m) s o(q_1) \cdots o(q_n) s.
\]
The next two lemmas describe the main properties of $I_q$.

LEMMA 2.8.3. Let $B \in \mathcal{R}^{[r]}[s]$, $q \in \mathbb{L}^+[\mathfrak{a}]$. Then: (i) for fixed $(\lambda, H) \in \mathcal{F}_I \times \mathfrak{a}_\alpha$, $m \mapsto I_q(B : \lambda : m : H)$ is of class $C^\infty$ on $M_{10}$ and its derivatives can be calculated.
by differentiating under the integral in (2.8.3). (ii) for fixed \((\lambda, m) \in \mathcal{F}_I \times M_0\) and \(\mu \in \mathcal{M}_0, \ H \mapsto I_q(\mathbf{B} : \lambda : m ; \mu : H)\) is an analytic function on \(\alpha_\varepsilon^+\). (iii) for any \(H \in \alpha_\varepsilon^+\) let

\[
(2.8.6) \quad \mathcal{F}_H = \left\{ \lambda : \lambda \in \mathcal{F}, \ |\text{Re}(s\lambda)(H)| < \frac{1}{4} \beta(H) \ \text{for all} \ s \in \mathfrak{m} \right\};
\]

then for fixed \(m \in M_0\) and any \(\mu \in \mathcal{M}_0\), the integral defining \(I_q(\mathbf{B} : \lambda : m ; \mu : H)\) exists for all \(\lambda \in \mathcal{F}_H\) and is holomorphic therein.

**Proof.** From the estimates for \(P_q\) obtained earlier we find that for \((\lambda, H) \in \mathcal{F}_I \times \alpha_\varepsilon^+\) and any compact set \(\gamma \subseteq M_0\),

\[
\int_{\mathbb{R}_+^{(s)}} \sup_{\tau \in \gamma} \left\| P_q(\mathbf{B} : \tau : \lambda : m ; \mu : H) \right\| \left\| e^{(\tau_1 + \cdots + \tau_n)\Gamma(I : H)} \right\| d\tau < \infty.
\]

This proves (i). In proving (ii) and (iii) we may assume that \(\mu = 1\), because, as is easily seen from (2.8.1) and Lemma 2.2.2, the linear space spanned by the functions \(P_q(\mathbf{C} : \cdots : \cdots)(\mathbf{C} \in \mathbb{R}^{(r)(s)})\) is invariant under differentiation by elements of \(\mathcal{M}_0\). For (ii), fix \(\lambda, m\) and write, for \(\tau \in \mathbb{R}_+^{(s)}\) and \(H' \in \alpha_0\),

\[
F_\tau(H' : \tau) = \prod_{1 \leq k \leq n} \exp\left(-\tau_k \sum_{j=1}^{n} q_j(H') e^{-\tau_k \Gamma(I : H')} B_{k, q_j}(m) e^{(\tau_1 + \cdots + \tau_n)\Gamma(I : H')} \right).
\]

For fixed \(\tau\), \(F_\tau(\cdot : \tau)\) is holomorphic on \(\alpha_0\). Now, the eigenvalues of the matrix \(\Gamma(\lambda' : H')\) are the numbers \((s\lambda')(H') (s \in \mathfrak{m})\) for any \(\lambda' \in \mathcal{F}, \ H' \in \alpha_0\) (cf. §2.4); so, by Lemma 60 of [3] we can select constants \(c_0 > 0, b_4 \geq 0\) such that for all \(u \in \mathbb{R}, \ \lambda' \in \mathcal{F}, \ H' \in \alpha_0\),

\[
(2.8.7) \quad \left\| e^{\tau \Gamma(I : H')} \right\| \leq c_0 [(1 + |u|)(1 + \|\lambda'\|)(1 + \|H'\|)]^{b_4} e^{\|u\| \max_{s \in \mathfrak{m}} |\text{Re}(s\lambda')(H')|}.
\]

Observing that for \(\lambda \in \mathcal{F}_I\) and \(H \in \alpha_\varepsilon^+\), \(\text{Re}(s\lambda)(H + H_i) = \text{Re}(s\lambda)(H_i)\) for all \(s \in \mathfrak{m}, H_i \in \alpha_0\), we conclude easily from (2.8.7) that for some \(\varepsilon_H > 0\),

\[
\int_{\mathbb{R}_+^{(s)}} \sup_{H \in \alpha_0, \|H\| \leq \varepsilon_H} \left\| F_\tau(H + H_1 : \tau) \right\| d\tau < \infty.
\]

Thus, \(H' \mapsto \int_{\mathbb{R}_+^{(s)}} F_\tau(H' : \tau) d\tau\) is holomorphic around \(H\), proving (ii). For (iii), fix \(m \in M_0, H \in \alpha_0\) and define \(G(\lambda : \tau)\) for any \(\lambda \in \mathcal{F}\) and \(\tau \in \mathbb{R}_+^{(s)}\) to be the product \(F_\tau(H : \tau)\) defined above. If \(\gamma\) is any compact set \(\subseteq \mathcal{F}_H\) we can find \(\varepsilon_0 > 0\) with \(0 < \varepsilon_0 < 1/4\) such that \(\left|\text{Re}(s\lambda)(H)\right| \leq \varepsilon_0 \beta(H)\) for \(s \in \mathfrak{m}, \ \lambda \in \gamma\). It then follows easily on using (2.8.7) that

\[
\int_{\mathbb{R}_+^{(s)}} \sup_{\tau \in \gamma} \left\| G(\lambda : \tau) \right\| d\tau < \infty.
\]

This proves (iii). The proof of the lemma is complete.

**Lemma 2.8.4.** Fix \(n \geq 1, \ B \in \mathbb{R}^{(r)(s)}\), \(q = (q_1, \cdots, q_n) \in \mathbb{L}^{(s)}\), and write \(q = q_1 + \cdots + q_n\). Let \(\mathcal{A}_q\) be the set of all \(f \in \mathcal{A}\) for which \(f(m \exp H) = \cdots\).
$$e^{-\varphi(m)}f(m)$$ for all \(m \in M_{10}, H \in \alpha_0\). Define the polynomial \(g_q\) on \(F \times \alpha_0\) by

$$g_q(\lambda; H) = \prod_{1 \leq i \leq r} \prod_{1 \leq a \leq n} (s^{-1}_{-i}\lambda - s^{-1}_i\lambda + q_a + \cdots + q_s)(H)$$

\((s_i, 1 \leq i \leq r)\) are as in §2.3). Then, given \(\mu \in M_{10}\), we can select \(f_1, \ldots, f_r\) in \(\mathcal{A}_q\) and \(r \times r\) matrices \(Q_1, \ldots, Q_r\) whose entries are polynomials on \(F \times \alpha_0\) such that for all \(\lambda \in F, m \in M_{10}, H \in \alpha_0^+\),

$$g_q(\lambda; H)I_q(B; \lambda; m; \mu; H) = \sum_{1 \leq i \leq r} f_i(m)Q_i(\lambda; H).$$

Moreover, for fixed \((m, H) \in M_{10} \times \alpha_0^+\), \(\lambda \mapsto I_q(B; \lambda; m; \mu; H)\) is the restriction to \(F\), of a \(\varphi\)-invariant rational function on \(F\).

**Proof.** As before we may assume \(\mu = 1\). For brevity we write \(I(\lambda; m; H)\) for \(I_q(B; \lambda; m; H)\). The \(\varphi\)-invariance of the entries of \(F(\cdot; H)\) implies that of \(I(\cdot; m; H)\). By Lemma 2.8.3, \(I(\lambda; m; H)\) is well defined by (2.8.3) for \(\lambda \in F_H\) and is holomorphic therein, for fixed \((m, H) \in M_{10} \times \alpha_0^+\). Now, by Lemmas 2.4.1 and 2.4.2,

$$e^{\varphi(I(\lambda; m; H))} = \sum_{1 \leq p \leq r} \exp(u(s^{-1}_{-p}\lambda)(H))F(s^{-1}_{-p}\lambda)$$

\((u \in \mathbb{R}, \lambda \in F' \cap F_H)\).

Substituting this in (2.8.3) we obtain, after a brief calculation, the following formula, for \(\lambda \in F' \cap F_H, m \in M_{10}, H \in \alpha_0^+\):

$$I(\lambda; m; H) = \sum \Lambda(p_1, \ldots, p_n+1; \lambda; H)\left(\prod_{1 \leq i \leq n} F(s_{-i}\lambda)B_{j, q_j}(m)\right)F(s_{n+1}^{-1}\lambda),$$

where the sum is over all \((p_1, \ldots, p_{n+1})\) with \(1 \leq p_1, \ldots, p_{n+1} \leq r\), and

$$\Lambda(p_1, \ldots, p_{n+1}; \lambda; H) = \prod_{1 \leq a \leq n} (s_{-a}\lambda - s_{n+1}^{-1}\lambda + q_a + \cdots + q_s)(H)^{-1}.$$

We should note that if \(s', s'' \in M, q' \in \mathbb{L}^+\), then \(|(s'\lambda - s''\lambda + q')(H)| \geq \frac{1}{2}\beta(H)\) for any \(H \in \alpha_0^+\) and \(\lambda \in F_H\), so that \(\Lambda\) is well-defined. Observing that the entries of \(\pi(\lambda)F(s^{-1}_{-p}\lambda)\) are polynomials in \(\lambda (1 \leq p \leq r, \text{cf. Lemma 2.4.2})\) and that for fixed \((p_1, \ldots, p_n, p_{n+1}; \ldots; \lambda)^{-1}\) divides \(g_q\), we obtain the following representation \((\lambda \in F' \cap F_H)\):

$$\pi(\lambda)^{n+1}g_q(\lambda; H)I(\lambda; m; H) = \sum_{1 \leq a \leq n} \sum_{1 \leq b \leq \beta} p_a(H)g_b(m)R_{ab}(\lambda),$$

where the \(p_a\) are polynomials on \(\alpha_0\), the \(g_b\) are elements of \(\mathcal{A}_q\), the \(R_{ab}\) are \(r \times r\) matrices whose entries are polynomials on \(F\). We may assume that the \(p_a\) and the \(g_b\) are respectively linearly independent. The formula (\(\ast\)) is valid for all \(\lambda \in F_H\) by continuity.

An elementary argument shows that if \(Q\) is a polynomial on \(F\) and \(s\) an integer \(\geq 0\) such that \(\pi^{-s}Q\) is holomorphic in an open neighborhood of 0, then \(\pi^s\) divides \(Q\). Applying this to \(\ast\) we see that \(\pi^{n+1}\) divides the entries of \(\sum_{a, b} p_a(H)g_b(m)R_{ab}'\) for each \(m \in M_{10}, H \in \alpha_0^+\). As the \(p_a\) and the \(g_b\) are linearly independent we may conclude that the entries of \(R_{ab} = \pi^{-[n+1]}R_{ab}'\) are all polynomials on \(F\). Lemma 2.8.4 follows easily from this.
2.9. The matrices $\Omega_{\kappa}$. Let $n$ be an integer $\geq 1$; $\eta, \mu \in \mathcal{W}_{10}$; $q \in L^+$. We then put, for $(\lambda, m, H) \in \mathcal{F}_I \times M_{10} \times a_+^\lambda,$

$$
(2.9.1) \quad \Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H) = \sum_{\gamma \in [r]^{|s|}} P_{\kappa, \eta}(\mu : B : H) \sum_{q \in L^+^{(m)}, q_1 + \cdots + q_n = q} \mathcal{I}_q(B : \lambda : m : H).
$$

Note that the sum over $L^+^{(m)}$ in (2.9.1) is always finite, and is even vacuous if $n > o(q)$, in which case we interpret it as 0. Since, for fixed $n, \eta$, supp $P_{\kappa, \eta}$ is finite, there is a finite set $M_{\kappa, \eta}$ such that $\Omega_{q}^{(\kappa, \eta)}(\mu : \cdots : \cdots) = 0$ if $\mu \in M_{\kappa, \eta}$. The significance of the matrices $\Omega_{q}^{(\kappa, \eta)}$ is clear from the following expression for

$$
e^{H(H)D} \Phi_{q}^{\kappa}(\lambda : m \exp H ; \eta : 1 : H)
$$

valid for $n \geq 0$, $\eta \in \mathcal{W}_{10}$, $\lambda \in \mathcal{F}_I$, $H \in a_+^\lambda$, $m \in M_{10}^+$, which is obtained at once from Lemma 2.8.2,

$$
\Theta(\lambda : m \exp H ; \eta) + \sum_{s \in s} \sum_{\mu \in \mathcal{W}_{10}} \sum_{q \in L^+} e^{\eta(H)} \Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H) \Theta(\lambda : m \exp H ; \mu).
$$

**Lemma 2.9.1.** Let $n \geq 1$, $\lambda \in \mathcal{F}_I$, $m \in M_{10}$, $H \in a_+^\lambda$, $H' \in a_0$, $\eta, \mu \in \mathcal{W}_{10}$, $t > 0$, $q \in L^+$. Then

$$
(2.9.2) \quad \Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : tH) = \Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H)
$$

$$
\Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m \exp H' : H) = e^{\eta(H')} \Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H).
$$

**Proof.** The first of these relations is an easy consequence of (2.8.3) and the fact that $P_{\kappa, \eta}(\mu : B : \cdot \cdot \cdot)$ is a homogeneous polynomial of degree $n$ on $a_{0 \kappa}$. The second follows from (2.8.4).

**Lemma 2.9.2.** Let $n$ be an integer $\geq 1$, $\eta \in \mathcal{W}_{10}$. Then we can find constants $C = C(n, \eta) > 0$ and $s = s(n, \eta) \geq 0$ such that for all $\mu \in \mathcal{W}_{10}$, $\lambda \in \mathcal{F}_I$, $m \in M_{10}^+$, $q \in L^+$, and $H \in a_+^\lambda$ with $\beta(H) \geq 1$,

$$
\|\Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H)\| \leq C(1 + \|H\|)(1 + \|\lambda\|)^{n+1} (1 + \|\gamma\|) \gamma(m)^{2n} o(q)^r.
$$

**Proof.** For $\mu, \lambda, m, q, H$ as above, $\|\Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H)\|$ is majorized by

$$
\sum_{\gamma \in [r]^{|s|}} P_{\kappa, \eta}(\mu : B : H) \sum_{q \in L^+^{(m)}, q_1 + \cdots + q_n = q} \|\mathcal{I}_q(B : \lambda : m : H)\|.
$$

In view of (2.8.5) and the fact that supp $P_{\kappa, \eta}$ is finite, we can find constants $C' = C'(n, \eta) > 0$ and $s' = s'(n, \eta) \geq 0$ such that for $\mu, \lambda, m, q, H$ as above, $\|\Omega_{q}^{(\kappa, \eta)}(\mu : \lambda : m : H)\|$ is majorized by

$$
C'(1 + \|H\|)(1 + \|\lambda\|)^{n+1} (1 + \|\gamma\|) \gamma(m)^{2n} \sum_{q \in L^+^{(m)}, q_1 + \cdots + q_n = q} o(q_1)^{r'} \cdots o(q_n)^{r'}.
$$

The sum over $q$ occurring in the last estimate is clearly at most $N o(q)^{nr}$ where $N$ is the number of $q = (q_1, \cdots, q_n) \in L^+^{(m)}$ with $o(q_j) \leq o(q)$ for $1 \leq j \leq n$; as $N$ is obviously $\leq 2^a o(q)^{nr}$, the lemma follows immediately.
We now define, for any integer \( n \geq 1 \), \( \gamma, \mu \in \mathcal{W}_{10} \), and \( q \in \mathbb{L}^+ \),

\[
\Omega_{q}^{[n,\gamma]} = \sum_{1 \leq s \leq n} \Omega_{q}^{(s,\gamma)}, \quad \Omega_{q}^{(\gamma)} = \sum_{1 \leq s < \infty} \Omega_{q}^{(s,\gamma)} .
\]

It is clear from the definition of \( \Omega_{q}^{(s,\gamma)} \) that the terms of the second sum above are 0 for \( s > o(q) \), so that \( \Omega_{q}^{[n,\gamma]} = \Omega_{q}^{(\gamma)} \) for \( n > o(q) \).

**Lemma 2.9.3.** Let \( q \in \mathbb{L}^+ \); \( \gamma, \mu \in \mathcal{W}_{10} \). Then, for each \( m \in M_{10} \) and \( H \in \alpha_0^+ \), the entries of the matrix \( \Omega_{q}^{(\gamma)}(\mu; \cdots; m; H) \) are \( \omega \)-invariant rational functions on \( F \) which are holomorphic on \( F_{H} \) (cf. (2.8.6)). Moreover, if we define the polynomial \( h_q \) on \( F \times \alpha_0^+ \) by

\[
h_q(\lambda; H) = \prod_{1 \leq i, j \leq r} \prod_{q' \in \mathbb{L}^+.q'^{-1}q \in \mathbb{L}} \left( \frac{s_i^{-1}\lambda - s_j^{-1}\lambda + q'}{H} \right)
\]

then we can find \( g_1, \ldots, g_r \in \mathcal{O} \) and \( r \times r \) matrices \( R_1, \ldots, R_r \) whose entries are polynomials on \( F \times \alpha_0^+ \) such that for all \( \lambda \in F_{F} \), \( m \in M_{10} \), \( H \in \alpha_0^+ \),

\[
h_q(\lambda; H)\Omega_{q}^{(\gamma)}(\mu; \lambda; m; H) = \sum_{1 \leq i \leq q} g_i(m)R_i(\lambda; H).
\]

**Proof.** If \( q = (q_1, \ldots, q_n) \in \mathbb{L}^+/\alpha_0^+ \) is such that \( q_1 + \cdots + q_n = q \), then \( g_q \) (cf. (2.8.8)) divides \( h_q \). The present lemma follows at once from Lemma 2.8.4.

2.10. The functions \( \Theta_q \). It is clear from Lemma 2.4.4 and the relation (2.8.6.1) that for all \( \lambda \in F_{F} \), \( m \in M_{10} \), \( H \in \alpha_0^+ \), \( \mu \in \mathcal{W}_{10} \),

\[
\Theta(\lambda; m \exp H; \mu) = \sum_{1 \leq p \leq r} \exp((s_p^{-1}\lambda)(H))F(s_p^{-1}\lambda)\Theta(\lambda; m; \mu).
\]

Let \( q \in \mathbb{L}^+ \), \( \gamma, \mu \in \mathcal{W}_{10} \), and \( n \geq 1 \). We define the \( r \times 1 \) vector function \( \Theta_{q, p}^{[n,\gamma]} \) and \( \Theta_{q, p}^{(\gamma)} \) (1 \( \leq p \leq r \)) on \( F_{F} \times M_{10} \times \alpha_0^+ \) as follows:

\[
\Theta_{q, p}^{[n,\gamma]}(\lambda; m; H) = \sum_{n \in \mathbb{W}_{10}} \Omega_{q}^{(n,\gamma)}(\mu; \lambda; m; H)F(s_p^{-1}\lambda)\Theta(\lambda; m; \mu),
\]

and \( \Theta_{q, p}^{(\gamma)} \) is defined by a similar formula with \( \Omega_{q}^{(\gamma)} \) replacing \( \Omega_{q}^{[n,\gamma]} \). Note that the sums over \( \mathbb{W}_{10} \) are finite and that for \( n > o(q) \), \( \Theta_{q, p}^{[n,\gamma]} = \Theta_{q, p}^{(\gamma)} \) (cf. § 2.9). It follows from the work of §§ 2.8 and 2.9 that \( \Theta_{q, p}^{[n,\gamma]} \) is an analytic function on \( \alpha_0^+ \) for fixed \( (\lambda, m) \in F_{F} \times M_{10} \), and is an analytic function on \( F_{F} \) for fixed \( (m, H) \in M_{10} \times \alpha_0^+ \). It is obvious from Lemma 2.9.1 that \( \Theta_{q, p}^{[n,\gamma]}(\lambda; m; tH) \) is independent of \( t \) (\( t > 0 \)), and that

\[
\Theta_{q, p}^{[n,\gamma]}(\lambda; m \exp H'; H) = \exp((s_p^{-1}\lambda - q(H'))\Theta_{q, p}^{[n,\gamma]}(\lambda; m; H),
\]

for all \( H' \in \alpha_0^+ \), \( \lambda, m, H \) being as above. We note finally that in this notation, for \( \lambda \in F_{F} \), \( m \in M_{10}^+ \), \( H \in \alpha_0^+ \), \( \gamma, \mu \in \mathcal{W}_{10} \),

\[
e^{-q(H)}\Phi_{q}(\lambda; m \exp H; \gamma; 1; H)
\]

becomes

\[
\Theta(\lambda; m \exp H; \gamma) + \sum_{\gamma, q \in \mathbb{L}^+} e^{-q(H)}\left\{ \sum_{1 \leq p \leq r} \exp((s_p^{-1}\lambda)(H))\Theta_{q, p}^{[n,\gamma]}(\lambda; m; H) \right\}.
\]

**Lemma 2.10.1.** Let \( n \geq 1 \), \( \gamma, \mu \in \mathcal{W}_{10} \). Then we can select constants \( C = C(n, \gamma) > 0 \) and \( s = s(n, \gamma) \geq 0 \) such that for \( \lambda \in F_{F} \), \( m \in M_{10}^+ \), \( q \in \mathbb{L}^+ \), \( 1 \leq p \leq r \), and \( H \in \alpha_0^+ \) with \( \beta(H) \geq 1 \).
\[ \|\pi(\lambda)\Theta_{\varphi, p}\|_{\mathcal{F}_m: H}\| \leq C[(1 + \|H\|)(1 + \|\lambda\|)]^r \gamma(m)^s \big(1 + \sigma(m)\big)^v \Xi(m) q(o(q)^s) \]

**Proof.** By Lemma 2.4.2 there are constants \( c' > 0 \) and \( s' \geq 0 \) such that
\[ \|\pi(\lambda)F(s^{-i}\lambda)\| \leq c'(1 + \|\lambda\|)^s' \text{ for all } \lambda \in F \text{ and } 1 \leq p \leq r. \]
So, if \( \lambda, m, H, q, p \) are as in the lemma, \( \|\pi(\lambda)\Theta_{\varphi, p}(\lambda: m : H)\| \) is majorized by
\[ c'(1 + \|\lambda\|)^s' \sum_{\eta \in \mathcal{M}_0} \sum_{1 \leq s \leq s'} \|\Omega_{\varphi, r}(\mu: \lambda: m : H)\| \|\Theta(\lambda: m : \mu)\| . \]
Now (cf. § 2.9), there is a finite set \( M_m \), such that \( \Omega_{\varphi, r}(\mu: \cdots : \cdots) = 0 \) if \( \mu \in M_m \).

This leads at once to the required estimate in view of (2.4.3) and Lemma 2.9.2.

**Lemma 2.10.2.** Let \( F'_i \) be the set of all \( \lambda \in F_i \) for which \( s_i^j\lambda | a_0 \) \( (1 \leq j \leq r) \) are all distinct. Then \( F'_i \) is a dense open subset of \( F_i \).

**Proof.** Let \( i \neq j \), \( 1 \leq i, j \leq r \) and \( A_{ij} = \{ \lambda: \lambda \in F'_j \text{ and } s_i^j\lambda | a_0 \neq s_i^j\lambda | a_0 \} \).
\( A_{ij} \) is open in \( F_i \). If \( A_{ij} \) is not dense, \( s_i^j\lambda = s_i^j\lambda \) on \( a_0 \) for all \( \lambda \in F_i \). Then \( H' = H \) for all \( H \in a_0 \) and \( t = s_i^j t_s \). By a theorem of Chevalley, this implies that \( t \in m_0 \) or \( s_i \in s_i m_0 \). As \( i \neq j \), this is a contradiction. Lemma 2.10.2 follows now immediately.

**Lemma 2.10.3.** Let \( q \in L^+ \), \( 1 \leq p \leq r \), \( \eta \in \mathcal{M}_0 \). Then, for any \( (\lambda, m) \in F_i \times M_0 \), \( \Theta_{\varphi, r}(\lambda: m : H) \) is independent of \( H \in a_0^+ \).

**Proof.** If \( m \in M_0 \) and \( H \in a_0^+ \), \( m \exp tH \in M_0^+ \) for all \( t > 0 \) sufficiently large. So, in view of (2.10.3), it is enough to prove that \( \Theta_{\varphi, r}(\lambda: m : H) \) is independent of \( H \in a_0^+ \) for all \( (\lambda, m) \in F_i \times M_0^+ \). By Lemma 2.10.2 and the smoothness and analyticity properties of \( \Theta_{\varphi, r} \), it is enough to find a nonempty open subset \( S \) of \( a_0^+ \) such that for \( 1 \leq p \leq r \), \( n > o(q) \),
\[ \Theta_{\varphi, r}(\lambda: m : H) \in \Theta_{\varphi, r}(\lambda: m : H') = 0 \]
for \( H, H' \in S \), \( (\lambda, m) \in F_i \times M_0^+ \). Let \( \Delta_{\varphi, r}(\lambda: m : H') \) denote this difference. We shall prove this for the open set \( S \) of all \( H \in a_0^+ \) with \( \beta(\lambda) > \max(1, \max_{1 \leq i \leq r} \alpha_i(\lambda)/2) \). Fix \( \eta \in \mathcal{M}_0 \), \( \lambda \in F_i \) and \( m \in M_0^+ \). Now, it is obvious that for almost all \( H \in a_0^+ \) one has (a) \( (s_i^{-1}\lambda)(\bar{H}) \); (b) the map \( q \mapsto q(\bar{H}) \) is injective on \( L^+ \). So, the set \( S \) of all \( (H, H') \in S \times S \) for which \( \bar{H} = H + H' \) satisfies (a) and (b) above, is dense in \( S \times S \). It is clearly enough to prove that \( \Delta_{\varphi, r}(\lambda: m : H') = 0 \) for \( 1 \leq p \leq r \). We shall fix \( (H, H') \in S \). All the constants entering the estimates in the proof are allowed to depend on \( \eta, \lambda, m, H, H' \). We write \( \Delta_{\varphi, r} \) for \( \Delta_{\varphi, r}(\lambda: m : H') \).

Note that \( (H', H) \in S \) also.

We begin first with the estimate obtained by replacing \( m \) and \( H \) in (2.7.1) by \( m \exp tH' \) and \( tH \) respectively, \( t \) being \( \geq 1 \). We can interchange \( H \) and \( H' \) in this and obtain a second estimate. Combining these two estimates we get the following result: given any integer \( n \geq 1 \), we can find con-
stated $A_n > 0$, $r_n \geq 0$ such that for all $t \geq 1$,
$$\|\sum_{q \in L^+} \sum_{i \leq p \leq r} \exp(t(s_p^{-1}\lambda - q)(H + H')) \Delta_{i,p}^{[\kappa]}\| \leq A_n(1 + t)^r e^{-2(1 + 1)(\beta(H) + \beta(H'))}.$$
In deriving this we should remember (2.10.3), the fact that $\Theta_{\gamma,\rho}^{[\kappa]}(\lambda: m: t\bar{H})$ is independent of $t$ ($\bar{H} \in \mathfrak{a}^+_\rho$), and replace $e^t \Phi_\lambda^n$ by the expression for it in terms of $\Theta$ and $\Theta_{\gamma,\rho}^{[\kappa]}$.

On the other hand, we see from Lemma 2.10.1 that given any integer $n \geq 1$, there are constants $B_n > 0$ and $s_n \geq 0$ such that $\sum_{i \leq p \leq r} \| \Delta_{i,p}^{[\kappa]} \| \leq B_n o(q)^{s_n}$ for all $q \in L^+$. So, if $k \geq 1$ is any integer, there is a constant $B_{n,k} > 0$ such that for all $t \geq 1$,
$$\sum_{q \in L^+, o(q) \geq 2k} \sum_{i \leq p \leq r} e^{-tq(H + H')} \| \Delta_{i,p}^{[\kappa]} \| \leq B_{n,k} e^{-2(1 + 1)(\beta(H) + \beta(H'))},$$
by Lemma 1 of the Appendix. Combining this with the earlier estimate we get the following result. Let $k \geq 1$ be any integer; then for any integer $n > k$, we can select constants $A = A_{n,k} > 0$, $r = r_{n,k} \geq 0$ such that for all $t \geq 1$,
$$\|\sum_{q \in L^+, o(q) \leq 2k} \sum_{i \leq p \leq r} \exp(t(s_p^{-1}\lambda - q)(H + H')) \Delta_{i,p}^{[\kappa]}\| \leq A(1 + t)^r e^{-2(1 + 1)(\beta(H) + \beta(H'))}.$$
We now observe that the numbers $q(H + H')$ ($q \in L^+$) are all positive and distinct. Moreover the numbers $(s_p^{-1}\lambda)(H + H')$ ($1 \leq p \leq r$) are pure imaginary and distinct. An elementary argument (cf. Lemma 56, [3]) now enables us to conclude from the last estimate the following result: fix $n, k$ with $n > k$; then $\Delta_{i,p}^{[\kappa]} = 0$ for $1 \leq p \leq r$ and for all $q \in L^+$ for which $o(q) \leq 2k$ and $q(H + H') < (2k + 1)(\beta(H) + \beta(H'))$. But, if $o(q) \leq k$,
$$q(H + H') \leq k \max_{1 \leq j \leq \rho} (\alpha_j(H) + \alpha_j(H')) < 2k(\beta(H) + \beta(H')).$$
So $\Delta_{i,p}^{[\kappa]} = 0$ if $1 \leq p \leq r$, $o(q) \leq k$, $n > k$. Since $k$ is arbitrary, the lemma is proved.

Lemma 2.10.3 permits us to write $\Theta_{\gamma,\rho}^{[\kappa]}(\lambda: m)$ instead of $\Theta_{\gamma,\rho}^{[\kappa]}(\lambda: m: H)$. We put $\Theta_{\gamma,\rho}(\lambda: m) = \Theta_{\gamma,\rho}^{[1]}(\lambda: m)$.

**Lemma 2.10.4.** For all $(\lambda, m) \in \mathcal{T} \times M_{10}$, $1 \leq p \leq r$, $q \in L^+$, $\eta \in \mathcal{M}_{10}$, we have $\Theta_{\gamma,\rho}^{[\kappa]}(\lambda: m) = \Theta_{\gamma,\rho}(\lambda: m; \eta)$.

**Proof.** The proof combines the techniques of Lemmas 2.2.2 and 2.10.3. It is enough to prove that $\Theta_{\gamma,\rho}^{[\kappa]}(\lambda: m: \bar{H}) = \Theta_{\gamma,\rho}^{[\kappa]}(\lambda: m: X: \bar{H})$ for all $(\lambda, m) \in \mathcal{T} \times M_{10}$, $\eta \in \mathcal{M}_{10}$, $X \in M_{10}$, for $q$, $p$ as above and $\bar{H} \in \mathfrak{a}^+_\rho$ suitably chosen. Fix $\lambda \in \mathcal{T}$, choose $\bar{H} \in \mathfrak{a}^+_\rho$ so that (i) $\beta(\bar{H}) > \max(1, 1/2 \max_{1 \leq j \leq \rho} \alpha_j(\bar{H}))$ (ii) $q \mapsto q(\bar{H})$ is injective on $L^+$ (iii) the numbers $(s_p^{-1}\lambda)(\bar{H})$ ($1 \leq p \leq r$) are distinct. In the proof to follow, $\lambda, m \in M_{10}$, $X, \eta, \bar{H}$ will be fixed and all the constants which enter the estimates are allowed to depend on these.
We select \( t_0 > 0 \) such that \( m \exp vX \in M_{10} \) for \( 0 \leq v \leq t_0 \). We then replace in the estimate (2.7.1), \( m, H, \) and \( \gamma \) by \( m \exp vX, tH, \) and \( X\gamma \) respectively, with \( 0 \leq v \leq t_0 \) and \( t \geq 1 \). An elementary argument based on Lemma 2.10.1 makes it clear that the resulting inequalities can be integrated term by term. On the other hand, we may use (2.7.1) to estimate \([\Phi(\lambda; m \exp vX \exp tH; \gamma)]^*\). Combining these two estimates we get the following result. Given any integer \( n \geq 1 \), there are constants \( A_n > 0 \) and \( r_n \geq 0 \) such that for all \( t \geq 1, 0 \leq v \leq t_0 \),

\[
\left\| \sum_{q \in L^+} \sum_{1 \leq p \leq r} \exp(t(s_{p}^{-1}\lambda - q)(H))\Psi_{q,p}^{(n)}(v) \right\| \leq A_n(1 + t)^{n} e^{-2(n+1)t\beta(H)}
\]

where, for \( q \in L^+ \) and \( 1 \leq p \leq r \),

\[
\Psi_{q,p}^{(n)}(v) = \int_{0}^{\infty} \Theta_{q,p}^{(n)}(\lambda; m \exp uX; H)du - [\Theta_{q,p}^{(n)}\lambda; m \exp uX; H]^*.
\]

We may now argue as in the proof of the previous lemma to conclude that

\[
\Psi_{q,p}^{(n)}(v) = 0 \text{ for } 1 \leq p \leq r, n > o(q), 0 \leq v \leq t_0. \text{ This implies that}
\]

\[
\Theta_{q,p}^{(n)}(\lambda; m; H) = \Theta_{q,p}^{(n)}(\lambda; m; X; H).
\]

This proves the lemma.

**Lemma 2.10.5**. For \( q \in L^+ \), let \( \Omega_{q;j} \) be the entries of the matrix \( \Omega_{q}^{(1)} \) (cf. § 2.9). Write \( \Theta_{q} \) for \( \Theta_{q,1} \). Then, for \( q \in L^+ \), \( 1 \leq p \leq r \), and \( \lambda, m \in S' \times M_{10} \), \( \Theta_{q,p}(\lambda; m) = \Theta_{q}(s_{p}^{-1}\lambda; m) \). Moreover, there is a finite set \( M_{q} \subseteq M_{10} \) such that the components \( \Theta_{q,p}(\cdot; \cdot)_{j} \) of \( \Theta_{q} \) are, for \( q \in L^+ \), \( \lambda, m \in S' \times M_{10} \), given by

\[
\Theta_{q}(\lambda; m)_{j} = \omega(\lambda) \sum_{\mu \in M_{q}} \sum_{1 \leq j \leq r} \Omega_{q;j}^{(1)}(\mu; \lambda; m; H)u_{j}(\lambda)\theta(\lambda; m; \mu); \text{ here } \omega, u_{j}, \theta \text{ are as in } \S\S 2.3, 2.4, \text{ and } H \in a_{+} \text{ is arbitrary.}
\]

**Proof**. The proofs of these assertions follow at once from (2.4.4) and (2.10.2) and the fact that \( \Omega_{q;j} \) are \( w \)-invariant.

Let \( \Xi \) be the linear space of all functions on \( S' \times M_{10} \) of the form \( \lambda, m \mapsto \theta(\lambda; m; \mu) \) (\( \mu \in M_{10} \)). For any \( q \in L^+ \) let \( \Xi_q \) be the linear space spanned by all functions on \( S' \times M_{10} \) of the form \( \lambda, m \mapsto g(m)\theta(\lambda; m; \mu) \) (\( g \in G_q, \mu \in M_{10} \); cf. Lemma 2.8.4 for definition of \( G_q \)). Since \( M_{10} \) is an open subset of \( M_{10} \) meeting each of its connected components, it follows that the elements of \( \Xi \) and \( \Xi_q \) are determined by their restrictions to \( S' \times M_{10}^+ \). It is clear that

\[
f(\lambda; m \exp H) = e^{q(\mu)}f(\lambda; m) \quad \text{and} \quad g(\lambda; m \exp H) = e^{q(-\gamma(\mu))}g(\lambda; m)
\]

for \( f \in \Xi, g \in \Xi_q \), where \( \lambda \in S' \), \( m \in M_{10} \), and \( H \in a_0 \).

**Lemma 2.10.6**. Fix \( \bar{H} \in a_0^+ \). For \( q \in L^+ \), let \( v_q \) be the \( \Xi \)-invariant polynomial on \( S \) defined by \( v_q(\lambda) = \prod_{q \in M_{10}} \h_q(s_{q}^{-1}\lambda; \bar{H}), \h_q \) being as in (2.9.4). Then, there is an \( \varepsilon_0 > 0 \), such that \( \inf_{s \in S' \times M_{10}} |v_q(\lambda)| > 0 \) for each \( q \in L^+ \), where \( S' \times M_{10} = 0 \) for any function \( f \).
\{ \lambda : \lambda \in F, |\lambda_R| < \varepsilon_0 \}; in particular, 1/\nu_q^\ast is holomorphic on F(\varepsilon_0) for each \q \in L^+. Moreover, we can select elements \theta_{q,j} \in G_q such that one has, for all 
(\lambda : m) \in F'_r \times M_{10},
\Theta_q(\lambda : m) = \omega(\lambda)\nu_q(\lambda)^{-1}\theta_{q,j}(\lambda : m)
for all q \in L^+, 1 \leq j \leq r.

Proof. Choose \varepsilon_0 so that F(\varepsilon_0) \subset F_H \subset F_H where F_H is as in (2.8.8). Since |
(s'\lambda - s'\lambda + q')(H) | \geq (1/2) \beta(H) \text{ for } s', s' \in 10, q' \in L^+, \text{ and } \lambda \in F_H, \text{ the assertions concerning } \nu_q \text{ follow immediately. Fix } q \in L^+. \text{ It follows from Lemma 2.9.3 that for each } \mu \in M_{10}, \text{ one can find an integer } a_\mu \geq 1, \text{ functions } g_{\mu;j,i:a_\mu} \in G_q \text{ and polynomials } r_{\mu;j,i:a_\mu} \text{ on } F \text{ such that } \nu_q(\lambda) \Omega_{\mu;j,i}(\mu : \lambda : m \approx H) \text{ can be expressed as the sum }

\sum_{1 \leq a \leq a_\mu} g_{\mu;j,i:a_\mu}(m)r_{\mu;j,i:a_\mu}(\lambda)

for 1 \leq j, l \leq r, (\lambda, m) \in F'_r \times M_{10}, \mu \in M_{10}. \text{ We may obviously assume that the } r_{\mu;j,i:a_\mu} \text{ are } m\text{-invariant. So } u_r r_{\mu;j,i:a_\mu} \text{ is } m\text{-invariant, and we can find } \xi_{\mu;j,i:a_\mu} \in G_q \text{ such that } \nu_{m_0}(\xi_{\mu;j,i:a_\mu}) = u_r r_{\mu;j,i:a_\mu}. \text{ Substituting in (2.10.4) we get the required formulae for } \Theta_q(\cdot : \cdot)_{j, q} \text{ where } \theta_{q,j} \text{ is defined for } (\lambda, m) \in F'_r \times M_{10} \text{ by }

\theta_{q,j}(\lambda : m) = \sum_{\mu} \sum_{1 \leq i \leq r} \sum_{1 \leq a \leq a_\mu} g_{\mu;j,i:a_\mu}(m)\theta(\lambda : m; \mu \xi_{\mu;j,i:a_\mu}).

2.11. Asymptotic expansions for \varphi. \text{ We shall now put together all the preceding results to obtain the fundamental asymptotic expansions of this paper. We need a final lemma. For uniformity of notation we write } \Theta_0(\lambda : m) = F(\lambda)\Theta(\lambda : m), F \text{ being as in } \S 2.4.

\textbf{Lemma 2.11.1.} \text{ Let } k \geq 0 \text{ be any integer and } \eta \in M_{10}. \text{ Then we can find constants } A_{k,\eta} > 0, r_{k,\eta} \geq 0 \text{ such that for all } \lambda \in F'_r, m \in M_{10}^+, \text{ and } H \in a_\eta^+ \text{ with } \beta(H) \geq 1,

(2.11.1) \quad \| \Phi(\lambda : m \exp H; \eta) - \sum_{q \in G_q} e^{-\gamma(\lambda)} \sum_{1 \leq j \leq r} \exp((s_q^\ast \lambda)(H)) \Theta_q(s_q^\ast \lambda : m; \eta) \| 
\leq A_{k,\eta} (1 + \| H \|)(1 + \| \lambda \|)^{r_{k,\eta}} \gamma(m)^2(1 + \sigma(m))^k \Xi_0(m)^{-\beta(H)}.

Proof. \text{ Let } n \geq 0 \text{ be an integer, } \eta \in M_{10}. \text{ Then, by (2.7.1) and the expression for } e^\ast \Phi_q^\ast \text{ obtained in } \S 2.9, \text{ it follows that there are constants } C_{\eta,\gamma} > 0, a_{\eta,\gamma} \geq 0 \text{ with the following property: for all } \lambda \in F'_r, m \in M_{10}^+, \text{ and } H \in a_\gamma^+ \text{ with } \beta(H) \geq 1, \text{ and writing } m_H \text{ for } m \exp H,

\| \Phi(\lambda : m_H; \eta) - \Theta(\lambda : m_H; \eta) - \sum_{\mu \in M_{10}} \sum_{q \in L^+} e^{-\gamma(\mu)} \Theta_q(s_q^\ast \lambda)(\mu : \lambda : m : H) \Theta(\lambda : m_H; \mu) \| 
\leq C_{\eta,\gamma} (1 + \| H \|)(1 + \| \lambda \|)^{r_{\eta,\gamma}} \gamma(m)^{2(n+1)} \Xi_0(m)^{-\beta(H)}(1 + \sigma(m))^n \Xi_0(m).

Let } k \geq 0 \text{ be any integer and let } E_k \text{ denote that part of the sum over } L^+.
occurring above for which \( o(q) > k \). We may now use Lemma 2.9.2 in conjunction with Lemma 1 of the Appendix to estimate \( E_k \). If we recall that there is a finite set \( M_{n,r} \) for which \( \Omega^{[n,r]}(\mu : \cdots : \cdots) = 0 \) for \( \mu \in M_{n,r} \), we obtain the following result: there are constants \( C = C_{n,k,r} > 0, b' = b_{n,k,r} \geq 0 \) such that for all \( \lambda \in \mathcal{F}_T, m \in M_{10}^+, \) and \( H \in a_0^+ \) with \( \beta(H) \geq 1 \),

\[
E_k \leq C[(1 + ||H||)(1 + ||\lambda||)]^{r' \gamma(m)^2(1 + \sigma(m))} \Xi_0(m) e^{-(k+1)\beta(H)}.
\]

If we combine this with the earlier estimate, for \( n = 2k \), we get the following result. Let \( k \geq 0 \) be an integer, \( \gamma \in \mathbb{M}_{10} \). Then there are constants \( A_{k,r} > 0, r_{k,r} \geq 0 \) such that for all \( \lambda \in \mathcal{F}_T, m \in M_{10}^+, \) and \( H \in a_0^+ \) with \( \beta(H) \geq 1 \), writing \( m_H \) for \( m \exp H \),

\[
||\Phi(\lambda : m_H; \gamma) - \Theta(\lambda : m_H; \gamma) - \sum_{\mu \in \mathbb{M}_{10}} \sum_{q \in \mathbb{L}^+, \phi(q) \leq k} e^{-q(H)} \Omega^{[q]}(\mu : \lambda : m : H) \Theta(\lambda : m_H; \mu)|| \leq A_{k,r}[(1 + ||H||)(1 + ||\lambda||)]^{r' \gamma(m)^2(1 + \sigma(m))} \Xi_0(m) e^{-(k+1)\beta(H)}.
\]

We shall now evaluate the terms within the summation in the above inequality. Fix \( \lambda \in \mathcal{F}_T, m \in M_{10}^+, H \in a_0^+ \). Then, for \( q \in \mathbb{L}^+ \),

\[
\sum_{\mu \in \mathbb{M}_{10}} \Omega^{[q]}(\mu : \lambda : m : H) \Theta(\lambda : m_H; \mu)
\]

reduces to

\[
\sum_{s \in \mathbb{Q}^{\mathbb{R}}} \sum_{\mu \in \mathbb{M}_{10}} \exp((s_p^{-1}\lambda)(H)) \Omega^{[q]}(\mu : \lambda : m : H) F(s_p^{-1}\lambda) \Theta(\lambda : m; \mu)
\]

which is just \( \sum_{s \in \mathbb{Q}^{\mathbb{R}}} \exp((s_p^{-1}\lambda)(H)) \Theta(s_p^{-1}\lambda : m; \gamma) \), in view of Lemmas 2.10.3, 2.10.4, and 2.10.5. Furthermore,

\[
\Theta(\lambda : m \exp H; \gamma) = \sum_{s \in \mathbb{Q}^{\mathbb{R}}} \exp((s_p^{-1}\lambda)(H)) \Theta(s_p^{-1}\lambda : m; \gamma) .
\]

Lemma 2.11.1 now follows at once.

It is clear from Lemma 2.10.6 that the function \( (\lambda, m) \mapsto \omega(\lambda)^{-1} \Theta_q(\lambda : m) \) extends to an analytic function on \( \mathcal{F}_T \times M_{10} \). We write \( \tilde{\psi}_q \) for this extension. Thus

\[
(2.11.2) \quad \Theta_q(\lambda : m) = \omega(\lambda) \tilde{\psi}_q(\lambda : m) \quad (\lambda, m) \in \mathcal{F}_T \times M_{10} .
\]

The following is then the main result of this chapter.

**Theorem 2.11.2.** The functions \( m \mapsto \tilde{\psi}_q(\lambda : m) \) are spherical on \( M_{10} \) for each \( \lambda \in \mathcal{F}_T, q \in \mathbb{L}^+ \). If \( \Xi_q \) and \( v_q \) \( (q \in \mathbb{L}^+) \) are as in Lemma 2.10.6, then there are elements \( \theta_q \in \Xi_q \) such that \( \theta_q(\lambda : m) = v_q(\lambda) \tilde{\psi}_q(\lambda : m) \) for all \( (\lambda, m) \in \mathcal{F}_T \times M_{10} \), \( q \in \mathbb{L}^+ \). In particular, \( \tilde{\psi}_q(\lambda : m \exp H) = e^{2q^{-1}(H)} \tilde{\psi}_q(\lambda : m) \).

Let \( k \geq 0 \) be an integer and \( \gamma \in \mathbb{M}_{10} \). Then we can find constants \( A_{k,r} > 0, r_{k,r} \geq 0 \) such that for all \( \lambda \in \mathcal{F}_T, m \in M_{10}^+, \) and \( H \in a_0^+ \) with \( \beta(H) \geq 1 \),
\[(2.11.3) \quad \varphi(\lambda; m \exp H; \gamma \circ d_c) - \sum_{s \in \mathbb{Z}^r \not\subset 0} \omega(s^{-1}\lambda) \exp((s^{-1}\lambda)(H)) \theta(s^{-1}\lambda; m; \gamma) - \sum_{q \in \mathbb{L}^+ \cup \{0\} \not\subset 0} e^{-q(H)} \sum_{s \in \mathbb{Z}^r \not\subset 0} \omega(s^{-1}\lambda) \exp((s^{-1}\lambda)(H)) \psi_q(s^{-1}\lambda; m; \gamma) \leq A_{k,\gamma}(1 + ||H||)(1 + ||\lambda||)^{\gamma(k)\gamma(m)^2(1 + \sigma(m))} \Xi_\varphi(m) e^{-(k+1)\beta(H)}.
\]

**Proof.** Excepting (2.11.3), the other statements are either obvious or have already been proved. (2.11.3) follows from (2.11.1) since the terms which enter above are the first components of the vectors which enter (2.11.1). This proves the theorem.

3. The spherical transform on \( \mathcal{S}(G) \)

3.1. **Summary.** In this chapter we shall use the asymptotic expansions of Chapter 2 to study the spherical transforms of rapidly decreasing functions on \( G \).

In §3.2 we introduce the tube domains \( \mathcal{T}^r \subseteq \mathcal{T} \). The corresponding domains for \( *\mathcal{T} \) are denoted by \( *\mathcal{T}^r \). The main result of this section is Lemma 3.2.6 which relates \( \mathcal{T}^r \) and \( *\mathcal{T}^r \). In §3.3 we examine some of the properties of \( c \) and \( c_0 \) considered as meromorphic functions on \( \mathcal{T} \). The function spaces \( \mathcal{Z}(\mathcal{T}^r) \) and \( \widetilde{\mathcal{Z}}(\mathcal{T}^r) \) are defined in §3.4. The main result here is Lemma 3.4.3. It sets up, using the relation between \( \mathcal{T}^r \) and \( *\mathcal{T}^r \), natural maps \( a \rightarrow A_{a,H} \) of \( \mathcal{Z}(\mathcal{T}^r) \) into \( \mathcal{Z}(*\mathcal{T}^r) \) \((H \in c_0)\), which are continuous, the continuity being uniform and rapidly decreasing in \( H \) in a suitable sense. This lemma plays an important role later on in the inductive step of proving that the wave packets (1.1.1) have the required decay at infinity on \( G \) for \( a \in \widetilde{\mathcal{Z}}(\mathcal{T}^r) \).

The spaces \( \mathcal{S}(G) \) \((0 < p < 2)\) are then introduced and we prove (Theorem 3.5.5) that the spherical transform is a continuous map of \( \mathcal{S}(G) \) into \( \widetilde{\mathcal{Z}}(\mathcal{T}^r) \) \((\varepsilon = 2/p - 1)\).

§§3.6 and 3.7 are devoted to an examination of the coefficients \( \psi_q \) of the asymptotic series for \( \varphi(\lambda; \cdot) \), when they are considered as functions of \( \lambda \).

We first obtain a formula for \( \psi_q(\lambda; h) \) for \( h \in A^+ \) (Lemma 3.6.2). We then prove the existence of open sets \( U \subset \mathcal{F}_R \) such that for each \( h \in A^+ \), \( \lambda \mapsto \pi_0(\lambda)\psi_q(\lambda; h) \) \((\lambda \in \mathcal{F}_I)\) extends to a slowly increasing holomorphic function on the tube \( T(U) \) (Lemmas 3.7.2 and 3.7.3). The open sets \( U \) are one-sided in a natural sense; they are of the form

\[ U = \{ \lambda \in \mathbb{C}^r : *\lambda \in *\mathcal{F}_R, ||\lambda|| < \delta, \lambda \in \mathcal{F}_R, (\alpha_i, \alpha_j) < \delta, 1 \leq j \leq r \} \]

for sufficiently small \( \delta > 0 \).

After some preliminary lemmas in §3.8 we come to §3.9 where we study the wave packets (1.1.1) for \( a \in \widetilde{\mathcal{Z}}(\mathcal{T}^r) \). The main result here is Lemma 3.9.6 which asserts that \( \varphi_a \in \mathcal{S}(G) \) and that \( a \mapsto \varphi_a \) is a continuous map of \( \widetilde{\mathcal{Z}}(\mathcal{T}^r) \) into \( \mathcal{S}(G) \). The proof is by induction on \( \dim(G) \) and Lemma 3.9.5 contains the main inductive step. We fix \( H_0 \) in \( \text{Cl}(a^+) \), \( H_0 \neq 0 \), and use the work of
Chapter 2 to approximate $d_\theta(h \exp H) \varphi_\theta(h \exp H)$ ($h \in A^+, H \in a^+_\theta$) by
\[
\int_{F_\theta} \theta(\lambda; h) a_\theta(\lambda) \tilde{\xi}(\lambda) e^{iH} d\beta_\theta(\lambda) + \sum_{q \in \mathbb{L}^{+}, a_q \subseteq k} \int_{F_\theta} \varphi_q(\lambda; h) a_\theta(\lambda) \tilde{\xi}(\lambda) e^{iH} d\beta_\theta(\lambda) .
\]
Here $a_\theta = a \cdot \pi/\pi_\theta$, $\tilde{\xi}$ is defined as in Lemma 3.3.2, $d\beta_\theta$ is the Plancherel measure for $M_{10}/K \cap M_{10}$, and $k \geq 0$ is an integer. Theorem 2.11.2 allows us to estimate the error in this approximation rather well. By the use of Lemmas 3.2.6 and 3.4.3 the first integral can be transformed into a suitable wave packet of elementary spherical functions on $M_{10}$. The required estimates for the first integral now follow from the induction hypothesis. The integrals of the $\varphi_q$ are handled similarly: there is a simplifying feature, namely, that the needed estimates already follow from the $F^*$ theory of $M_{10}$ and one does not require the induction hypothesis. We then construct a compact neighborhood $W_{H_0}$ of $H_0$ in $\text{Cl}(a^+)$ such that if $S[W_{H_0}]$ is the corresponding sectorial region (cf. (3.9.5)), the above estimates for suitable choices of $h$ and $H$ lead to the following result: given $u \in \mathcal{M}_{10}$ and $l \geq 0$, there is a continuous seminorm $\zeta_{u,l}$ on $\tilde{\mathcal{Z}}(F^*)$ such that
\[
|\varphi_\theta(h; u)| \leq \zeta_{u,l}(a) e^{-l/p} e^{l \log k} (1 + \|\log h\|)^{-l}
\]
for all $h \in S[W_{H_0}]$, $a \in \tilde{\mathcal{Z}}(F^*)$. Since $\text{Cl}(a^+) \setminus \{0\}$ can be covered by finitely many such sectorial regions, the main results of the section follow quickly. These results are then put together in §3.10 to yield the main result of this paper (Theorem 3.10.1). We conclude the section with some remarks which complement the foregoing theory from various points of view. There is an appendix which contains some results from analysis that are used often in the paper.

3.2. The tubes $F^*$, $*F^*$. We now go back to $g$. For any $\varepsilon > 0$ we define $F^*$ and $F^*_R$ by setting $F^*_R = F^* \cap F_R$ and
\[
F^* = \{ \lambda : \lambda \in F^*, |\text{Re}(s\lambda)(H)| \leq \varepsilon \rho(H) \ \forall H \in a^+, s \in \mathfrak{m} \} .
\]
Then $F^*$ is a tube in $F$ based on $F^*_R \cdot F^* \subseteq F^*$, $F^*_R \subseteq F^*$ for $0 < \varepsilon' < \varepsilon$, and $0 \in F^*_R \cdot F^*_R = -F^*_R$ is a compact, convex, $\mathfrak{m}$-invariant subset of $F_R$. For any $a > 0$, $aF^* = F^*_a$. We fix $\varepsilon > 0$ throughout this section.

Lemma 3.2.1. Let $\lambda \in F^*_R$. Then $\lambda \in F^*$ if and only if $(s\lambda)(H) \leq \varepsilon \rho(H)$ ($H \in a^+, s \in \mathfrak{m}$). If $\lambda \in F^*_R$ is such that $H_2 \in \text{Cl}(a^+)$, then $\lambda \in F^*$ if and only if $\lambda(H) \leq \varepsilon \rho(H)$ for all $H \in a^+$.

Proof. Let $s_0 \in \mathfrak{m}$ be such that $s_0 \Delta^+ = -\Delta^+$. Then $s_0^2 = 1$, $s_0 \rho = -\rho$, and $H \mapsto -s_0 H$ is a bijection of $a^+$ onto itself. If $\lambda \in F^*_R$ satisfies the first set of con-
ditions for \( s \in m, H \in a^+ \), as \((s\lambda)(H) = -(s,s\lambda)(-s,H)\), we have \((s\lambda)(H) \geq -\varepsilon \rho(s,H) = -\varepsilon \rho(H)\). Suppose \( \lambda \in \mathcal{F}_K \) is such that \( H \subseteq \mathrm{Cl}(a^+) \) and the second set of conditions is satisfied. Then, for \( H \in a^+ \) and \( s \in m \),
\[
(s\lambda)(H) = (sH, H) \leq (H, H) = \lambda(H)
\]
by Corollary 1 to Lemma 35 of [3]; so \((s\lambda)(H) \leq \varepsilon \rho(H)\).

**Lemma 3.2.2.** We have \( \text{Int} \mathcal{F}' = \bigcup_{\delta < \epsilon', < \mathcal{F}'}, \text{Int} \) denoting interior.

**Proof.** Let us write \( \text{Int} \mathcal{F}'_K \) for the interior of \( \mathcal{F}'_K \) considered as a subset of \( \mathcal{F}_K \). It is enough to prove that \( \text{Int} \mathcal{F}'_K = \bigcup_{\delta < \epsilon'} \mathcal{F}'_K \). We shall prove first that \( 0 \subseteq \text{Int} \mathcal{F}'_K \). Since \( \mathcal{F}'_K \) is convex, this reduces to proving that for any \( \lambda \in \mathcal{F}_K \), there exists a \( \tau > 0 \) such that \( \tau \lambda \in \mathcal{F}'_K \). Fix \( \lambda \in \mathcal{F}_K \). Let \( S^+ \) be the compact set of all \( H \in \text{Cl}(a^+) \) with \( \| H \| = 1 \). It is obvious that for some constant \( a > 0 \), \( \rho(H) \geq a > 0 \) for all \( H \in S^+ \). Then \( \lambda H \in \mathcal{F}'_K \) for some \( \tau > 0 \).

Now, if \( C \) is a convex subset of \( \mathcal{F}_K \), \( \lambda \in \text{Int} C \), and \( \lambda' \in C \), then \( a\lambda + (1 - a)\lambda' \in \text{Int} C \) for \( 0 < a \leq 1 \). Let \( 0 < \varepsilon' < \varepsilon \) and \( \lambda \in \mathcal{F}'_K \). If \( \tau = \varepsilon / \varepsilon' > 1 \), then \( \tau \lambda \in \mathcal{F}_K \) and so \( \lambda = \tau^{-1}(\tau \lambda) \in \text{Int} \mathcal{F}_K \). Suppose \( \lambda \in \text{Int} \mathcal{F}_K \). Then we can select \( \tau > 1 \) such that \( \tau \lambda \in \mathcal{F}_K \). If \( \varepsilon' = \varepsilon \tau^{-1} \), then \( 0 < \varepsilon' < \varepsilon \) and \( \lambda \in \mathcal{F}'_K \). This proves the lemma.

Let \( H_0 \neq 0 \) be any element of \( \text{Cl}(a^+) \). We shall fix it for the rest of this section. We use the notation of §§ 2.1 and 2.1. Define \( H_j \) as in § 2.1 and
\[
\beta_j \in a^* \quad \text{by} \quad \beta_j(H) = (H_j, H) \quad (H \in a, 1 \leq j \leq \nu).
\]
The \( \beta_j \) vanish on \( *a = m_a \cap a \). If \( \mu \in a^* \) and \( \mu \left| *a = 0 \right. \), then
\[
\mu = \sum_{1 \leq j \leq \nu} \mu_j \beta_j.
\]

**Lemma 3.2.3.** We have \( (H_i, H_j) \geq 0 \) for \( 1 \leq i, j \leq \nu \). If we write \( \beta_i = \sum_{1 \leq j \leq \nu} m_{ij} \alpha_j \), then \( m_{ij} \geq 0 \) for \( 1 \leq i \leq \nu \), \( 1 \leq j \leq d \).

**Proof.** Since \( H_j \in \text{Cl}(a^+) \) for \( 1 \leq j \leq \nu \), the first assertion follows from Lemma 35 of [3]. By the same token, \( \beta_i(H) = (H_i, H) \geq 0 \) for \( 1 \leq i \leq \nu \), \( H \in a^+ \). This implies the second assertion.

**Lemma 3.2.4.** We have \( (1) \ 0 \leq \rho_i(H) \leq \rho(H) \) for all \( H \in a^+ \) \( (2) \ \rho_0, \alpha_j \geq (\rho, \alpha_j) > 0 \) for \( 1 \leq j \leq \nu \) \( (3) \ H_{\rho_0} \in a^+_+ \) \( (4) \ (\ast a)^+ \) is the set of all \( H' \in \ast a \) such that \( \alpha_k(H') > 0 \) for \( \nu < k \leq d \), then \( -H_{(\alpha_j, \ast a)} \in \text{Cl}(\ast a)^+ \) for \( 1 \leq j \leq \nu \).

**Proof.** Since \( \rho = \rho_0 + *\rho \), we have (1). If \( 1 \leq j \leq \nu < k \leq d \), then \( (\alpha_k, \alpha_j) \leq 0 \) so that \( (\ast \rho, \alpha_j) \leq 0 \). This proves (2); since \( \rho_0 \left| *a = 0 \right. \) and \( (\rho_0, \alpha_j) \geq (\rho, \alpha_j) > 0 \), we have (3) also. With \( j, k \) as above, \( 0 \geq (\alpha_k, \alpha_j) = \alpha_k(H_{\alpha_j}) = \alpha_k(H_{\alpha_j, \ast a}) \) since \( \alpha_k \left| a_o = 0 \right. \) and \( H_{(\alpha_j, \ast a)} \) is the projection of \( H_{\alpha_j} \) on \( *a, \mod a_o \). This proves (4).

We now introduce two mutually orthogonal subspaces of \( \mathcal{F} \) of importance for us in the sequel. Let
\( \forall F = \{ \lambda : \lambda \in \mathcal{F}, \lambda | a_0 = 0 \}, \quad \forall^* F = \{ \lambda : \lambda \in \mathcal{F}, \lambda | a = 0 \} \),

\begin{align*}
\forall^* F_R &= \forall F \cap \mathcal{F}_R, & \forall^* F_I &= \forall F \cap \mathcal{F}_I, \\
\forall^* F_R &= \forall^* F \cap \mathcal{F}_R, & \forall^* F_I &= \forall^* F \cap \mathcal{F}_I. 
\end{align*}

We also define \( \forall^* F_R \) and \( \forall^* F_I \) by \( \forall^* F_r = \forall F \cap \mathcal{F}_r \) and

\begin{equation}
\forall^* F_I = \{ \lambda : \lambda \in \mathcal{F}, | \text{Re}(s'\lambda)(H)| \leq \varepsilon^* \rho(H) \forall H \in \{a^+, a'\}, s' \in \mathcal{W} \}. 
\end{equation}

**Lemma 3.2.5.** If \( \lambda, \lambda' \in \forall^* F_R \) are such that for \( 1 \leq j \leq \nu, 0 \leq (\lambda, \alpha_j) \leq \varepsilon(\rho, \alpha_j) \) and \( 0 \leq (\lambda', \alpha_j) < \varepsilon(\rho, \alpha_j) \), then \( \lambda \in \mathcal{F}_R \) and \( \lambda' \in \mathcal{F}'_R \). In particular, \( \varepsilon \rho_0 \in \mathcal{F}_R \) while \( \varepsilon' \rho_0 \in \mathcal{F}'_R \) for \( 0 < \varepsilon' < \varepsilon \).

**Proof.** Clearly \( H_i \in \text{Cl}(a^+) \) and so we need only prove that \( \lambda(H) \leq \varepsilon \rho(H) \) for all \( H \in a^+ \) (Lemma 3.2.1). Fix \( H \in a^+ \). Then

\[ \lambda(H) = \sum_{i \leq j \leq \nu} (\lambda, \alpha_j) \beta_j(H) \leq \varepsilon \sum_{i \leq j \leq \nu} (\rho_0, \alpha_j) \beta_j(H) = \varepsilon \rho(H) \]

as \( \beta_j(H) \geq 0 \) for \( 1 \leq j \leq \nu \) by Lemma 3.2.3. So \( \lambda(H) \leq \varepsilon \rho_0(H) \leq \varepsilon \rho(H) \).

For \( \lambda', \lambda'' \), we can choose \( \varepsilon' \) with \( 0 < \varepsilon' < \varepsilon \) such that \( 0 \leq (\lambda', \alpha_j) \leq \varepsilon'(\rho_0, \alpha_j) \) for \( 1 \leq j \leq \nu \). Then \( \lambda' \in \mathcal{F}'_R \subseteq \mathcal{F}'_R \) (Lemma 3.2.2).

**Lemma 3.2.6.** Let \( \lambda, \lambda' \in \forall^* F_R \) be such that

\begin{equation}
\begin{align*}
\varepsilon[(\rho_0, \alpha_j) - (\rho, \alpha_j)] &\leq (\lambda', \alpha_j) \leq \varepsilon(\rho_0, \alpha_j), \\
\varepsilon[(\rho_0, \alpha_j) - (\rho, \alpha_j)] &\leq (\lambda', \alpha_j) < \varepsilon(\rho_0, \alpha_j). 
\end{align*}
\end{equation}

Then \( \forall^* F_R \) is \( \forall \subseteq \mathcal{F}_R \) and \( \text{Int} \forall^* F_R \) is \( \forall' \subseteq \mathcal{F}'_R \). For any \( \alpha \in \Delta^+ \) with \( \alpha = m_1 \alpha_1 + \cdots + m_d \alpha_d \), let \( \forall \alpha = m_1 \alpha_1 + \cdots + m_d \alpha_d \). Then, for \( \forall \lambda \in \mathcal{F}_R \), \( (\forall \lambda + \forall \alpha, \alpha) \geq \varepsilon(\rho, \forall \alpha) - (\varepsilon \rho_0, \forall \alpha) \geq 0 \).

**Proof.** For the first relation concerning \( \forall \lambda \) it suffices to prove that if \( \forall \lambda \in \mathcal{F}_R \), then \( \forall \lambda + \forall \alpha \in \mathcal{F}_R \). Select \( s' \in \mathcal{W} \) such that \( (s' \lambda, \alpha_k) \geq 0 \) for \( \nu < k \leq d \). Let \( \mu = s' \lambda, \lambda = \forall \lambda + \forall \alpha \). Then \( \lambda' = s' \lambda = \mu + \forall \alpha \) and it is enough to prove that \( \lambda' \in \mathcal{F}_R \). We claim that \( (\lambda', \alpha_p) \geq 0 \) for \( 1 \leq p \leq d \). For \( \nu < p \leq d \), \( (\lambda', \alpha_p) = (\mu, \alpha_p) \geq 0 \). On the other hand, let \( 1 \leq p \leq \nu \). As \( \mu \in \mathcal{F}_R \), by (4) of Lemma 3.2.4, \( \mu(-H(\forall \alpha_{1\cdots d})) \leq \varepsilon \rho(-H(\forall \alpha_{1\cdots d})) \) or \( (\mu, \alpha_p) \geq \varepsilon(\forall \rho, \alpha_p) \).

\[ (\lambda', \alpha_p) \geq \varepsilon(\forall \rho, \alpha_p) + \varepsilon[(\rho_0, \alpha_p) - (\rho, \alpha_p)] = 0. \]

This proves our claim. So, by Lemma 3.2.1, it remains only to check that \( \lambda'(H) \leq \varepsilon \rho(H) \) for all \( H \in a^+ \). Let \( H \in a^+ \). Then \( \mu(H) \leq \varepsilon \rho(H) \). On the other hand, \( \forall \lambda(H) \leq \varepsilon \rho_0(H) \) as we saw in the proof of Lemma 3.2.5. So \( \lambda'(H) \leq \varepsilon \rho(H) \). For proving the last statement let \( \alpha \in \Delta^+ \) also. Then \( \beta = s' \alpha \in \Delta^+ \) also, and \( \forall \beta = \forall \alpha \). Moreover, as \( (\mu, \alpha_k) \geq 0 \) for \( \nu < k \leq d \), \((\mu, \beta) \geq (\mu, \forall \beta) = (\mu, \forall \alpha) \) while \( (\forall \lambda, \beta) = (\forall \lambda, \forall \beta) = (\forall \lambda, \forall \alpha) \). Hence \( (\lambda, \alpha) = (\lambda', \beta) \geq (\mu, \forall \alpha) + (\forall \lambda, \forall \alpha) \). But we saw earlier that \( (\mu, \alpha_j) \geq \varepsilon(\forall \rho, \alpha_j) \) for \( 1 \leq j \leq \nu \), so that \( (\mu, \forall \alpha) \geq \varepsilon(\forall \rho, \forall \alpha) \). Hence
\( (\lambda, \alpha) \geq \varepsilon(\rho, \alpha) + (\lambda, \alpha) = \varepsilon(\rho, \alpha) - (\rho_0 - \lambda, \alpha) \).

Further \( \varepsilon(\rho, \alpha) - (\rho_0 - \lambda, \alpha) = (\lambda, \alpha) - \varepsilon(\rho_0 - \rho, \alpha) \geq 0 \).

It remains to prove that \( \text{Int} \ast F'_r \pm \alpha' \subseteq \text{Int} F'_r \). Exactly as before, it is enough to prove that if \( \lambda \in \text{Int} \ast F'_r \), then \( \lambda + \alpha' \in \text{Int} F'_r \). By Lemma 3.2.2 we can choose \( \varepsilon' \) such that \( 0 < \varepsilon' < \varepsilon \) and \( \lambda \in \ast F'_r \). Increasing \( \varepsilon' \) we may also assume that \( (\lambda', \alpha_j) \leq \varepsilon'(\rho_0, \alpha_j) \) for \( 1 \leq j \leq \nu \). Then \( \lambda + \alpha' \in F'_r \subseteq \text{Int} F'_r \).

**Corollary 3.2.7.** We have \( \ast F'_r \pm \epsilon \rho_0 \subseteq F'_r \). Moreover, for \( \lambda \in \ast F'_r \), \( \alpha \in \Delta^+ \setminus * \Delta^+ \), \( (\lambda + \varepsilon \rho_0, \alpha) \geq \varepsilon(\rho, \alpha) > 0 \).

**3.3. The function \( \xi \).** For any set \( Q \subseteq \Delta^+ \) and any \( \delta > 0 \) let us define
\[
U(Q, \delta) = \{ \lambda : \lambda \in F'_r, (\lambda, \alpha) < \delta \text{ for all } \alpha \in Q \}.
\]

\( U(Q, \delta) \) is open in \( F'_r \), connected and contains 0. We use the notation and results of the Appendix concerning tube domains and holomorphic functions on them.

**Lemma 3.3.1.** Let \( \Gamma \) denote the classical gamma function. Let \( \alpha \in \Delta^+, \alpha, a', b \) numbers \( > 0 \) and \( \delta = b^{-1} \min(a, a') \). Then the function
\[
\gamma_{a, a', b} : \lambda \mapsto \Gamma(a + b(\lambda, \alpha))|\Gamma(a' + b(\lambda, \alpha))|
\]
is well-defined on \( T(-U(\alpha, \delta)) \) and belongs to \( \xi(U(\alpha, \delta)) \).

**Proof.** Write \( U = U(\alpha, \delta), \gamma = \gamma_{a, a', b} \). If \( \lambda \in -U \), then \( b \Re(\lambda, \alpha) + a \) and \( b \Re(\lambda, \alpha) + a' \) are both \( > 0 \). So \( \gamma \) is well-defined and holomorphic on \( T(-U) \). Now it is easy to see from the properties of the classical gamma function (cf. Hille [9, p. 238]) that for \( B, a, a' > 0 \) fixed, we can choose constants \( c > 0, l \geq 0 \) such that \( |\Gamma(z + a, \alpha)/\Gamma(z + a')| \leq c(1 + |z|^l) \) for all \( z \in \mathbb{C} \) with \( 0 < \Re z < B \). It follows quickly from this that \( \gamma \in \xi(U(\alpha, \delta)) \).

We now consider an \( H_0 = \pm \) in \( \text{Cl}(a^+) \) and use earlier notation. Let \( c \) and \( \mathfrak{c} \), be as in Chapter 2. We write, for \( \lambda \in F'_r \), \( b(\lambda) = \pi(\lambda)c(\lambda) \), \( b_0(\lambda) = \pi_0(\lambda)c(\lambda) \). Then Gindikin and Karpelevic [1] determined the functions \( b \) and \( b_0 \) explicitly. To describe their result, let \( \Sigma \subseteq \Delta^+ \) be such that for each root \( \alpha \in \Delta^+ \), \( \Sigma \cap R \cdot \alpha \) has exactly one element. Let \( \ast \Sigma = \Sigma \cap * \Delta^+ \). For any \( \alpha \in \Delta^+ \) let \( \mathcal{P}_\alpha \) be the set of all functions of the form \( \gamma_{a, a', b}^{\alpha}(a, a', b > 0) \). Then for suitable nonzero constants \( c, c_0 \) and elements \( f_{a, j} \in \mathcal{P}_\alpha \), \( \alpha \in \Sigma, 1 \leq j \leq m(\alpha) \),
\[
\mathbf{b} = c \prod_{\alpha \in \Sigma} \prod_{1 \leq j \leq m(\alpha)} f_{a, j}, \quad \mathbf{b}_0 = c_0 \prod_{\alpha \in \Sigma} \prod_{1 \leq j \leq m(\alpha)} f_{a, j}
\]
on \( F'_r \). We use (3.3.2) to extend \( \mathbf{b} \) and \( \mathbf{b}_0 \) as meromorphic functions on \( \mathcal{F} \). The following lemma and its corollary are then obvious (cf. § 2.4).

**Lemma 3.3.2.** There exists a \( \delta_0 > 0 \) such that (i) \( \mathbf{b} \) and \( \mathbf{b}^{-1} \) (resp. \( \mathbf{b}_0 \) and \( \mathbf{b}_0^{-1} \)) are holomorphic on \( T(-U(\Sigma, \delta_0)) \) (resp. \( T(-U(* \Sigma, \delta_0)) \)) and belong to...
\[ S^{-U}(\Sigma, \delta_0) \text{ (resp. } S^{-U}(\Sigma, \delta_0)') \] (ii) \exists unique \( \xi \) in \( S^{-U}(\Sigma, \delta_0) \) such that for \( \lambda \in U(\Sigma, \delta_0) \),
\[ \xi(\lambda) = b_0(\lambda) / b(-\lambda). \]

**Corollary 3.3.3.** Let \( \delta_0 > 0 \) be as above and \( V \), a bounded subset of \( U(\Sigma \setminus \Sigma, \frac{1}{2}\delta_0) \). Then there is \( l \geq 0 \) such that for all \( u \in S(F) \),
\[ \sup_{t \in T(F)} (1 + ||\lambda||^2)^{-l} |\xi(\lambda; \partial(u))| < \infty. \]

Define
\[ \mathcal{U} = \{(*\lambda, \delta_0) : *\lambda \in \mathcal{F}_R, \delta_0 \in \mathcal{F}_R, *\lambda + \delta_0 \in U(\Sigma \setminus \Sigma, \frac{1}{2}\delta_0) \}. \]

Then \( s : (*\lambda, \delta_0) \mapsto \xi(*\lambda + \delta_0) \) is holomorphic on \( T(\mathcal{U}) \). Moreover, if \( \mathcal{V} \) is any bounded subset of \( \mathcal{U} \), there is \( p \geq 0 \) such that for all \( *u \in S(*F) \) and \( \delta_0 \in \mathcal{U} \),
\[ \sup_{(*\lambda, \delta_0) \in T(\mathcal{V})} [(1 + ||\lambda||^2)(1 + ||\lambda||^2)^{-p} |s(*\lambda; \partial(*u) : \delta_0; \partial(\delta_0))|] < \infty. \]

Note that \( T(-U(\Sigma, \delta_0)) \) is invariant under translations by elements of \( \mathcal{F}_R \) and \( b_0(*\lambda + \delta_0) = b_0(*\lambda) \) for all \( *\lambda \in T(-U(\Sigma, \delta_0)) \) and \( \delta_0 \in \mathcal{F}_R \).

We shall now introduce certain open subsets of \( \mathcal{F}_R \) and \( \mathcal{F}_R' \) which will be of use in the sequel. For any \( \delta > 0 \) let
\[ \mathcal{F}_R'(\delta) = \{ *\lambda : *\lambda \in \mathcal{F}_R, ||*\lambda|| < \delta \} \quad \text{for } l \leq j \leq \nu \}
\[ *\mathcal{F}_R(\delta) = \{ *\lambda : *\lambda \in \mathcal{F}_R, ||*\lambda|| < \delta \}, \quad U^\delta = \mathcal{F}_R(\delta) \times U^\delta. \]

Further, for any \( \varepsilon > 0 \), let
\[ \mathcal{G}_R^\varepsilon = \{ *\lambda : *\lambda \in \mathcal{F}_R, 0 < (\lambda + \varepsilon \rho_0), \alpha_j \} \subset \mathcal{F}_R, \lambda \leq j \leq \nu \}. \]

Clearly \( \mathcal{G}_R^\varepsilon \) is a nonempty bounded open subset of \( \mathcal{F}_R \) and \( -\varepsilon \rho_0 \in \text{Cl}(\mathcal{G}_R^\varepsilon) \).

Furthermore, it follows from Lemma 3.2.6 that \( \text{Int} *\mathcal{F}_R + \mathcal{G}_R^\varepsilon \subseteq \text{Int} \mathcal{F}_R \), \( *\mathcal{F}_R + \text{Cl}(\mathcal{G}_R^\varepsilon) \subseteq \mathcal{F}_R \), and \( \lambda, \alpha \leq 0 \) for all \( \lambda \in *\mathcal{F}_R \) and \( \text{Cl}(\mathcal{G}_R^\varepsilon), \alpha \in \Delta^+ \setminus \Delta^0. \)

**Lemma 3.3.4.** We have, for any \( \varepsilon > 0 \), \( *\mathcal{F}_R \times \text{Cl}(\mathcal{G}_R^\varepsilon) \subseteq \mathcal{U} \). Further, there exists a \( \delta > 0 \) such that \( U^\delta \subseteq \mathcal{U} \).

**Proof.** The first relation follows from the above remarks. For the second, let \( \bar{\delta} > 0 \). Then, for \( (*\lambda, \alpha) \in U^\varepsilon \) and \( \alpha \in \Delta^+ \setminus \Delta^0 \), we have \( (*\lambda + \varepsilon \rho_0, \alpha) \leq \bar{\delta}(||\alpha|| + \delta(\alpha)) \). So \( U^\varepsilon \subseteq \mathcal{U} \) if \( \bar{\delta} > 0 \) is small enough.

3.4. **The spaces \( \mathbb{Z}(F^i), \mathbb{Z}(*F^i) \).** For any \( \varepsilon > 0 \) we define \( \mathbb{Z}(F^i) \) to be the space of all complex-valued functions \( F \) such that (i) \( F \) is defined on Int \( F^i \), holomorphic there (ii) For any \( u \in S(F) \) and any integer \( l \geq 0 \),
\[ \mathbb{Z}_{\varepsilon, l}(F) = \sup_{*u \in \text{Int} F^i} (1 + ||\lambda||^2)^j |F(\lambda; \partial(u))| < \infty. \]

\( \mathbb{Z}(F^i) \) is an algebra (without 1). The \( \mathbb{Z}_{\varepsilon, l}(F) \) are seminorms on \( \mathbb{Z}(F^i) \) and convert it into a Fréchet algebra i.e., a Fréchet space with multiplication being jointly continuous. From (ii) above, it follows that for any \( F \in \mathbb{Z}(F^i) \) and \( u \in S(F) \), \( \partial(u)F \) is uniformly bounded on Int \( F^i \). Since Int \( F^i \) is convex, this implies
that for each \( u \in S(\mathcal{F}) \), \( \partial(u)F \) is uniformly continuous on \( \text{Int} \mathcal{F} \), thus extending to a continuous function on \( \mathcal{F} \); for \( \lambda \in \mathcal{F} \) we write \( F(\lambda; \partial(u)) \) for the value of this extension at \( \lambda \). The maps \( F' \mapsto F(\lambda; \partial(u)) \) \((F' \in Z(\mathcal{F}'), u \in S(\mathcal{F}), \lambda \in \mathcal{F}')\) are continuous linear functionals on \( Z(\mathcal{F}') \). We denote by \( \tilde{Z}(\mathcal{F}') \) the closed subalgebra of \( \mathfrak{m} \)-invariant elements of \( Z(\mathcal{F}') \).

**Lemma 3.4.1.** Let \( S(\mathcal{F}) \) denote as usual the Schwartz space of the real Hilbert space \( \mathcal{F} \). Fix \( \varepsilon > 0 \). If \( F \in Z(\mathcal{F}') \) and \( \xi \in \text{Int} \mathcal{F}'_\varepsilon \), the function \( F(\xi + \lambda) \) lies in \( S(\mathcal{F}) \). If \( \mu \) is any continuous seminorm on \( S(\mathcal{F}) \), there is a continuous seminorm \( \zeta \) on \( Z(\mathcal{F}') \) such that \( \mu(F) \leq \zeta(F) \) for all \( F \in Z(\mathcal{F}') \) and \( \xi \in \text{Int} \mathcal{F}'_\varepsilon \).

**Proof.** This is obvious on noting that \( \| \xi + \lambda \|^2 = \| \xi \|^2 + \| \lambda \|^2 \geq \| \lambda \|^2 \) for \( \xi \in \mathcal{F}'_\varepsilon \), \( \lambda \in \mathcal{F} \).

Let \( S(\mathcal{A}) \) and \( S(\mathcal{F}) \) be the Schwartz spaces of \( \mathcal{A} \) and \( \mathcal{F} \). Then the Fourier transform maps \( S(\mathcal{A}) \) bijectively onto \( S(\mathcal{F}) \). For any \( \varepsilon > 0 \) let \( S'(\mathcal{A}) \) be the space of all complex valued \( \mathfrak{m} \)-invariant \( C^\infty \) functions on \( \mathcal{A} \) with the following property: for any \( \nu \in \mathfrak{A} \) and any \( l \geq 0 \),

\[
\sigma_{\nu, l}(\mathcal{A}(f)) = \sup_{h \in \mathfrak{A}} e^{i\rho(\log h)}(1 + \sigma(h))^l |f(h; \nu)| < \infty;
\]

here \( \sigma \) is the spherical function on \( G \) introduced in §2.4. An elementary variation of the classical Paley-Wiener argument shows that the Fourier transform maps \( S'(\mathcal{A}) \) onto the space of restrictions of the elements of \( \tilde{Z}(\mathcal{F}') \) to \( \mathcal{F} \). In particular, the subspace of elements of \( \tilde{Z}(\mathcal{F}') \) which extend to entire functions on \( \mathcal{F} \), is dense in \( \tilde{Z}(\mathcal{F}') \).

From now on we shall fix an \( \varepsilon > 0 \).

**Lemma 3.4.2.** Define \( \mathcal{G}^\varepsilon \) by (3.3.6), and let \( \mathcal{G} = \text{Int} * \mathcal{F}'_\varepsilon \times \mathcal{G}^\varepsilon \). Then, for any \( \alpha \in \mathcal{Z}(\mathcal{F}') \), the function \( G_\alpha(\lambda, \nu) \mapsto a(\lambda + \nu) \xi(\lambda + \nu) \) is well defined and holomorphic on the tube \( T(\mathcal{G}) \). Moreover, given any integer \( l \geq 0 \), \( \star \nu \in S(\mathcal{F}) \) and \( \circ \nu \in S(\mathcal{F}) \), we can select a continuous seminorm \( \zeta_{\nu, l}(\mathcal{F}) \) such that for all \( a \in \mathcal{Z}(\mathcal{F}') \)

\[
\sup_{(\nu, \lambda) \in \mathcal{G}, |T(\mathcal{G})|} [(1 + \| \lambda \|^2)(1 + \| \nu \|^2)]^l |G_\alpha(\lambda, \nu) \Xi(\lambda + \nu)| \leq \zeta_{\nu, l}(\mathcal{F}),
\]

**Proof.** For \( a \in \mathcal{Z}(\mathcal{F}') \) let \( a(\lambda, \nu) \star a(\lambda + \nu) \). By Lemma 3.2.6 (cf. §3.3), \( a \) is well defined and holomorphic on \( T(\mathcal{G}) \). Moreover, \( s(\lambda, \nu) \mapsto \xi(\lambda + \nu) \) is well defined and holomorphic on \( T(\mathcal{G}) \), by Corollary 3.3.3 and Lemma 3.3.4. The rest of the proof is a routine followup of Corollary 3.3.3.

We now come to the basic result of this section. We denote by \( \mathcal{Z}(\mathcal{F}') \) the space of complex valued functions defined and holomorphic on \( \text{Int} * \mathcal{F}' \) and satisfying the growth conditions which are the obvious analogues of those defining \( \mathcal{Z}(\mathcal{F}) \). \( \mathcal{Z}(\mathcal{F}') \) is the space of \( \mathfrak{m}_\nu \)-invariant elements in \( \mathcal{Z}(\mathcal{F}') \).
Let \( a \in \mathcal{Z}(\mathcal{F}_t) \). Then \( \lambda \mapsto a(\lambda)\xi(\lambda) \) is an element of \( \mathcal{S}(\mathcal{F}_t) \). Hence for any \( \lambda \in \mathcal{F}_t \), and any \( H \in a_\alpha \), the integral \( \int_{\mathcal{F}_t} a(\lambda + \alpha)\xi(\lambda + \alpha) e^{i\lambda H} d\alpha \) will converge absolutely. Here \( d\alpha \) is a Lebesgue measure on \( \mathcal{F}_t \).

**Lemma 3.4.3.** For any \( a \in \mathcal{Z}(\mathcal{F}_t) \) and \( H \in a_\alpha \), there is exactly one element \( A_{a,H} \in \mathcal{Z}(\mathcal{F}_t) \) such that for all \( \lambda \in \mathcal{F}_t \),

\[
A_{a,H}(\lambda) = e^{i\rho(H)} \int_{\mathcal{F}_t} a(\lambda + \alpha)\xi(\lambda + \alpha) e^{i\lambda H} d\alpha .
\]

Moreover, given any continuous seminorm \( *\zeta \) on \( \mathcal{Z}(\mathcal{F}_t) \) and any integer \( l \geq 0 \), we can select a continuous seminorm \( \zeta \) on \( \mathcal{Z}(\mathcal{F}_t) \) such that

\[
*\zeta(A_{a,H}) \leq (1 + ||H||)^{-l} \zeta(a)
\]

for all \( a \in \mathcal{Z}(\mathcal{F}_t) \) and all \( H \in a_\alpha \).

**Proof.** Let \( a \in \mathcal{Z}(\mathcal{F}_t) \), \( H \in a_\alpha \) and let us write, for \( \lambda \in \mathcal{F}_t \),

\[
I(a;H;\lambda) = e^{i\rho(H)} \int_{\mathcal{F}_t} a(\lambda + \alpha)\xi(\lambda + \alpha) e^{i\lambda H} d\alpha .
\]

Clearly, there is at most one function holomorphic on \( \text{Int} \mathcal{F}_t \), and coinciding with \( I(a;H;\cdot) \) on \( \mathcal{F}_t \). We shall now analytically continue \( I(a;H;\cdot) \).

Let \( \delta > 0 \) and let

\[
\mathcal{F}_\delta^* = \{ \lambda : \lambda \in \mathcal{F}_R, |(\lambda, \alpha_j)| < \delta \text{ for } 1 \leq j \leq \nu \} \\
\bigcup \{ \lambda : \lambda \in \mathcal{F}_R, -\epsilon(\alpha_j) < (\lambda, \alpha_j) < 0 \text{ for } 1 \leq j \leq \nu \}.
\]

If \( \delta > 0 \) is small enough, the first member of the union in (3.4.3) is contained in \( \text{Int} \mathcal{F}_R \), while the second member is \( \subseteq \text{Int} \mathcal{F}_R \) by Lemma 3.2.5. So \( \mathcal{F}_\delta^* \subseteq \text{Int} \mathcal{F}_R \) for sufficiently small \( \delta > 0 \). It now follows from Corollary 3.3.3 that for \( \delta > 0 \) small enough and \( \forall \lambda \in \mathcal{F}_t \),

\[
G_{a;\delta}: \lambda \mapsto \alpha(\lambda + \alpha)\xi(\lambda + \alpha) e^{i\lambda H}
\]

belongs to \( \mathcal{L}_{\mathcal{F}_\delta} \). Therefore, as \( \mathcal{F}_\delta^* \) is a connected open set in \( \mathcal{F}_R \) containing \( 0 \),

\[
I(a;H;\lambda) = e^{i\rho(H)} \int_{\mathcal{F}_t} a(\lambda + \alpha + \mu)\xi(\lambda + \alpha + \mu) e^{i\lambda H} d\alpha ,
\]

for any \( \mu \in \mathcal{F}_\delta^* \) (cf. Lemma 3, Appendix). Since \( \mathcal{F}_\delta^* \subseteq \mathcal{F}_\delta \) for any \( \delta > 0 \), we get, for \( \lambda \in \mathcal{F}_t \), \( \mu \in \mathcal{F}_\delta \),

\[
I(a;H;\lambda) = e^{i\rho(\mu) H} \int_{\mathcal{F}_t} G_{a}(\lambda + \alpha + \mu) e^{i\lambda H} d\alpha .
\]

By Lemma 3.4.2, for any \( \nu \in S(\mathcal{F}_t) \) and \( \nu \in S(\mathcal{F}) \), the integral

\[
\int_{\mathcal{F}_t} |G_{a}(\lambda + \nu) d\lambda |
\]

uniformly converges for \( \lambda, \mu \in \mathcal{F}_t \times \mathcal{G} \). So writing
(3.4.5) \[ A_{a,H}(\varepsilon \lambda; \varepsilon \mu) = e^{(\varepsilon \rho_0 + \varepsilon \rho_1)(H)} \int_{\mathfrak{T}_{\mathfrak{T}}} G_a(\varepsilon \lambda; \varepsilon \mu) e^{\varepsilon \lambda (H)} d\varepsilon \lambda \]

for \((\varepsilon \lambda, \varepsilon \mu) \in \text{Int } \mathfrak{T} \times \mathfrak{T}^d\), then \(A_{a,H}(\varepsilon \lambda; \varepsilon \mu)\) is holomorphic on \(\text{Int } \mathfrak{T} \times \mathfrak{T}^d\), and its derivatives with respect to \(\varepsilon \lambda\) can be calculated by differentiating under the integral sign. On the other hand, the formula for \(I\) shows that \(A_{a,H}(\varepsilon \lambda; \varepsilon \mu)\) is independent of \(\varepsilon \mu\) for \(\varepsilon \lambda \in \mathfrak{T}^d\). Hence the same is true for \(\varepsilon \lambda \in \text{Int } \mathfrak{T}^d\). So we can choose a function \(A_{a,H}\), defined and holomorphic on \(\text{Int } \mathfrak{T}^d\), such that for \(\varepsilon \lambda \in \text{Int } \mathfrak{T}^d\), \(\varepsilon \mu \in \mathfrak{T}^d\),

\[ A_{a,H}(\varepsilon \lambda) = e^{(\varepsilon \rho_0 + \varepsilon \rho_1)(H)} \int_{\mathfrak{T}^d} G_a(\varepsilon \lambda; \varepsilon \mu) e^{\varepsilon \lambda (H)} d\varepsilon \lambda . \]

By elementary Fourier transform theory we can finally choose, given any polynomial \(u\) on \(a\), an element \(v \in S(\mathfrak{T}^d)\) that depends only on \(u\), such that for any \(v \in S(\mathfrak{T}^d), \varepsilon \lambda \in \text{Int } \mathfrak{T}^d, \varepsilon \mu \in \mathfrak{T}^d, H \in a\),

\[ (u(H)A_{a,H}(\varepsilon \lambda; \partial v)) = e^{(\varepsilon \rho_0 + \varepsilon \rho_1)(H)} \int_{\mathfrak{T}^d} G_a(\varepsilon \lambda; \partial v; \varepsilon \mu; (\partial v)) e^{\varepsilon \lambda (H)} d\varepsilon \lambda . \]

Let \(l, l' \geq 0\) be integers. Select \(u\) such that \((1 + ||H||)^l \leq \varepsilon u(H)\) for all \(H \in a\). Define \(v\) corresponding to this \(u\) as above. By Lemma 3.4.2, we can select a continuous seminorm \(\zeta\) on \(S(\mathfrak{T}^d)\) such that for all \(a \in S(\mathfrak{T}^d), H \in a\), and \(\varepsilon \lambda \in \text{Int } \mathfrak{T}^d\),

\[ \left| \int_{\mathfrak{T}^d} G_a(\varepsilon \lambda; \partial v; \varepsilon \mu; (\partial v)) e^{\varepsilon \lambda (H)} d\varepsilon \lambda \right| \leq \zeta(a)(1 + ||\varepsilon \lambda||^2)^{-l'} . \]

Consequently, for all \(\varepsilon \mu \in \mathfrak{T}^d\), and \(a, H, \varepsilon \lambda\) as above,

\[ (1 + ||H||)^l |A_{a,H}(\varepsilon \lambda; \partial v))| \leq \zeta(a)(1 + ||\varepsilon \lambda||^2)^{-l'} e^{(\varepsilon \rho_0 + \varepsilon \rho_1)(H)} . \]

As \(-\varepsilon \rho_0 \in C_l(\mathfrak{T}^d)\), we can let \(\varepsilon \mu \rightarrow -\varepsilon \rho_0\) in this estimate. The resulting conclusion is the one asserted in the lemma.

**Lemma 3.4.4.** Let \(f\) be a polynomial on \(\mathfrak{T}\) such that \(\inf_{\lambda \in \mathfrak{T}} |f(\lambda - \varepsilon \rho_0)| > 0\), where \(0 < \varepsilon < \varepsilon'\). Then, for any \(a \in S(\mathfrak{T}^d)\), the function

\[ \tau_{f,a}(\lambda) \mapsto a(\lambda - \varepsilon \rho_0) \xi(\lambda - \varepsilon \rho_0) / |f(\lambda - \varepsilon \rho_0)| \]

is an element of \(S(\mathfrak{T})\). Moreover, \(a \mapsto \tau_{f,a}\) is a continuous map of \(S(\mathfrak{T}^d)\) into \(S(\mathfrak{T})\).

**Proof.** By Lemma 3.2.5, \(-\varepsilon \rho_0 \in \text{Int } \mathfrak{T}^d\). So, by Lemma 3.4.1, the function \(a_{-\varepsilon \rho_0} : \lambda \mapsto a(\lambda - \varepsilon \rho_0)\) lies in \(S(\mathfrak{T}^d)\), and \(a \mapsto a_{-\varepsilon \rho_0}\) is a continuous map of \(S(\mathfrak{T}^d)\) into \(S(\mathfrak{T})\). On the other hand, if \(\lambda \in \mathfrak{T}^d\), Re(\(\lambda - \varepsilon \rho_0, \alpha\)) = \(-\varepsilon'\rho_0, \alpha\) < 0 for \(\alpha \in \Delta^+ \setminus \mathfrak{T}^d\) so that \(\lambda - \varepsilon \rho_0 \in T(U(\Sigma \setminus \mathfrak{T}^d, 1/\delta_0))\). The lemma follows easily from these observations and Corollary 3.3.3.
3.5. The space $\mathcal{F}_\rho(G)$. We shall now introduce the space $\mathcal{F}_\rho(G)$ of spherical functions on $G$. Let $\Xi$ and $\sigma$ be as in §2.4.

**Lemma 3.5.1.** Let $\varepsilon > 0$. Then, for $\lambda \in \mathcal{F}$ and $h \in A^+$,

$$|\varphi(\lambda; h)| \leq e^{\varepsilon \rho(||h||_G)} \Xi(h).$$

**Proof.** We have $|\varphi(\lambda; x)| \leq \varphi(\lambda_{-a}^+; x)$ for $\lambda \in \mathcal{F}$, $x \in G$. Fix $\lambda \in \mathcal{F}$; and select $s \in w$ such that $s H_{-a} \in \text{Cl}(a^+)$. Write $\mu = s \lambda_{-a}$. Then $\mu(H(hk)) \leq \mu(\log h)$ for all $h \in A^+$ and $k \in K$ by Lemma 35 of [3], so that we obtain $e^{\rho(||H(hk)||_G)} \leq e^{\varepsilon \rho(\log h)}$ on noting that $\mu \in \mathcal{F}_h$. This leads to the desired estimate.

**Lemma 3.5.2.** There exists a constant $c > 0$ such that for $x \in G$, $k \in K$,

$$||H(xk)|| \leq c \sigma(x).$$

**Proof.** It is enough to find $c > 0$ such that $||H(hk)|| \leq c \sigma(h)$ for all $h \in A^+$, $k \in K$. Let $s_0 \in w$ be such that $s_0 \cdot a^+ = -a^+$. Then, by Lemma 35 of [3] and its corollaries, we have $(s_0(\log h), H) \leq (H(hk), H) \leq (\log h, H)$ for all $h \in A^+$, $k \in K$, $H \in \text{Cl}(a^+)$. So $|(H(hk), H)| \leq ||H|| ||\log h|| = ||H|| \sigma(h)$ for such $h$, $k$, $H$. This leads at once to the desired conclusion.

**Lemma 3.5.3.** For $u \in S(\mathcal{F})$ let $d_u = \deg(u)$, $\varphi_{\lambda, u}(x) = \varphi(\lambda; \partial(u); x)$ ($\lambda \in \mathcal{F}$, $x \in G$). Then, if $q \in \mathcal{D}$, $(q - \nu(q)(\lambda))^{d_u} \varphi_{\lambda, u} = 0$ for all $\lambda \in \mathcal{F}$. Furthermore, given $a \in \mathcal{G}$, there are constants $c = c_{a, u} > 0$ and $l = l_{a, u} \geq 0$ such that for all $x \in G$, $\lambda \in \mathcal{F}$,

$$|\varphi(\lambda; \partial(u); x; a)| \leq c [(1 + ||\lambda||)(1 + \sigma(x))]^{l(1 + 1)}. $$

**Proof.** Fix $\lambda \in \mathcal{F}$. Then, arguing as in Lemma 3 of [3], we get

$$\varphi(\lambda; \partial(u); x; q) = \int K \varphi_1(H(xk)) e^{-\rho(||H(xk)||_G)} dk$$

where $\varphi_1 = (\partial(\nu(q)) \cdot u)e^q$. Now, we can select $u_j \in S(\mathcal{F})$ with $\deg(u_j) < \deg(u)$ and $\nu_j \in \mathcal{H}$ ($1 \leq j \leq s$) such that $\partial(\nu(q)) \cdot u = u \partial(\nu(q)) + \sum_{1 \leq j \leq s} u_j \partial(\nu_j)$. This implies at once that $(q - \nu(q)(\lambda))\varphi_{\lambda, u} = \varphi_{\lambda, u_j}(\lambda)\varphi_{1, u_j}$. The first result now follows by induction on $\deg(u)$.

Fix $a \in \mathcal{G}$. Then there are $a_j \in \mathcal{G}$ and analytic functions $\beta_j (1 \leq j \leq s)$ on $K$ such that $a^{k-1} = \sum_{1 \leq j \leq s} \beta_j(k) a_j$ for all $k \in K$. Moreover, as $\mathcal{G}$ is the direct sum of $\mathcal{A}_\mathcal{H}$ and $\mathcal{A}_\mathcal{H}^\prime$, we can select $b_i \in \mathcal{H}$, $w_{i,j} \in \mathcal{H}$ such that $a_j = \sum_{1 \leq i \leq s} w_{i,j} b_i (mod \mathcal{G}_n)$ for $1 \leq j \leq s$. So, if we write $F(y) = u(H(y)) e^{2 - \rho(\log y)}$ ($y \in G$), we get, for all $x \in G$,

$$\varphi(\lambda; \partial(u); x; a) = \sum_{1 \leq j \leq s} \sum_{1 \leq i \leq r} \int K \beta_j(k; u^*_j) F(xk; b_i) dk.$$ 

Select polynomials $p_{im}$ on $a_e$ and elements $c_m \in \mathcal{A}$ ($1 \leq m \leq t$) such that $\partial(b_i) \cdot u = \sum_{1 \leq m \leq t} p_{im}(c_m)$ for $1 \leq i \leq r$. Then
\[ \varphi(\lambda; \partial(u) : x; a) = \sum_{j, i, m} c_m(\lambda - \rho) \int_{x_k} \beta_j(k; u^*_x) \rho_{ij}(H(xk)) e^{i \lambda x^i} d \lambda d \rho \left( H(xk) \right) . \]

So, by Lemma 3.5.2, there are constants \( c_1 > 0, r_1 \geq 0 \) such that for all \( \lambda \in \mathcal{F}, x \in G \), \( |\varphi(\lambda; \partial(u) : x; a)| \) is majorized by \( c_1[(1 + \| \lambda \|)(1 + \sigma(x))]^{r_1} \varphi(\lambda_x : x) \). It now follows quickly from (2.4.1) and Lemma 3.5.1 that for suitable constants \( c_2 > 0, r_2 \geq 0 \), \( \varphi(\lambda_x : x) \leq c_2(1 + \sigma(x))^{r_2} \Xi(x)^{-l+1} \), for all \( x \in G, \lambda \in \mathcal{F} \). This leads to the desired estimate.

Let \( p > 0 \). We define \( \mathcal{B}^p(G) \) to be the space of all \( C^\infty \) spherical functions \( f \) on \( G \) such that for any \( a \in \mathcal{S} \) and any integer \( l \geq 0 \),

\[ \mu_{c, t}^p(f) = \sup_{x \in G} (1 + \sigma(x))^{l/2} \Xi(x)^{-l/p} |f(x; a)| < \infty. \]

For \( p = 2 \), we get the space \( \mathcal{B}(G) \) of Harish-Chandra [5]. The seminorms \( \mu_{c, t}^p \) convert \( \mathcal{B}^p(G) \) into a Fréchet space. Now, there is an \( r_0 \geq 0 \) such that \( \Xi(1 + \sigma)^{-r_0} \in \mathcal{S}^p(G) \) (cf. [5]). So, for \( f \in \mathcal{B}^p(G), a \in \mathcal{S} \) and any integer \( l \geq 0 \), \( (1 + \sigma)^l |af|^p \in \mathcal{S}^l(G) \). We denote by \( \mathcal{B}^c(G) \) the space of all spherical functions in \( C^\infty_c(G) \). \( \mathcal{B}^c(G) \) is dense in \( \mathcal{B}^p(G) \); the proof of this is similar to the corresponding proof for the case \( p = 2 \) (cf. [5] §13). The following lemma is then an easy consequence of these remarks.

**Lemma 3.5.4.** Let \( \varphi \) be a Borel function on \( G \) such that

\[ |\varphi(x)| \leq C \Xi(x)^{3/2}(1 + \sigma(x))^l \]

for all \( x \in G, C \) and \( l \) being constants \( > 0 \). Then \( L(f) = \int_G f(x; a) \varphi(x) dx \) converges absolutely for all \( f \in \mathcal{B}^p(G), a \in \mathcal{S}, \) and \( L \) is a continuous linear functional on \( \mathcal{B}^p(G) \). If \( \varphi \in C^\infty_c(G) \), and both \( \varphi \) and \( a^* \varphi \) satisfy estimates of the above form, then for all \( f \in \mathcal{B}^p(G), \int_G af \cdot \varphi dx = \int_G f \cdot a^* \varphi dx \).

**Theorem 3.5.5.** Let \( 0 < p < 2 \) and let \( \varepsilon = (2/p) - 1 \). Then, for \( f \in \mathcal{B}^p(G) \), the integral \( \hat{f}(\lambda) = \int_G f(x) \varphi(-\lambda; x) dx \) converges absolutely for all \( \lambda \in \mathcal{F}' \). The function \( \hat{f} \) lies in \( \mathcal{S}(\mathcal{F}') \) and \( f \rightarrow \hat{f} \) is a continuous map of \( \mathcal{B}^p(G) \) into \( \mathcal{S}(\mathcal{F}') \).

**Proof.** It is clear that \( \varepsilon > 0 \). The absolute convergence of the integral for \( \lambda \in \mathcal{F}' \) is clear from Lemmas 3.5.3 and 3.5.4. The same arguments show that for any \( u \in \mathcal{S}(\mathcal{F}) \) we can find an integer \( l_u \geq 0 \) and a continuous seminorm \( \mu_u \) on \( \mathcal{B}^p(G) \) such that for all \( \lambda \in \text{Int} \mathcal{F}', f \in \mathcal{B}^p(G), |\hat{f}(\lambda; \partial(u))| \leq (1 + \| \lambda \|)^{l_u} \mu_u(f) \). On the other hand, for \( \lambda, f \) as above, we have, for any \( q \in \mathcal{S}, \)

\[ \int_G f(x; (q^* - \nu(q)(-\lambda))^{l_u+1}) \varphi(-\lambda; \partial(u^*): x) dx = 0; \]

here \( u^* \) is the adjoint of \( u \). Expanding the derivative hitting \( f \) and using the estimate above we get, for \( \lambda, f \) as above
\[ |v(q)(-\lambda)|^{d_u+1} |\hat{f}(\lambda; \partial(u))| \leq 2^{d_u+1} (1 + ||\lambda||)^{d_u} \sum_{\ell \leq \ell_u \leq d_u+1} |v(q)(-\lambda)|^{d_u+1-\ell} \mu_{u,\ell}(q^{*}f). \]

From this, and the fact that \( A \) is a finite module over \( v(\mathcal{O}) \), we easily obtain the following result: given \( u \in S(F) \), there is \( l_u \geq 0 \), and for each \( v \in A \), a continuous seminorm \( \mu_{u,\ell} \) on \( \mathcal{S}(G) \) such that

\[ |v(\lambda)| |\hat{f}(\lambda; \partial(u))| \leq (1 + ||\lambda||)^{l_u} \mu_{u,\ell}(f) \]

for all \( f \in \mathcal{S}(G), \lambda \in \text{Int } F^\circ \). Since \( l_u \) does not depend on \( v \), this leads us at once to the conclusion that \( \hat{f} \in \mathcal{S}(F^\circ) \) and that the map \( f \mapsto \hat{f} \) is continuous. Since \( \hat{f} \) is \( w \)-invariant, we are through.

3.6. A formula for \( \psi_v \). We undertake in this and the next section a close study of the functions \( \psi_v \) which enter the asymptotic series for \( \phi_{\lambda} \) (cf. Chapter 2). The present section will be devoted to obtaining a formula for \( \psi_v \) (3.6.5), that will display its analyticity properties rather explicitly.

Let \( D^+ \) and \( L^+ \) be as in \( \S \, 2.1 \). Let

\[ M = \{m_0, \ldots, \alpha_{r+1} + \cdots + m_i \alpha_i: m_i \text{ is an integer } \geq 0 \text{ for } v < k \leq d \}. \]

For each \( n \in D^+ \) let \( \Pi_n \) be the (affine) hyperplane in \( F: \Pi_n = \{\lambda: \lambda \in F, 2(\lambda, n) = (n, n)\} \). We put \( \mathcal{F}' = F \setminus \bigcup_{n \in D^+} \Pi_n \). Each compact subset of \( F \) meets only finitely many of the \( \Pi_n \). So it is clear that \( \mathcal{F}' \) is a dense, connected open subset of \( F \). Further \( \mathcal{F}' \) contains a tube \( T(U) \) for some open neighborhood \( U \) of \( 0 \) in \( F \).

It follows from Harish-Chandra's work [3] that there are rational functions \( c_n(n \in D) \) on \( F \) with the following properties: (i) \( c_0 = 1; c_n \) is holomorphic in \( \mathcal{F}' \) for all \( n \in D^+ \) (ii) if \( E \) is any compact set \( \subseteq \mathcal{F}' \) and \( \delta > 0 \), we can select a constant \( C(E, \delta) > 0 \) such that for all \( n \in D, \sup_{\ell \in E} |c_n(\lambda)| \leq C(E, \delta) e^{c|\lambda|} \) (iii) for all \( \lambda \in \mathcal{F}'_1 \) and all \( h \in A^+ \),

\[ e^{(\log k)\phi}(\lambda: h) = \sum_{x \in D} c(x^{-1}\lambda) \sum_{n \in D} c(n \lambda) e^{(x^{-1} - n)(\log k)}. \]

Here \( c \) is as in \( \S \, 2.4, 3.3 \). Note that the estimate for \( c_n \) guarantees absolute convergence in (3.6.2).

Let \( q \in L \). Write, for \( \lambda \in \mathcal{F}', h \in A^+ \)

\[ \chi_{\ell}(\lambda: h) = \sum_{x \in M} c_{x+q}(\lambda) e^{(\ell - q - q') (\log k)}. \]

**Lemma 3.6.1.** For any \( \lambda \in \mathcal{F}', \chi_{\ell}(\lambda: \cdot) \) is well defined and is a \( C^\infty \) function on \( A^+ \). Let \( v \in A \). Then, for any \( h \in A^+, \chi_{\ell}(\cdot: h; v) \) is holomorphic in \( \mathcal{F}' \), and \( \forall(\lambda, h) \in \mathcal{F}' \times A^+, \chi_{\ell}(\lambda, h; v) \) (3.6.4)

\[ \chi_{\ell}(\lambda: h; v) = \sum_{x \in M} v(\lambda - q - q') c_{x+q}(\lambda) e^{(\ell - q - q') (\log k)}. \]

**Proof.** It follows from the estimate (3.6.4) that for any \( v \in A \), the series
\[
\sum_{q' \in \mathcal{M}} |v(\lambda - q - q')| \cdot |c_{q+q'}(\lambda)| \cdot |e^{(2-q-q')(\log h)}| < \infty
\]
for all \((\lambda, h) \in \mathcal{F}^\prime \times A^+,\) the convergence being uniform on compact subsets of \(\mathcal{F}^\prime \times A^+.\) This leads to the present lemma.

The main justification for introducing the functions \(\chi_q\) is contained in the following lemma.

**Lemma 3.6.2.** Let \(q \in L^+.\) Then, for any \(h \in A^+\) and any \(\lambda \in \mathcal{F},\) with \(\pi_0(\lambda) \neq 0,\) we have

\[
(3.6.5) \quad e^{\sigma(\log h)} \psi_q(\lambda; h) = \sum_{s' \in \mathfrak{W}_0} c_0(s^{-1}\lambda) \chi_q(s^{-1}\lambda; h).
\]

**Proof.** Fix \(h \in A^+.\) \(c_0\) is certainly well defined and continuous on the set of all \(\lambda \in \mathcal{F}\) with \(\pi_0(\lambda) \neq 0.\) As \(\mathcal{F}\) contains \(\mathcal{F}^\prime,\) we see that the right side of (3.6.5) is continuous on \(\{\lambda: \lambda \in \mathcal{F},\ \pi_0(\lambda) \neq 0\}.\) Moreover, we know from the work of Chapter 2 that \(\psi_q(\cdot; h)\) is continuous on \(\mathcal{F}^\prime.\) So it is enough to prove (3.6.5) for a conveniently chosen dense subset of \(\mathcal{F}^\prime.\) In view of Lemma 2.10.2, it is sufficient to establish (3.6.5) on \(\mathcal{F}^\prime.\) Fix \(\lambda \in \mathcal{F}^\prime,\) \(h \in A^+.\) Note that \(c = \pi c\) and \(b_0 = \pi c_0\) vanish at no point of \(\mathcal{F},\) and so \(\omega(\lambda) \neq 0.\) In the estimates to follow, we shall allow the constants to depend on \(\lambda\) and \(h\) without explicit mention.

Write \(\delta = \frac{1}{2} \min_{1 \leq p \leq 4} \alpha_p(\log h).\) Then, we can find a constant \(C' > 0\) such that for \(a \in D, s \in \mathcal{W} |c_0(s^{-1}\lambda)| \leq C' e^{-\delta \sigma(n)}.\) So, for all \(s \in \mathcal{W}, n \in D, H \in \text{Cl}(a_0^\circ),\) we have

\[
|c_0(s^{-1}\lambda) e^{-\delta \sigma(n)}| \leq C' e^{-\delta \sigma(n)}.
\]

If we write \(C = C' \sum_{n \in D} e^{-\delta \sigma(n)},\) we then obtain the estimate \(|\chi_q(s^{-1}\lambda; h)| \leq C\) \((q \in L, s \in \mathcal{W}).\)

Now, \(h \exp H \in A^+\) for all \(H \in a_0^\circ.\) On the other hand, as \(\omega\) is invariant under \(a_0^\circ,\) \(c_0(s^{-1}s_p^{-1}\lambda) = c_0(s^{-1}s_p^{-1}\lambda) \omega(s_p^{-1}\lambda)\) for \(s' \in \mathcal{W}, 1 \leq p \leq r.\) Hence, from (3.6.2), we obtain, after a simple calculation, the following formula, valid for all \(H \in a_0^\circ:\)

\[
e^{\sigma(\log h)} \varphi(\lambda; h \exp H) d_0(h \exp H) = \sum_{q \in L} e^{-\theta(H)} \sum_{1 \leq p \leq r} \sum_{s' \in \mathcal{W}_0} \omega(s_p^{-1}\lambda) c_0(s^{-1}s_p^{-1}\lambda) \exp((s_p^{-1}\lambda)(H)) \chi_q(s^{-1}s_p^{-1}\lambda; h).
\]

From the estimate for \(\chi_q\) and Lemma 1 of the Appendix we then get the following result. Let \(k \geq 0\) be any integer; then there is a constant \(C_k > 0\) such that for all \(H \in a_0^\circ\) with \(\beta(H) \geq 1\) and all \(t \geq 1,\)

\[
|e^{\sigma(\log h)} \varphi(\lambda; h \exp tH) d_0(h \exp tH) - \sum_{q \in L,\gamma(q) \leq k} e^{-\delta \sigma(H)} \sum_{1 \leq p \leq r} \sum_{s' \in \mathcal{W}_0} \omega(s_p^{-1}\lambda) \exp(t(s_p^{-1}\lambda)(H)) c_0(s^{-1}s_p^{-1}\lambda) | \chi_q(s^{-1}s_p^{-1}\lambda; h)| \leq C_k e^{-\delta k+1}(h)(H).
\]

We now compare this with the estimate (2.11.3) specialized to the case \(\gamma = 1,\)

\(m = h,\) and \(H\) replaced by \(tH.\) Let
\[ \Delta_{q,p}(\lambda; h) = \omega(s_{q}^{-1}(\lambda)) \sum_{e' \in \mathbb{A}_{q}} c_{q}(s_{q}^{-1}(\lambda)) \Psi_{q}(s_{q}^{-1}(\lambda); h) - e^{-\delta_{q}(\log h)} \Psi_{q}(s_{q}^{-1}(\lambda); h) \] .

Then, arguing as in Lemma 2.10.3, we conclude that \( \Delta_{q,p}(\lambda; h) = 0 \) for \( q \in \mathbb{L}, 1 \leq p \leq r \). Taking \( p = 1 \) and remembering that \( \omega(\lambda) \neq 0 \), we get (3.6.5).

### 3.7. Analyticity and growth properties of \( \psi_{q} \)

We use the notation of §§2.10 and 2.11. We choose \( \tilde{H} \in \mathbb{A}_{q}^{+} \) and fix it throughout this section. For \( q \in \mathbb{L}^{+} \), let \( h_{q} \) and \( v_{q} \) be the polynomials on \( \mathcal{F} \) defined in Lemma 2.10.6. Then there is an \( \varepsilon_{0} > 0 \) such that \( 1/v_{q} \) is holomorphic on the tube \( \{ \lambda; \lambda \in \mathcal{F}, ||\lambda|| < \varepsilon_{0} \} \) for all \( q \in \mathbb{L}^{+} \). It follows from Theorem 2.11.2 that given \( \eta \in \mathbb{M}_{10}^{+} \), we can select \( g_{j, \eta} \in \mathcal{A}_{q}^{+} \), \( \mu_{j, \eta} \in \mathbb{M}_{10}^{+} (1 \leq j \leq l_{q}) \) such that for all \( (\lambda, m) \in \mathcal{F}_{1} \times \mathbb{M}_{10}^{+} \),

\[ v_{q}(\lambda)\psi_{q}(\lambda; m; \eta) = \theta_{q}(\lambda; m; \eta) = \sum_{i \leq j \leq l_{q}} g_{j, \eta}(m)\theta(\lambda; m; \mu_{j, \eta}) . \]

**Lemma 3.7.1.** There exists a \( \delta > 0 \) with the following property. Let \( \ast \mathbb{U}^{+} \) and \( \ast \mathcal{F}_{R}(\delta) \) be defined by (3.3.5). Then, with \( \mathcal{F} \) as in §6.3,

\[ U_{s' \in \mathbb{M}_{10}^{+}} s' \cdot T(\ast \mathcal{F}_{R}(\delta) + \ast \mathbb{U}^{+}) \subseteq \mathcal{F} . \]

**Proof.** Let \( \gamma = \inf_{s' \in \mathbb{D}^{+}} ||n|| \). Then \( \gamma > 0 \). Choose \( \delta > 0 \) such that

\[ 2\delta(||\beta_{1}|| + \cdots + ||\beta_{v}|| + 1) < \gamma . \]

Let \( \lambda \in \mathcal{F} \) such that \( \lambda_{R} \in \ast \mathcal{F}_{R}(\delta) + \ast \mathbb{U}^{+} \). We shall prove that \( s' \lambda \in \Pi_{n} \) for \( s' \in \mathbb{M}_{10}^{+}, n \in \mathbb{D}^{+} \). Fix \( s' \in \mathbb{M}_{10}^{+}, n \in \mathbb{D}^{+} \). It is enough to verify that \( \text{Re}(s' \lambda, n) < \frac{1}{2}(n, n) \). Write \( \lambda_{R} = \lambda^{*} + \delta \lambda \) where \( \lambda^{*} \in \ast \mathcal{F}_{R}(\delta) \) and \( \delta \lambda \in \ast \mathbb{U}^{+} \). Now \( ||s' \lambda^{*}|| = ||\lambda^{*}|| < \delta \) so that \( (s' \lambda^{*}, n) < \delta ||n|| \). Further, \( \delta \lambda = \delta \lambda \) so that \( (\delta \lambda, n) = \sum_{i \leq j \leq v} (\delta \lambda_{i}, \alpha_{j})(n, \beta_{j}) \). But \( (n, \beta_{j}) \geq 0 \) and \( (\delta \lambda_{i}, \alpha_{j}) < \delta \) for \( 1 \leq j \leq v \). Hence \( (\delta \lambda, n) < \delta \sum_{i \leq j \leq v} (\delta \lambda_{i}, \beta_{j}) \leq \delta (||\beta_{1}|| + \cdots + ||\beta_{v}||) ||n|| \). Therefore

\[ \text{Re}(s' \lambda, n) = (s' \lambda_{R}, n) = (s' \lambda^{*}, n) + (s' \delta \lambda, n) < \delta (||\beta_{1}|| + \cdots + ||\beta_{v}|| + 1) ||n|| < \frac{\gamma}{2} ||n|| , \]

showing that \( \text{Re}(s' \lambda, n) < \frac{1}{2}(n, n) \).

We are now in a position to obtain the main result on the analytic continuation of the \( \psi_{q} \). Let \( \mathbb{U}^{+} \) be as in (3.3.5).

**Lemma 3.7.2.** We can select \( \delta > 0 \) with the following property. Let \( q \in \mathbb{L}^{+} \), \( \eta \in \mathbb{M}_{10}^{+}, h \in \mathbb{A}^{+} \) be arbitrary, but fixed. Then there exists a function \( F_{q,v}(h; \cdots) \) defined and holomorphic on \( \mathbb{T}(\mathbb{U}^{+}) \) such that for all \( (*\lambda, \delta \lambda) \in \ast \mathcal{F}_{R} \times \ast \mathcal{F}_{R}, F_{q,v}(h; *\lambda, \delta \lambda) = \pi_{v}(\lambda) \psi_{q}(\ast \lambda + \delta \lambda; h; \eta) \). \( F_{q,v}(h; \cdots) \) is uniquely determined by this condition and

\[ v_{q}(\ast \lambda + \delta \lambda) F_{q,v}(h; *\lambda, \delta \lambda) = \pi_{v}(\ast \lambda) \theta_{q}(\ast \lambda + \delta \lambda; h; \eta) \]

for all \( (*\lambda, \delta \lambda) \in \mathbb{T}(\mathbb{U}^{+}) \) (cf. (3.7.1)).

**Proof.** We choose \( \delta > 0 \) such that (i) Lemma 3.7.1 is valid and (ii) if \( \lambda \in \ast \mathcal{F}_{R} \) and \( ||\lambda|| < \delta \), then \( \lambda \in -U(*\Sigma, 1/2, \delta) \), \( \delta_{0} > 0 \) being as in Lemma 3.3.2.
TRANSFORMS ON LIE GROUPS

287

Then, for any \( \gamma \in \mathfrak{g} \), \( b_0 \) is holomorphic at \( \gamma + *\gamma \) whenever \( ||*\gamma|| < \delta \), and \( b_0(\gamma + *\gamma) = b_0(*\gamma) \). We shall prove the lemma for this choice of \( \delta \).

We consider first the case when \( \gamma = v \in \mathfrak{u} \). Let \( v' = e^{*\rho} \circ v \circ e^{-*\rho} \). We define, for \( q \in \mathbb{L}^+ \), \( h \in A^+ \),

\[
F_{\theta, \rho}(h: *\gamma; : \lambda) = e^{-*\rho(\log h)} \sum_{s' \in \mathfrak{m}_0} \varepsilon(s') b_0(s'^{-1})\chi(s'^{-1}; h; v')
\]

for all \( (\gamma, *\gamma) \in T(U') \); here \( \lambda = *\gamma + \gamma \), and the \( \chi \), as are as in \( \S \) 3.6. By the choice of \( \delta \), \( s'^{-1} \in \mathfrak{f} \) for all \( s' \in \mathfrak{m}_0 \) while the function \( (\gamma, *\gamma) \to b_0(s'^{-1}(*\gamma + \gamma)) \)

is well defined and holomorphic on \( T(U') \) for all \( s' \in \mathfrak{m}_0 \). So \( F_{\theta, \rho}(h: \cdots) \) is well defined and holomorphic on \( T(U') \). We shall now prove the first relation. Since both sides of it are continuous functions on \( \mathfrak{f} \times \mathfrak{f} \), it is enough to prove it for all \( (\gamma, *\gamma) \in \mathfrak{f} \times \mathfrak{f} \) with \( \pi_0(*\gamma) \neq 0 \). Fix such a \( (\gamma, *\gamma) \), and write \( \lambda = *\gamma + \gamma \). Then

\[
\varepsilon(s') b_0(s'^{-1}) = \pi_0(\gamma) c_0(s'^{-1})
\]

and hence (3.6.5) implies that \( F_{\theta, \rho}(h: *\gamma; \lambda) = \pi_0(\gamma) \psi_\lambda(h; v) \). The uniqueness is obvious as \( T(U') \) is connected. This proves the lemma when \( \gamma = v \in \mathfrak{u} \).

We now come to the general case when \( \gamma \in \mathfrak{m}_0 \). We can select [3] analytic functions \( f_j \) on \( A^+ \) and elements \( v_j \) of \( \mathfrak{u} \) (\( 1 \leq j \leq t \)) such that for all \( C^\infty \)

spherical functions \( \varphi \) on \( M_0 \), and \( h \in A^+ \), \( \varphi(h'; \eta) = \sum_{i \in \mathfrak{f}_j} f_j(h)\varphi(h'; v_j) \). We then define, for \( (\gamma, *\gamma) \in T(U') \),

\[
F_{\theta, \rho}(h: *\gamma; : \lambda) = \sum_{i \in \mathfrak{f}_j} f_j(h)F_{\theta, \rho}(h: *\gamma; : \lambda).
\]

It is obvious that \( F_{\theta, \rho}(h: \cdots) \) is well defined and holomorphic on \( T(U') \). Now, the \( \psi_\lambda(\lambda: \cdots) \) are spherical functions on \( M_0 \) for each \( \lambda \in \mathfrak{f} \); so, we have, for \( \lambda \in \mathfrak{f} \), \( \psi_\lambda(h; \eta) = \sum_{i \in \mathfrak{f}_j} f_j(h)\psi_\lambda(h; v_j) \). Since the lemma has already been proved for all elements of \( \mathfrak{u} \), the desired conclusions follow at once.

The next two lemmas give the main estimates on the growth of \( \psi_\lambda \).

**Lemma 3.7.3.** Let \( \delta > 0 \) be as in Lemma 3.7.2. Fix \( q \in \mathbb{L}^+ \), \( \gamma \in \mathfrak{m}_0 \), \( h \in A^+ \) and let \( F_{\theta, \rho}(h: \cdots) \) be as above. Then \( F_{\theta, \rho}(h: \cdots) \in \mathcal{S}^{\delta} \).

**Proof.** Given any \( \mu \in \mathfrak{m}_0 \) we can find constants \( c_\mu > 0 \) and \( l_\mu \geq 0 \) such that \( |\hat{\theta}(\lambda: \mu; \mu)| \leq c_\mu(1 + ||\lambda||)^l_\mu \hat{\theta}(\lambda: \mu; \mu) \) for all \( \lambda \in \mathfrak{f} \) and \( \mu \in \mathfrak{m}_0 \) (cf. Lemma 46 of [3]). This estimate shows that for fixed \( m \in M_{10} \) and \( \mu \in \mathfrak{m}_0 \), the function \( \theta(\mu; \cdot; \mu) \) lies in \( \mathcal{S}^{\delta} \). Let \( \delta > 0 \) be as in Lemma 3.7.2. For \( (\gamma, *\gamma) \in \mathfrak{f} \times \mathfrak{f} \), write \( H_q(\gamma: \lambda; \gamma) = \psi_q(*\gamma + \gamma) \) and for \( (\gamma, *\gamma) \in T(U') \), write

\[
H_{\theta, \rho}(h: *\gamma; : \lambda) = \Lambda_{\theta}(\gamma: \gamma)F_{\theta, \rho}(h: *\gamma; : \lambda).
\]

The observations made just now together with Lemma 3.7.2 imply that \( H_{\theta, \rho}(h: \cdots) \) belongs to \( \mathcal{S}^{\delta} \). We shall now apply Lemma 4 of the Appendix to conclude that \( F_{\theta, \rho}(h: \cdots) \) lies in \( \mathcal{S}^{\delta} \). For this we must verify that the polynomial \( \Lambda_q \) satisfies the conditions imposed on \( \Lambda \) in that lemma. Now, \( \Lambda_q \) is a product of functions of the form \( \Lambda_{s', s'^{\prime}} \) for \( s', \gamma' \in \mathfrak{m}, \gamma' \in \mathbb{L}^+ \), where
\[ \Lambda_{s',s';q'} = \Lambda_{s',s';q'}' + q'(\bar{H}) \]
and \( \Lambda_{s',s'} \) is the linear function on \( *F \times *F \) given by
\[ \Lambda_{s',s'}'((s' + \lambda, s' + \lambda)(\bar{H})) = \left( (s' + \lambda, s' + \lambda) \right) \in *F 	imes *F \right). \]
Since \( q'(\bar{H}) > 0 \), \( \Lambda_{s',s'} \equiv 0 \), while it is obvious that \( \Lambda_{s',s';q'} \) is real valued on \( *F \times *F \). This proves the lemma.

**Lemma 3.7.4.** There is \( \varepsilon_0 > 0 \) with the following property. For any \( q \in L^+, m \in M_0, \eta \in M_{10}, \psi_q(\cdot; m; \eta) \) is the restriction of \( F \) to \( F \) for a function in \( \mathcal{L} \) where \( U = \{\lambda: \lambda \in F, ||\lambda|| < \varepsilon_0\} \); moreover, given any compact set \( \omega \subseteq M_{10}, \)
we can select constants \( c = c_{q,\varepsilon,\omega} > 0 \) and \( l = l_{q,\varepsilon,\omega} \geq 0 \) such that \( |\psi_q(\lambda; m; \eta)| \leq c(1 + ||\lambda||)^l \) for all \( m \in \omega, \lambda \in F \) with \( ||\lambda|| < \varepsilon_0 \).

**Proof.** Follows from (3.7.1).

**Lemma 3.7.5.** Fix \( q \in L^+ \). Then \( P_q \) is the set of all real numbers \( t \) such that \( \inf_{x \in F} \|v_q(\lambda - t\rho_0)\| = 0 \). Then \( P_q \) is finite, \( 0 \in P_q \), and for \( t \in R \setminus P_q, \)
\[ \inf_{x \in F} \|v_q(\lambda - t\rho_0)\| > 0. \]

**Proof.** Fix \( s', s'' \in w, q' \in L^+ \). Let \( f_{s',s';q'} \) be the function on \( F \) defined by \( f_{s',s';q'}(\lambda) = (s' + s'' + q')(\bar{H}) \). For \( \lambda \in F \) and \( t \in R \), \( \Re f_{s',s';q'}(\lambda - t\rho_0) = t(s''\rho_0 - s'\rho_0)(\bar{H}) + q'(\bar{H}) \). Now \( q'(\bar{H}) > 0 \) and so \( t(s''\rho_0 - s'\rho_0)(\bar{H}) + q'(\bar{H}) \) can vanish for at most one value of \( t \), and this value cannot be zero. If \( t \) is not equal to this exceptional value, we have, \( \inf_{x \in F} \|f_{s',s';q'}(\lambda - t\rho_0)\| \geq \|t(s''\rho_0 - s'\rho_0) + q'(\bar{H})\| > 0 \). Since \( v_q \) is a product of finitely many of the \( f_{s',s';q'} \), the lemma follows at once.

### 3.8. Two lemmas. We recall that \( *A = \exp *a, A = \exp a \). Also, if \( *\lambda \in *F, \lambda \in \mathcal{F}, m \in M_{0,0}, a \in A_0, \eta_1 \in M_0, \eta_2 \in M_0, \) then
\[ \theta((*\lambda + \lambda; ma; \eta_1, \eta_2) = \theta((*\lambda; m; \eta_1) e^{2(\log a)} \eta_2(\lambda)). \]
We select Lebesgue measures \( d(*\lambda) \) and \( d(\lambda) \) on \( *F \) and \( \mathcal{F} \) respectively such that \( d\lambda \sim d(*\lambda) \). Let \( \pi = \pi_\beta \). For brevity of notation we write \( \gamma(\lambda) = c_{\lambda}(\lambda)^{-1} c_{\overline{\lambda}}(\lambda)^{-1} \) and \( \gamma(\lambda) = c_{\ast}(\lambda)^{-1} c_{\overline{\lambda}}(\lambda)^{-1} \).

**Lemma 3.8.1.** Fix \( \varepsilon > 0 \). Then for \( h_1 \in *A, h_2 \in A_0, H = a_0, \eta_1 \in M_{0,0}, \eta_2 \in M_{0,0} \) and any \( a \in Z(F) \),
\[ \int \theta((\lambda; h_1, h_2; \eta_1, \eta_2) a(\lambda) \xi(\lambda) \pi^+(\lambda) e^{i(\overline{H})} \gamma(\lambda) d\lambda = e^{-r_{p(\beta)} + \log a} \int \theta((\lambda; h_1; \eta_1) \mathcal{A}_{a_1}(H + \log a \xi(\lambda) \gamma(\lambda) d\lambda. \]
Here \( a_1 = \eta_2 \pi^+ a \), \( A_{a_1, H + \log a} \) is defined by Lemma 3.4.3, and both sides converge absolutely.
Proof. It is obvious that the two integrands lie in the respective Schwartz spaces. Hence both integrals converge absolutely. Moreover, using (3.8.1) and the Fubini theorem the left side of (3.8.2) reduces to

\[ \int_{F_I} \theta(*\lambda; h; \eta) c_0(\lambda) c_0(-\lambda)^{-1} I(\lambda) d\star \lambda \]

where, for \(*\lambda \in F_I^*,

\[ I(\lambda) = \int_{0 \times F_I} a_1(\lambda + \lambda') \xi(\lambda + \lambda') e^{\rho_0(H + \log h_0)} d\lambda \]

and \(a_1 = \eta \pi^* a \in \mathcal{Z}(F^*)\). By Lemma 3.4.3, for \(*\lambda \in F_I^*

\[ I(\lambda) = e^{-i\rho_0(H + \log h_0)} A \tau_{\pi^* a \pi^*} \]

Lemma 3.8.1 now follows at once.

Let \(\theta_q (q \in L^+)\) be as in § 3.7. Let \(P_q\) be the finite subset of \(R\) considered in Lemma 3.7.5.

**Lemma 3.8.2.** Fix \(\varepsilon > 0\). Let \(q \in L^+, h \in A^+, H \in \pi, \eta \in M^\pi\). Suppose \(\varepsilon'\) is such that \(0 < \varepsilon' < \varepsilon\) and \(\varepsilon' \in R\setminus P_q\). Then for any \(a \in \mathcal{Z}(F^*)

\[ \int_{F_I} \psi_q(\lambda; h; \eta) a(\lambda) \xi(\lambda) \pi^*(\lambda) e^{\lambda H} \gamma(\lambda) d\lambda 
\]

\[ = e^{-i\rho_0(H)} \int_{F_I} \theta_q(\lambda - \varepsilon' \rho_0; h; \eta) \tau_{\pi^* a \pi^*} \gamma(\lambda) d\lambda \]

here \(\tau_{\pi^* a \pi^*}\) is defined as in Lemma 3.4.4, and both integrals converge absolutely.

Proof. In view of Lemma 3.7.4 it is obvious that the integrand in the left of (3.8.3) lies in \(S(F_I^*)\). On the other hand, since \(\varepsilon' \in R\setminus P_q\), \(v_q\) satisfies the conditions imposed on \(f\) in Lemma 3.4.4. So, by that lemma, \(\tau_{\pi^* a \pi^*}\) lies in \(S(F_I^*)\). It follows easily that the integrand on the right also belongs to \(S(F_I^*)\). By Lemma 3.7.2 and the Fubini theorem, the integral on the left of (3.8.3) reduces, putting \(a' = \pi^* a\), to

\[ \int_{F_I} \pi_0(-\lambda) b_0(\lambda)^{-1} b_0(-\lambda)^{-1} I(\lambda) d\star \lambda \]

where, for \(*\lambda \in F_I^*,

\[ I(\lambda) = \int_{0 \times F_I} F_{\pi^* a \pi^*} (h; \lambda; \eta) a'(-\lambda + \lambda') \xi(\lambda + \lambda') e^{\rho_0(H)} d\lambda \]

Let \(\delta > 0\) and let \(0 V^\delta\) be the open subset of \(0 F^*_R\) defined by (3.4.3). We saw during the course of the proof of Lemma 3.4.3 that \(0 V^\delta \subseteq \text{Int} F^*_R\) if \(\delta\) is small enough, and is connected. Choose \(\delta > 0\) so that (1) \(0 V^\delta \subseteq F^*_R\), (2) \(U^\delta \subseteq q\) (cf. (3.3.3)) and (3) Lemmas 3.7.2 and 3.7.3 are applicable. The work of §§ 3.3
and 3.7 now reveals that Lemma 3 of the Appendix is applicable. We therefore conclude, as \(-\varepsilon'\rho_0 \in V^s\), that for all \(*\lambda \in \ast F_I,\)

\[
I(*\lambda) = e^{-\varepsilon'\rho_0(h)} \times \int_{\text{aff} \mathcal{F}_I} F_{\mathfrak{a}, \eta}(h; *\lambda : \nu \lambda - \varepsilon' \rho_0) \tau_{\mathfrak{a}}(\nu \lambda + \varepsilon' \rho_0) \epsilon(\nu \lambda + \varepsilon' \rho_0) e^{2i(H)} d\nu \lambda.
\]

In view of Lemma 3.4.4 and 3.7.2 we find, for \(*\lambda \in \ast F_I,\)

\[
I(*\lambda) = e^{-\varepsilon'\rho_0(h)} \pi_0(*\lambda) \int_{\text{aff} \mathcal{F}_I} \theta_{\mathfrak{a}}(\nu \lambda + \varepsilon' \rho_0; h; \eta) \tau_{\mathfrak{a} + \varepsilon' \rho_0} \epsilon(\nu \lambda + \varepsilon' \rho_0) e^{2i(H)} d\nu \lambda.
\]

This leads at once to (3.8.3).

3.9. Formation of wave packets. We saw in §3.5 that if \(0 < p < 2\), then for any \(f \in \mathcal{S}^p(G), \hat{f}\) lies in \(\mathcal{S}(\mathcal{F})\) (\(\varepsilon = (2/p) - 1\)) and that \(f \mapsto \hat{f}\) is a continuous map of \(\mathcal{S}^p(G)\) into \(\mathcal{S}(\mathcal{F})\). We shall devote the remainder of the paper to the proof of the fact that this map is a linear topological isomorphism. This is our main result.

Suppose \(a \in \mathcal{S}(\mathcal{F})\). We write \(\varphi_a\) for the "wave packet" given by

\[
(3.9.1) \quad \varphi_a(x) = \int_{\mathcal{F}} a(\lambda) \varphi(\lambda : x) \epsilon(\lambda) \nu (-\lambda) d\lambda \quad (x \in G).
\]

It then follows from Harish-Chandra's work [4] that \(\varphi_a \in \mathcal{S}(G)\) and that there is a nonzero constant \(c_0\) such that \(\varphi_a = c_0 \sum_{a \in \mathfrak{a}} a\). \(\forall a \in \mathcal{S}(\mathcal{F})\). Our aim in this section is to obtain a proof that \(\varphi_a \in \mathcal{S}^p(G)\) for all \(a \in \mathcal{S}(\mathcal{F})\) and that \(a \mapsto \varphi_a\) is a continuous map of \(\mathcal{S}(\mathcal{F})\) into \(\mathcal{S}^p(G)\). This proof will go by induction on \(\dim G\).

To begin with, we shall assume the induction hypothesis and derive two crucial estimates. Note that \(\varphi_a = \varphi_{\bar{a}}\) where \(\bar{a} = [iv]^{-1} \sum_{a \in \mathfrak{a}} a\).

**Lemma 3.9.1.** Fix \(0 < p < 2\) and let \(\varepsilon = (2/p) - 1\). Let the induction hypothesis hold. Fix \(H_0 \in \text{Cl}(a^+)\), \(H_0 \neq 0\), and use earlier notation. Then, given any \(\eta \in \mathcal{M}_{10}\) and any integer \(l \geq 0\), we can find a continuous seminorm \(\zeta = \zeta_{\eta, l}\) on \(\mathcal{S}(\mathcal{F})\) such that

\[
(3.9.2) \quad \left| e^{-\rho_0(\log h + H)} \int_{\mathcal{F}} \theta(\lambda : h; \eta) \varphi(\lambda) \pi(\nu \lambda) \epsilon(\nu \lambda) e^{2i(h)} \gamma(\lambda) d\lambda \right| \leq \zeta_{\eta, l}(a) e^{-(2/p)\rho(\log h + H)} (1 + \|\log h + H\|)^{-l}
\]

for all \(h \in \text{Cl}(A^+), H \in a_0, a \in \mathcal{S}(\mathcal{F})\).

**Proof.** Since both sides are continuous functions of \(h\) on \(A\), it is sufficient to prove the estimate (3.9.2) for all \(h \in A^+\). We may also assume that \(\eta = \eta_1 \eta_2\) with \(\eta_1 \in \mathcal{M}_0, \eta_2 \in \mathcal{M}_0\). Let \(a \in \mathcal{S}(\mathcal{F}), H \in a_0\) and \(h_1 \in A, h_2 \in A_0\). Then, by Lemma 3.8.1,
\[ 
\int_{\mathcal{F}} \theta(\lambda; h, h_2; \eta_1, \eta_2) a(\lambda) \xi(\lambda) \pi^+(\lambda) e^{i(H/\gamma)} \gamma(\lambda) d\lambda 
\]
\[ = e^{-\rho_0(H + \log h_2)} \int_{\mathcal{F}} \theta^*(\lambda; h_1; \eta_1) A_{a_1, H + \log h_2}^*(\lambda) \gamma(\lambda) d\lambda \]

where \( a_1 = \eta_2 \pi^+ a \). By Lemma 3.4.3, \( A_{a_1, H + \log h_2} \in \mathfrak{Z}^*(\mathcal{F}^i) \) so that the induction hypothesis is applicable. For any \( f \in \mathfrak{Z}^*(\mathcal{F}^i) \) let

\[ \theta_f(m) = \int_{\mathcal{F}} \theta^*(\lambda; m) f(\lambda) \gamma(\lambda) d\lambda \quad (m \in M_0). \]

Then (cf. [4, Theorem 3]) for any \( \eta' \in \mathcal{W}_0 \),

\[ \theta_f(m; \eta') = \int_{\mathcal{F}} \theta^*(\lambda; m; \eta') f(\lambda) \gamma(\lambda) d\lambda \quad (m \in M_0). \]

By the induction hypothesis, \( \theta_f \in \mathcal{B}^0(M_0) \) and \( f \mapsto \theta_f \) is a continuous map of \( \mathfrak{Z}^*(\mathcal{F}^i) \) into \( \mathcal{B}^0(M_0) \). So, given any \( \eta' \in \mathcal{W}_0 \) and any integer \( l' \geq 0 \), we can find a continuous seminorm \( *z^{(v, l')} \) on \( \mathfrak{Z}^*(\mathcal{F}^i) \) such that

\[ |\theta_f(m; \eta')| \leq z^{(v, l')}(f) E_0(m)^{2/p} (1 + \sigma(m))^{-l'} \]

for all \( m \in M_0 \) and \( f \in \mathfrak{Z}^*(\mathcal{F}^i) \). Hence

\[ e^{-\rho_0(H + \log h_1 h_2)} \int_{\mathcal{F}} \theta(\lambda; h, h_2; \eta_1, \eta_2) a(\lambda) \xi(\lambda) \pi^+(\lambda) e^{i(H/\gamma)} \gamma(\lambda) d\lambda \]

\[ \leq e^{-\rho_0(H + \log h_1 h_2)} \cdot e^{-\rho_0(H + \log h_2)} E_0(h_1)^{2/p} (1 + \sigma(h_1))^{-l'} z^{(v, l')}(A_{a_1, H + \log h_2}) \]

for all \( a \in \mathfrak{Z}^*(\mathcal{F}^i) \), \( H \in \alpha_0 \), \( h_1 \in *A \), \( h_2 \in A_0 \). We now select a continuous seminorm \( z^{(v, l')}_1 \) on \( \mathfrak{Z}^*(\mathcal{F}^i) \) such that (Lemma 3.4.3) \( z^{(v, l')}_1(A_{a_1, H'}) \leq z^{(v, l')}_1(a') (1 + \|H'\|)^{-l'} \) for all \( a' \in \mathfrak{Z}^*(\mathcal{F}^i) \) and \( H' \in \alpha_0 \). On the other hand, as \( a \mapsto \eta_2 \pi^+ a = a_1 \) is a continuous endomorphism of \( \mathfrak{Z}^*(\mathcal{F}^i) \), we can find a continuous seminorm \( z^{(v, l')}_2 \) on \( \mathfrak{Z}^*(\mathcal{F}^i) \) such that for all \( a \in \mathfrak{Z}^*(\mathcal{F}^i) \), \( z^{(v, l')}_2(a') \leq z^{(v, l')}_2(a) \). Finally, we select a constant \( c_0 > 0 \) and an integer \( c_0 \geq 0 \) such that \( E_0(h') \leq c_0 e^{-\rho_0(\log h')^p} (1 + \sigma(h'))^c \) for all \( h' \in (*A)^+ \); here \( (*A)^+ \) is the set of all \( h' \in *A \) such that \( \alpha(\log h') > 0 \) for all \( \alpha \in *A^+ \). Taking into account all of this and noting that if \( h_1 h_2 \in A^+ \), then \( h_1 \in (*A)^+ \), we then have the following result. Given \( \eta \) and \( l \) as in the lemma, we can find a continuous seminorm \( z^{(v, l)}_1 \) on \( \mathfrak{Z}^*(\mathcal{F}^i) \) such that for all \( h = h_1 h_2 \in A^+ \) with \( h_1 \in *A \), \( h_2 \in A_0 \), \( H \in \alpha_0 \), and \( a \in \mathfrak{Z}^*(\mathcal{F}^i) \), the left side of (3.9.2) is majorized by

\[ z^{(v, l)}_1(a) e^{-\rho_0(H + \log h_1 h_2) - \rho_0(\log h_1 h_2)^p} (1 + \sigma(h))^{-l} (1 + \|H + \log h_2\|)^{-l}. \]

We now note that \( (1 + \|\log h_1\|)(1 + \|H + \log h_2\|) \geq 1 + \|\log h_1\| + \|H + \log h_2\| \geq 1 + \|H + \log h_1 h_2\| \) while \( \rho_0(H + \log h_1 h_2) + \rho_0(\log h_1) + (2/p) \rho(\log h_1) = (2/p) \rho(H + \log h_1 h_2) \). If we replace \( h_1 h_2 \) by \( h \), we are led to (3.9.2). This proves the lemma.
LEMMA 3.9.2. Fix $p, \varepsilon$ as above. Let $q \in L^+, \eta \in M_{10}$ and let $\varepsilon' \leq \varepsilon$ and $\varepsilon' \in R \backslash P_q$ where $P_q$ is as in Lemma 3.7.4. Let $l \geq 0$ be an integer. Then we can select a continuous seminorm $\xi_{q, \varepsilon'}^{(l)}$ on $Z(\mathcal{F})$ such that

\[
(3.9.4) \quad \left| e^{-\rho_0(1 + \log h)} \int_{\mathcal{F}} \psi_q(\lambda; h; \eta) a(\lambda) \pi^+(\lambda) \xi(\lambda) e^{i(H)\tau}(\lambda) d\lambda \right| \\
\leq \xi_{q, \varepsilon'}^{(l)}(a) e^{-\rho_0(1 + \log h)} (1 + \|H + \log h\|)^{-l} e^{(\varepsilon' \rho_0)(1 + \log h)} + \varepsilon' \rho_1(\log h) \\
\text{for all } h \in C(A^+), H \in A_0 \text{ and } a \in Z(\mathcal{F}).
\]

Proof. It follows from Lemma 3.7.4 that the integral in the left of (3.9.4) is well defined and continuous for all $h \in A$. Hence, in view of the continuity, it is enough to obtain (3.9.4) for $h \in A^+$. By Lemma 3.8.2, for $h \in A^+$ and $H \in A_0$, writing $\tau = \tau_{q, \varepsilon'}^{+a, a'}$,

\[
\int_{\mathcal{F}} \psi_q(\lambda; h; \eta) a(\lambda) \pi^+(\lambda) \xi(\lambda) e^{i(H)\tau}(\lambda) d\lambda \\
= e^{-\rho_0(1 + \log h)} \int_{\mathcal{F}} \theta_q(\lambda - \varepsilon' \rho_0; h; \eta) \tau(\lambda) e^{i(H)\tau}(\lambda) d\lambda \\
= e^{-\rho_0(1 + \log h)} \sum_{i \in i, j, l} g_{q, i, j}(h) \int_{\mathcal{F}} \theta(\lambda - \varepsilon' \rho_0; h; \mu_q, i, j) \tau(\lambda) e^{i(H)\tau}(\lambda) d\lambda \\
= e^{-\rho_0(1 + \log h)} \sum_{i \in i, j, l} g_{q, i, j}(h) \int_{\mathcal{F}} \theta(\lambda; h \exp H; d_{i, j}^o \mu_q, i, j^o d_{i, j}^o) \tau(\lambda) \gamma(\lambda) d\lambda
\]

here we are using the easily proved identities

\[
\theta(\lambda; ma; \mu) = \theta(\lambda; m; \mu) e^{(\log a)} \quad (m \in M_{10}, a \in A_0) \\
\theta(\lambda - \varepsilon' \rho_0; m_1; \mu) = d_{i, j}^o \theta(\lambda; m_1; \mu) e^{(\log a)} \quad (m_1 \in M_{20})
\]

$d_{i, j}^o$ being as in §1.2. Since $g_{q, i, j} \in G$, we can (cf. (2.2.2)) select a constant $C_{q, \eta} > 0$ such that for all $h \in A^+$, $1 \leq j \leq l_{q, \eta}$, $|g_{q, i, j}(h)| \leq C_{q, \eta}$. Using these, we obtain, for all $a \in Z(\mathcal{F})$, $h \in A^+$, $H \in A_0$,

\[
\left| e^{-\rho_0(1 + \log h)} \int_{\mathcal{F}} \psi_q(\lambda; h; \eta) a(\lambda) \pi^+(\lambda) \xi(\lambda) e^{i(H)\tau}(\lambda) d\lambda \right| \\
\leq C_{q, \eta} e^{-\rho_0(1 + \log h)} (1 + \|H + \log h\|)^{-l} e^{(\varepsilon' \rho_0)(1 + \log h)} \\
\times \sum_{i \in i, j, l} \left| \int_{\mathcal{F}} \theta(\lambda; h \exp H; d_{i, j}^o \mu_q, i, j^o d_{i, j}^o) \tau(\lambda) \gamma(\lambda) d\lambda \right|.
\]

Now, by Theorem 3 of [4], we can select a continuous seminorm $\psi_{q, \varepsilon'}^{(l)}$ on $S(\mathcal{F}_l)$ such that

\[
\sum_{i \in i, j, l} \left| \int_{\mathcal{F}} \theta(\lambda; h \exp H; d_{i, j}^o \mu_q, i, j^o d_{i, j}^o) \tau(\lambda) \gamma(\lambda) d\lambda \right| \\
\leq \psi_{q, \varepsilon'}^{(l)}(f) \Xi_0(h \exp H)(1 + \|H + \log h\|)^{-l(1 + \rho_0)},
\]

for all $f \in S(\mathcal{F}_l)$, $h \in A^+$, $H \in A_0$, $e_0$ being as in the proof of the previous lemma. By Lemma 3.4.4 and the continuity of the endomorphism $a \mapsto \pi^+a$ of $Z(\mathcal{F})$,
we can select a continuous seminorm $\zeta_{s, t}^{i, j}$ on $\mathcal{Z}(F')$ such that $\nu_{s, t}^{i, j}(\tau) \leq \zeta_{s, t}^{i, j}(a)$ for all $a \in \mathcal{Z}(F')$. Moreover, if we write $h = h_1h_2$ with $h_1 \in \ast A$, $h_2 \in A_0$, then $h \in A^+$ implies that $h_1 \in \ast A^+$ and so

$$
\Xi_0(h \exp H) = \Xi_0(h_1) \leq c_0 e^{-\rho(\log h_1)(1 + ||\log h_1||)}.
$$

At the same time, as $\ast a$ and $a_0$ are mutually orthogonal, $||\log h_1|| \leq ||\log h_1 + (H + \log h_2)|| = ||H + \log h||$. Hence, using the relations proved just now, we get the following result. There is a continuous seminorm $\zeta_{s, t}^{i, j}$ on $\mathcal{Z}(F')$ such that the left side of (3.9.4) is majorized by

$$
\zeta_{s, t}^{i, j}(a) e^{-(1/2)p\rho(\log h_1)(1 + ||\log h_1 + (H + \log h_2)||)}(1 + ||H + \log h||)^{-1}.
$$

(3.9.4) follows at once from this.

Let $W$ be a compact subset of $\Cl(a^+)$. We shall now define the sectors $S[ W]$ by

$$
S[ W] = \{\exp tH': t \geq 0, H' \in W\}.
$$

For any $a \in \mathcal{Z}(F')$ let $\varphi_a$ be defined by (3.9.1). Then (cf. [4, Theorem 3]) for any $b \in \mathcal{S}$ and all $x \in G$,

$$
\varphi_a(x; b) = \int \gamma_1 \varphi_2(x; b) a(\lambda) e(\lambda)^{-1} c(-\lambda)^{-1} d\lambda.
$$

**Lemma 3.9.3.** Let $\varepsilon$, $p$ be as above. Suppose $W$ is a compact subset of $\Cl(a^+)$ and $t_0 > 0$. Then given any $b \in \mathcal{S}$ and any integer $l \geq 0$ we can select a continuous seminorm $\zeta_{s, t}^{i, j}$ on $\mathcal{Z}(F')$ such that

$$
\sup_{H' \in W} \Xi(\exp tH')^{1/2}(1 + \sigma(\exp tH'))^l |\varphi_a(\exp tH'; b) | \leq \zeta_{s, t}^{i, j}(a)
$$

for all $a \in \mathcal{Z}(F')$.

**Proof.** Let $L = \{\exp tH': H' \in W, 0 \leq t \leq t_0\}$. Then $L$ is compact and we can find a constant $C > 1$ such that $C^{-1} \leq \Xi(x) \leq C, C^{-1} \leq 1 + \sigma(x) \leq C$ for all $x \in L$ (cf. [5]). For $f \in \mathcal{S}(G)$ let

$$
\nu(f) = C^{3/2} \sup_{x \in C} \Xi(x)^{-1} |f(x; b)|.
$$

Since $\nu$ is continuous on $\mathcal{S}(G)$, we can find a continuous seminorm $\nu'$ on $\mathcal{S}(F'_i)$ such that $\nu(\varphi_a) \leq \nu'(a)$ for all $a \in \mathcal{Z}(F')$. The lemma follows at once from Lemma 3.4.1.

Consider the finite sets $P_s \ (q \in L^+)$ of Lemma 3.7.5. For any integer $k \geq 0$ let $P^{(k)} = \bigcup_{q \leq k} \mathcal{S}(G) P_q$. $P^{(k)}$ is finite for all $k$. Write $P^{(\infty)} = \bigcup_{q \leq k} \mathcal{S}(G) P^{(k)}$. $P^{(\infty)}$ is at most countable. Fix $H_0 \in \Cl(a^+)$, $H_0 \neq 0$. We shall construct a compact neighborhood $W_{H_0}$ of $H_0$ in the following manner. First we select $\varepsilon'$ such that $0 < \varepsilon' < \varepsilon$, $(\varepsilon - \varepsilon') \rho_0(H_0) < \beta(H_0)$, $\varepsilon' \in \mathbb{R} \setminus P^{(\infty)}$. We then define $\kappa$ by

$$
(\varepsilon - \varepsilon') \rho_0(H_0) = (1 - \kappa) \beta(H_0).
$$
Clearly \( 0 < \kappa < 1 \). We then choose \( \kappa' > 0 \) such that \((1 - \kappa)(1 + \kappa') < 1 - \kappa/2\). Now, for \( H' \) sufficiently close to \( H_0 \) and belonging to \( \mathrm{Cl}(\alpha^+) \), \( \alpha(H') \geq (1 - \kappa/2)\alpha(H_0) \) for all \( \alpha \in \Delta^+ \) while \( \|H'\| \leq 2 \|H_0\| \). Moreover, if \( H' \in \mathrm{Cl}(\alpha^+) \) and \( H' \to H_0, (\varepsilon - \varepsilon')\rho_\alpha(H') + \varepsilon^* \rho(H') \rightarrow (\varepsilon - \varepsilon')\rho_\alpha(H_0) \); so, for \( H' \) in \( \mathrm{Cl}(\alpha^+) \) sufficiently near \( H_0 \), \( (\varepsilon - \varepsilon')\rho_\alpha(H') + \varepsilon^* \rho(H') \leq (1 + \kappa')(\varepsilon - \varepsilon')\rho_\alpha(H_0) \). We now define \( W_{H_0} \) as the set of all \( H' \in \mathrm{Cl}(\alpha^+) \) such that

\[
\begin{align*}
(i) & \quad \|H'\| \leq 2 \|H_0\| \\
(ii) & \quad \alpha(H') \geq \left(1 - \frac{\kappa}{2}\right)\alpha(H_0) \quad \text{for all } \alpha \in \Delta^+ \\
(iii) & \quad (\varepsilon - \varepsilon')\rho_\alpha(H') + \varepsilon^* \rho(H') \leq (1 + \kappa')(\varepsilon - \varepsilon')\rho_\alpha(H_0) .
\end{align*}
\]

(3.9.7)

It is clear from what we said above that \( W_{H_0} \) is a compact neighborhood of \( H_0 \) in \( \mathrm{Cl}(\alpha^+) \).

**Lemma 3.9.4.** Fix \( H_0 \in \mathrm{Cl}(\alpha^+), H_0 \neq 0 \). Define \( W_{H_0} \) as above. For any \( H' \in W_{H_0} \) let

\[
H'_1 = H' - \left(1 - \frac{\kappa}{2}\right)H_0, \quad H'_2 = \left(1 - \frac{\kappa}{2}\right)H_0 .
\]

Then \( H'_1 \in \mathrm{Cl}(\alpha^+) \) and

\[
(\varepsilon - \varepsilon')\rho_\alpha(H') + \varepsilon^* \rho(H') < \beta(H'_2) \quad \forall H' \in W_{H_0} .
\]

**Proof.** It is obvious that \( H'_1 \in \mathrm{Cl}(\alpha^+) \). Moreover, it is clear that \( \beta(H'_2) = (1 - \kappa/2)\beta(H_0) > (\varepsilon - \varepsilon')\rho_\alpha(H') + \varepsilon^* \rho(H') \) if we remember the definitions of \( \kappa \) and \( \kappa' \).

**Lemma 3.9.5.** Let \( p, \varepsilon \) be as above. Assume the induction hypothesis. Fix \( H_0 \in \mathrm{Cl}(\alpha^+), H_0 \neq 0 \). Define \( W_{H_0} \) as in (3.9.7). Then, for any \( u \in \mathcal{G} \) and any integer \( l \geq 0 \), we can select a continuous seminorm \( \zeta_{u,l} \) on \( \mathbb{Z}(F') \) such that

\[
|\varphi_a(h; u)| \leq \zeta_{u,l}(a) e^{-(2/p)\rho(\log h)}(1 + \|\log h\|)^{-l}
\]

for all \( a \in \mathbb{Z}(F') \) and \( h \in S[W_{H_0}], S[W_{H_0}] \) being defined by (3.9.5).

**Proof.** We write \( S = S[W_{H_0}] \) and for any \( \tau \geq 0 \) let

\[
S_\tau = \{\exp tH' : H' \in W_{H_0}, t \geq \tau\} .
\]

Since \( a \mapsto \tilde{a} \) is a continuous map of \( \mathbb{Z}(F') \) onto \( \tilde{\mathbb{Z}}(F') \), it is enough to prove our estimate for \( a \in \tilde{\mathbb{Z}}(F') \). By Lemma 3.9.3 it suffices to select a continuous seminorm \( \zeta_{u,l} \) on \( \mathbb{Z}(F') \) and a \( t_0 > 0 \) such that \( \forall a \in \tilde{\mathbb{Z}}(F') \)

\[
|\varphi_a(h; u)| \leq \zeta_{u,l}(a) e^{-(2/p)\rho(\log h)}(1 + \|\log h\|)^{-l} \quad (\forall h \in S_{t_0}) .
\]

We shall prove this for a suitable \( \zeta_{u,l} \) and \( t_0 = [(1 - \kappa/2)\beta(H_0)]^{-1} \).

We shall consider first the case when \( u \in \mathcal{M}_{10} \). Let us write \( \eta = d_0 \circ u \circ d_0^{-1} \), \( d_0 \) being as in §1.2. For any integer \( k \geq 0 \) let us write
\[ P_k(\lambda : m : H) = d_\theta (m \exp H)^{-1} \]
\[ \times \sum_{\ell \leq p \leq r} \omega (s^-_{\ell \lambda}) \exp ((s^-_{\ell \lambda})(H)) \theta (s^-_{\ell \lambda} : m ; \eta) \]
\[ + d_\theta (m \exp H)^{-1} \sum_{\ell \leq P \leq k} e^{-q(H)} \]
\[ \times \sum_{\ell \leq p \leq r} \omega (s^-_{\ell \lambda}) \exp ((s^-_{\ell \lambda})(H)) \psi (s^-_{\ell \lambda} : m ; \eta) \]
for all \( \lambda \in \mathcal{F}'_1 \), \( m \in M_{10}^\omega \), \( H \in a_0^\omega \). Put
\[ \varphi (\lambda : m \exp H ; u) = P_k(\lambda : m : H) + P_k(\lambda : m : H) \]
where \( R_k \) is defined by this equation. Now,
\[ \omega (\lambda) c(\lambda)^{-1} c(-\lambda)^{-1} = \pi_0 (\lambda) \pi_0 (-\lambda) b_0 (\lambda)^{-1} b_0 (-\lambda)^{-1} \rho_0 (-\lambda) \xi (\lambda) \]
for all \( \lambda \in \mathcal{F}'_1 \). This shows that the left side of this relation is the restriction to \( \mathcal{F}'_1 \) of a function in \( S^\omega \), \( U \) being some neighborhood of 0 in \( \mathcal{F}_R \). In view of the estimates in §3.7 and the \( m \)-invariance of the function \( \lambda \mapsto a(\lambda) c(\lambda)^{-1} c(-\lambda)^{-1} \)
and the fact that each term in (3.9.8) as well as \( R_k \) are integrable with respect to \( a(\lambda) c(\lambda)^{-1} c(-\lambda)^{-1} d\lambda \). We thus have
\[ \varphi (\lambda : m \exp H ; u) = \int_{\mathcal{F}_1} P_k(\lambda : m : H) a(\lambda) c(\lambda)^{-1} c(-\lambda)^{-1} d\lambda \]
\[ + \int_{\mathcal{F}_1} R_k(\lambda : m : H) a(\lambda) c(\lambda)^{-1} c(-\lambda)^{-1} d\lambda \]
for all \( a \in \tilde{S}(\mathcal{F}) \), \( m \in M_{10}^\omega \), \( H \in a_0^\omega \). Furthermore,
\[ \int_{\mathcal{F}_1} P_k(\lambda : m : H) a(\lambda) c(\lambda)^{-1} c(-\lambda)^{-1} d\lambda \]
\[ = r^* d_\theta (m \exp H)^{-1} \theta (\lambda : m) + \sum_{\ell \leq P \leq k} e^{-q(H)} \psi (\lambda : m) \]
for all \( a \), \( m \), \( H \) as above; here \( r^* = (-1)^{[\mathfrak{a}^\omega]} r \) and
\[ \theta (\lambda : m ; \eta) = \int_{\mathcal{F}_1} \theta (\lambda : m ; \eta) a(\lambda) \pi_0 (\lambda) \xi (\lambda) e^{iH} \gamma (\lambda) d\lambda \]
\[ \psi (\lambda : m ; \eta) = \int_{\mathcal{F}_1} \psi (\lambda : m ; \eta) a(\lambda) \pi_0 (\lambda) \xi (\lambda) e^{iH} \gamma (\lambda) d\lambda \].

We shall now estimate these terms. First we use Theorem 2.11.2. It follows from that theorem that
\[ |R_k(\lambda : m : H)| \leq d_\theta (m \exp H)^{-1} A_{k, \gamma} [(1 + ||H||)(1 + ||\ell||)]^{r_k} \]
\[ \times \gamma (m)^3 \Xi (m) (1 + \sigma (m))^\delta e^{-q(H)} \]
for all \( \lambda \in \mathcal{F}'_1 \), \( m \in M_{10}^\omega \), \( H \in a_0^\omega \) with \( \beta (H) \geq 1 \). So, if we write
\[ \zeta_k (a) = A_{k, \gamma} \int_{\mathcal{F}_1} |a(\lambda)| (1 + ||\ell||)^{r_k} |c(\lambda)^{-1} c(-\lambda)^{-1}| d\lambda \]
then \( \zeta_k \) is a continuous seminorm on \( \tilde{S}(\mathcal{F}) \) and we have the estimate
\[ \left| \int_{\mathcal{F}} R_k(\lambda; h; H) a(\lambda) e(\lambda)^{-1} c(-\lambda)^{-1} d\lambda \right| \leq \zeta_k(a)(1 + \|H\|)^{\delta_k \gamma(h)} (1 + \|\log h\|)^{\delta_k \Xi_k(h)} e^{-(k+1)H - \rho_0(H + \log h) \|h\|} \]

for all \(a \in \mathcal{Z}(\mathcal{F})\), \(H \in a_\mathcal{F}^+\) with \(\beta(H) \geq 1\) and \(h \in \text{Cl}(A^+)_q\). We now take up the estimation of \(\theta_{\mathcal{F}, \gamma}\) and \(\psi_{\mathcal{F}, a, \gamma}\). This is done with the help of Lemmas 3.9.1 and 3.9.2. By Lemma 3.9.1 we can select, corresponding to \(\gamma \in \mathcal{M}_0\), and integer \(l \geq 0\), a continuous seminorm \(\zeta_{0, l, \mathcal{F}}\) on \(\mathcal{Z}(\mathcal{F})\) such that

\[ |r^* e^{-\rho_0(H + \log h)} \theta_{\mathcal{F}, \gamma}(h; H)| \leq \zeta_{0, l, \mathcal{F}}(a) e^{-(2/p)H + \log h + \|H + \log h\|^{-1}} \]

for all \(h \in \text{Cl}(A^+)_q\), \(H \in a_0\), \(a \in \mathcal{Z}(\mathcal{F})\). On the other hand, as \(q(H) \geq \beta(H)\) for \(q \in L^+\) and \(H \in a^+\), we have the following consequence of Lemma 3.9.2. Let \(0 < \epsilon' < \epsilon\) and \(\epsilon' \in \mathbb{R}\setminus \mathbb{P}^{(\infty)}\); then we can find a continuous seminorm \(\zeta_{k, l, \mathcal{F}}\) on \(\mathcal{Z}(\mathcal{F})\) such that

\[ |r^* e^{-\rho_0(H + \log h)} \sum_{q \in \mathbb{Z}} \psi_{\mathcal{F}, a, \gamma}(h; H)| \leq \zeta_{k, l, \mathcal{F}}(a) \]

\[ \times e^{-(2/p)H + \log h + \|H + \log h\|^{-1}} e^{(s - \epsilon')H + \log h + \|H + \log h\|^{-1}} \]

for all \(h \in \text{Cl}(A^+)_q\), \(H \in a_0\), \(a \in \mathcal{Z}(\mathcal{F})\).

So far \(k\) has been arbitrary. We now select \(k\) to be an integer such that

\(k(1 - \kappa/2)\beta(H_\mathcal{F}) > (3/p) \sup_{H' \in \mathcal{H}_H} \rho(H')\), and put \(t_0 = [(1 - \kappa/2)\beta(H_\mathcal{F})]^{-1}\). Consider now an element \(\exp tH\) of \(A\) with \(H' \in \mathcal{H}_H\) and \(t \geq t_0\). If \(H'\) and \(H\) are as in Lemma 3.9.4, \(\exp tH' = \exp tH\) if \(t(1 - \kappa/2)H_\mathcal{F} \leq 1\). Since \(\exp tH' \in \text{Cl}(A^+)_q\) and \(\beta(t(1 - \kappa/2)H_\mathcal{F}) \geq 1\) we can substitute \(h = \exp tH'\), \(H = t(1 - \kappa/2)H_\mathcal{F}\) in (3.9.10). If we now recall that \(\gamma(h) \leq 1\), \(\Xi_{\mathcal{F}}(h) \leq 1\), and \(e^{-\rho_0(H + \log h) \|h\|} \leq 1\) while \(\|H'\| \leq 3 \|H\|\), we get the following result. There is a continuous seminorm \(\zeta_k\) on \(\mathcal{Z}(\mathcal{F})\) such that with \(k\), \(t_0\) as above,

\[ \left| \int_{\mathcal{F}} R_k(\lambda; \exp tH'; t(1 - \kappa/2)H_\mathcal{F}) a(\lambda) e(\lambda)^{-1} c(-\lambda)^{-1} d\lambda \right| \leq \zeta_k(a) e^{-(3/p)\rho(tH')} \]

for all \(a \in \mathcal{Z}(\mathcal{F})\), \(H' \in \mathcal{H}_H\) and \(t \geq t_0\). Furthermore with \(H' \in \mathcal{H}_H\) and \(t \geq t_0\), we substitute \(h = \exp tH\), \(H = t(1 - \kappa/2)H_\mathcal{F}\) in (3.9.11) and (3.9.12). By Lemma 3.9.4, the right side of (3.9.12) is majorized by

\[ \zeta_{k, l, \mathcal{F}}(a) e^{-(2/p)\rho(tH')} (1 + \|tH'\|)^{-l} \]

while the right side of (3.9.11) is majorized by

\[ \zeta_{0, l, \mathcal{F}}(a) e^{-(2/p)\rho(tH')} (1 + \|tH'\|)^{-l} \].

If we combine all these estimates we see that there is a continuous seminorm \(\zeta_{u, l, \mathcal{F}}\) on \(\mathcal{Z}(\mathcal{F})\) such that

\[ |\varphi_u(h; u)| \leq \zeta_{u, l, \mathcal{F}}(a) e^{-(2/p)\rho(\log h)(1 + \|\log h\|)^{-l}} \]
for all \( a \in \Xi(F) \) and \( h \in S_{t_0} \). As mentioned earlier, this proves the lemma for \( u \in \mathcal{M}_{10} \).

We finally take up the case of an arbitrary \( u \in \mathfrak{G} \). By Lemma 2.2.1 we can choose \( u_0, u_1, \ldots, u_s \in \mathcal{M}_{10} \) and functions \( g_1, \ldots, g_s \) in the ring \( \mathcal{R} \) of Chapter 2 such that for all \( C^\infty \) spherical functions \( \varphi \) on \( G \) and all \( m \in M_{10}^\circ \),

\[
\varphi(m; u) = \varphi(m; u_0) + \sum_{1 \leq j \leq s} g_j(m) \varphi(m; u_j) .
\]

Now, if \( H' \in W_{H_0} \), \( \alpha(H') \geq (1 - \kappa/2) \beta(H_0) = t_0^{-1} \) for all \( \alpha \in \Delta^+ \setminus \Delta^+ \) and hence \( \gamma(\exp tH') \leq \frac{1}{2} \) for \( t \geq t_0 \). In particular \( \exp tH' \in M_{10}^\circ \) for \( H' \in W_{H_0} \) and \( t \geq t_0 \). Furthermore the above estimate implies, in view of (2.2.2), that \( \sup_{h' \in S_{t_0}} |g(h')| < \infty \) for any \( g \in \mathcal{R} \). From these we conclude that there is a constant \( C_a > 0 \) with the following property: for any \( C^\infty \) spherical function \( \varphi \) on \( G \),

\[
(3.9.14) \quad |\varphi(h'; u)| \leq C_a \sum_{0 \leq j \leq s} |\varphi(h'; u_j)|
\]

for all \( h' \in S_{t_0} \). We now observe that for any \( a \in \Xi(F) \), the wave packet \( \varphi_a \) is a \( C^\infty \) spherical function on \( G \). So, as we have already proved the lemma for all \( u' \in \mathcal{M}_{10} \), the estimate (3.9.14) implies the lemma for arbitrary \( u \in \mathfrak{G} \). This completes the proof of the lemma.

We are now in a position to obtain the fundamental result of this section.

**Lemma 3.9.6.** Let \( G \) be a group of class \( \mathcal{K} \). Let \( 0 < p < 2 \) and \( \varepsilon = 2/p - 1 \). For any \( a \in \Xi(F) \) let \( \varphi_a \) be defined by (3.9.1). Then \( \varphi_a \in \mathcal{A}^p(G) \) and the map

\[
a \mapsto \varphi_a
\]

is continuous from \( \Xi(F) \) into \( \mathcal{A}^p(G) \).

**Proof.** We shall prove this by induction on \( \dim(G) \). There is no problem in starting the induction; for, the case when \( \dim(a) = 0 \) is utterly trivial. So we may assume \( \dim(a) > 0 \) and the induction hypothesis. We now apply Lemma 3.9.5. Let \( u \in \mathfrak{G} \) and \( l \), an integer \( \geq 0 \). By a compactness argument we select finitely many elements \( H_0^{(i)} \neq 0 \) (\( 1 \leq i \leq s \)) in \( \text{Cl}(a^+) \) such that

\[
\text{Cl}(A^+) \subseteq \bigcup_{1 \leq i \leq s} S[ W_{H_0^{(i)}}] ,
\]

the sectors \( S[ W_{H_0^{(i)}}] \) being defined by (3.9.5). We apply Lemma 3.9.5 to each \( H_0^{(i)} \), \( 1 \leq i \leq s \). We then obtain the following result. Let \( u' \in \mathfrak{G} \) and \( l' \) an integer \( \geq 0 \); then, we can select a continuous seminorm \( \zeta_{u', l'} \) on \( \Xi(F) \) such that

\[
(3.9.15) \quad |\varphi_a(h; u')| \leq \zeta_{u', l'}(a) e^{-[(2/p)l\log h]}(1 + ||\log h||)^{-l'}
\]

for all \( a \in \Xi(F) \) and all \( h \in \text{Cl}(A^+) \).

Select analytic functions \( \beta_1, \ldots, \beta_p \) on \( K \) and elements \( u_1, \ldots, u_p \in \mathfrak{G} \).
such that
\[ u^k = \sum_{1 \leq j \leq p} \beta_j(k) u_j \quad (k \in K). \]

Then, for any $C^\infty$ spherical function $\varphi$ on $G$, we have
\[ \varphi(k, hk_j; u) = \sum_{1 \leq j \leq p} \beta_j(k) \varphi(h; u_j) \]
for all $k, k_j \in K$, $h \in \text{Cl}(A^+)$. Let $C = \max_{1 \leq j \leq p} \sup_{h \in K} |\beta_j(k)|$. Write
\[ \zeta_{u, l} = C \sum_{1 \leq j \leq p} \zeta_{u, l, j}. \]

Then $\zeta_{u, l}$ is a continuous seminorm on $\mathcal{Z}(\mathcal{F}^l)$; moreover, as the $\varphi_a$ are $C^\infty$ spherical functions, we get the following result from (3.9.15): for any $h \in \text{Cl}(A^+)$, $k, k_j \in K$, and any $a \in \mathcal{Z}(\mathcal{F}^l)$,
\[ |\varphi_a(k, hk_j; u)| \leq \zeta_{u, l}(a) e^{-\frac{3}{2} p \rho \log h} (1 + \| \log h \|)^{-l}. \]

Now, $e^{-\rho \log h} \leq \mathcal{E}(k, hk_j)$, $\| \log h \| = \sigma(k, hk_j)$ and $G = K \text{Cl}(A^+) K$. So, from the last estimate we have
\[ |\varphi_a(x; u)| \leq \zeta_{u, l}(a) \mathcal{E}(x) \sigma(x)^{-l} \]
for all $x \in G$. In view of (3.5.1), this means
\[ \mu_{u, l}^p(\varphi_a) \leq \zeta_{u, l}(a) \quad (a \in \mathcal{Z}(\mathcal{F}^l)). \]

In other words, $\varphi_a \in \mathcal{S}^p(G)$ and $a \mapsto \varphi_a$ is a continuous map. This completes the proof of the lemma.

3.10. Spherical transforms on $\mathcal{S}^p(G)$. We are now in a position to formulate and prove the fundamental result of this paper.

**Theorem 3.10.1.** Let $0 < p < 2$ and $\varepsilon = 2/p - 1$. Let $G$ be a group of class $\mathcal{K}$. Then $\mathcal{S}^p(G)$ is a convolution algebra, and the spherical transform map $f \mapsto \hat{f}$ is a linear topological isomorphism of $\mathcal{S}^p(G)$ onto $\mathcal{Z}(\mathcal{F}^l)$.

**Proof.** We know by Theorem 3.5.5 that for $f \in \mathcal{S}^p(G)$, $\hat{f} \in \mathcal{Z}(\mathcal{F}^l)$, and that the map $f \mapsto \hat{f}$ is continuous. We now notice that $\mathcal{S}^p(G) \subseteq \mathcal{S}^p(G)$. So, in view of Harish-Chandra’s result [5] that the spherical transform map is injective on $\mathcal{S}^p(G)$, we see that the map $f \mapsto \hat{f}$ is injective on $\mathcal{S}^p(G)$. On the other hand, it follows from the work of Harish-Chandra in [4] that we can select a non-zero constant $c$ with the following property: $\hat{\varphi}_a(\lambda) = ca(\lambda)$ for all $\lambda \in \mathcal{F}^l$ and $a \in \mathcal{Z}(\mathcal{F}^l)$. Taking into account these observations and Lemma 3.9.6 we conclude that the map $a \mapsto c^{-1} \varphi_a$ is the inverse of the map $f \mapsto \hat{f}$. So $f \mapsto \hat{f}$ is a linear topological isomorphism of $\mathcal{S}^p(G)$ onto $\mathcal{Z}(\mathcal{F}^l)$. Now $\mathcal{S}^2(G)$ is known to be a convolution algebra. So, as $\mathcal{Z}(\mathcal{F}^l)$ is closed under multiplication, we may conclude that $\mathcal{S}^p(G)$ is a convolution algebra. This proves the theorem.

**Remarks 1.** In the proof of the theorem we have used the fact that the spherical transform map $f \mapsto \hat{f}$ is injective. The injectivity of $f \mapsto \hat{f}$ on $\mathcal{S}^2(G)$
was conjectured by Harish-Chandra in [4]. He proved this in [5] using his
type of the discrete series. Recently, Gangolli [12]* has obtained a direct
proof of this result, based on a very elegant method of estimating the ele-
mentary spherical functions (cf. also Helgason [8]). If $G$ is either complex
or of real rank 1, the injectivity had been proved earlier by Helgason [7].

We wish to point out that the work of § 3.9 leads to another proof of the
injectivity of the spherical transform, thus making the proof of Theorem
3.10.1 self-contained. We shall sketch the argument very briefly. In fact,
in view of the work of [4] (cf. especially Corollary 2 to Theorem 5) it is suf-
ficient if we prove the following: there is a nonzero constant $c$ such that
$\varphi_{\varepsilon} = cf$ for all $f \in \mathcal{S}_c^\infty(G)$ (= the space of $C^\infty$ spherical functions with com-
act supports). Now we can select a nonzero constant $c$ such that for any
$m$-invariant $a \in \mathcal{S}(\mathcal{F}_1)$, $\hat{\varphi}_{a} = ca$ [4]. In particular, for any $\varepsilon > 0$ and any
$a \in \mathcal{S}(\mathcal{F}_1)$, $\hat{\varphi}_{a} = ca$ on $\mathcal{F}_1$. Let $f \in \mathcal{S}_c^\infty(G)$. The $\hat{f}$ lies in all the $\mathcal{S}(\mathcal{F}_1)$ and so,
by Lemma 3.9.6, $g = \varphi_{\varepsilon}$ lies in $\mathcal{S}_c^p(G)$ for all $p$, $0 < p < 2$. Write $h = g - cf$.
Then $h$ lies in all $\mathcal{S}_c^p(G)$ ($0 < p < 2$) and $\hat{h} = 0$ on $\mathcal{F}_1$. As $\hat{h}$ is entire, $\hat{h} = 0$.
In particular, $h \in \mathcal{S}(G)$ and $\int_{G} h\varphi \, dx = 0$ for all elementary spherical func-
tions $\varphi$ coming from irreducible unitary representations of $G$. A standard
argument now gives us $h = 0$ or $\varphi_{\varepsilon} = cf$.

2. In analogy with $\mathcal{S}_c^p(G)$, one may raise the question of determining the
spherical transforms of elements of $\mathcal{S}_c^p(G)$. This problem has been settled by
Gangolli [12]. For $G = SL(2, \mathbb{R})$ this was done by Ehrenpreis and Mautner
[10], in their papers I, II. In the case $G$ is either complex or of real rank 1,
this was done by Helgason [7] (cf. also [8]).

3. In the theory of the asymptotic expansions in Chapter 2 the element
$H_0 \in \text{Cl}(a^+)$ is arbitrary. If we take $H_0 \in a^+$, the series for $\varphi(\lambda : \cdot \cdot)$ that we get
is the one obtained by Harish-Chandra [3]. Thus the work in Chapter 2 leads
to an alternative method of constructing the series for $\varphi(\lambda : \cdot \cdot)$ on $A^+$.

Appendix.

In this appendix we collect together a few elementary lemmas in analysis.

**Lemma 1.** Let $Z^+$ be the set of all non-negative integers, $Z^{+(s)} = Z^+ \times \cdots \times Z^+$
(s factors). For any $m = (m_1, \ldots, m_s) \in Z^{+(s)}$, let $|m| = m_1 + \cdots + m_s$. If
$l \geq 0$ and $k \geq 0$ are integers, then there is a constant $C = C(s, l, k) > 0$ such
that for any $\varepsilon > 0$,

\[
\sum_{|m| > \varepsilon} m^{l} e^{-\varepsilon|m|} \leq C\varepsilon^{-(k+1)(s+1)} (1 - e^{-\varepsilon})^{-(s+1)(s+1)}.
\]

* We are indebted to Professor Gangolli for sending us a preprint of this work.
Proof. For any \( u > 0 \), \( \sum |m| \leq i \leq 1 \leq (1 + u)^i \). Hence the sum in (1) is majorized by \( \sum_{r \geq i+1} r^i (r+1)^e e^{-\rho r} \). (1) now follows in an elementary manner.

For the next lemma, \( V \) is a real vector space of dimension \( s \); \( V \), its complexification; \( V^* \), the dual of \( V \) over \( \mathbb{R} \). Let \( \mu_1, \ldots, \mu_s \) be a basis of \( V^* \); \( \mathcal{Q} \), the set of all elements of the form \( m_1 \mu_1 + \cdots + m_s \mu_s \), where the \( m_i \) are all integers \( \geq 0 \). For \( q = m_1 \mu_1 + \cdots + m_s \mu_s \), we write \( o(q) = m_1 + \cdots + m_s \). \( V^+ \) is the set of all \( v \in V \) such that \( \mu_j(v) > 0 \), \( 1 \leq j \leq s \); and for \( v \in V^+ \), let \( \sigma(v) = \min_{1 \leq j \leq s} \mu_j(v) \). Suppose \( c_q (q \in \mathcal{Q}) \) are constants such that

\[
\sup_{q \in \mathcal{Q}} (1 + \eta)^{-\sigma(q)} |c_q| < \infty
\]

for every \( \eta > 0 \). Then the series \( \sum_{q \in \mathcal{Q}} |c_q e^{-\sigma(q)}| \) converges for all \( v \in V^+ + (-1)^{1/2} V \), the convergence being uniform in any region of the form \( \{ v : \sigma(\text{Re} v) \geq \varepsilon > 0 \} \). If \( F(v) = \sum_{q \in \mathcal{Q}} c_q e^{-\sigma(q)} \) \( (v \in V^+ + (-1)^{1/2} V) \), \( F \) is holomorphic.

**Lemma 2.** Let notation be as above. Suppose \( v \in V^+ \) is such that \( F(tv) = 0 \) for all sufficiently large \( t > 0 \). Then, for every \( u \geq 0 \), \( \sum_{q \in \mathcal{Q}, q(v) = u} c_q = 0 \). In particular, if \( F = 0 \) on an open nonempty subset of \( V^+ \), then \( c_q = 0 \) for all \( q \in \mathcal{Q} \).

Proof. Let \( F(tv) = 0 \) for \( t \geq t_0 > 0 \). For any \( u \geq 0 \), the set \( \{ q : q \in \mathcal{Q}, q(v) \leq u \} \) is finite. Hence \( \{ q(v) : q \in \mathcal{Q} \} = \{ a_n : n \geq 0 \} \) where \( 0 = a_0 < a_1 < a_2 < \cdots \). If \( \gamma_n = \sum_{q \in \mathcal{Q}, q(v) = a_n} c_q \), then

\[
F(tv) = \sum_{n \geq 0} \gamma_n e^{-i a_n} \quad (t > 0),
\]

and \( \sum_{n \geq 0} |\gamma_n| e^{-i a_n} < \infty \). As \( \gamma_n \geq 0 \), the series \( \sum_{n \geq 0} \gamma_n e^{-i a_n} \) converges uniformly for \( t \geq t_0 \) to \( \gamma_0 = 0 \). Letting \( t \to +\infty \), we find \( \gamma_0 = 0 \). Suppose \( N \geq 1 \) and \( \gamma_n = 0 \) for \( 0 \leq n \leq N - 1 \). Then \( \gamma_N + \sum_{n > N} \gamma_n e^{-i (a_n - a_N)} = 0 \) \( (t \geq t_0) \). The series is again uniformly convergent, and so, letting \( t \to +\infty \), we find \( \gamma_N = 0 \). So, by induction on \( n \), \( \gamma_n = 0 \) for all \( n \geq 0 \).

Suppose \( F = 0 \) on a nonempty open subset of \( V^+ \). As \( F \) is analytic, \( F = 0 \) on \( V^+ \). Select \( v \in V^+ \) such that \( v \) does not lie on any of the hyperplanes \( \{ u : u \in V, m_1 \mu_1(u) + \cdots + m_s \mu_s(u) = 0 \} \), \( (m_1, \cdots, m_s) \in \mathbb{Z}^{<s} \setminus \{0\} \). Then \( q \mapsto q(v) \) is injective on \( \mathcal{Q} \). By the preceding result, \( c_q = 0 \) for all \( q \). This proves the lemma.

In most applications there will exist an integer \( r \geq 0 \) such that \( c_q = O(o(q)^r) \). This will permit us to use the above lemma. We also note that for almost all \( v \in V^+ \), the map \( q \mapsto q(v) \) is injective on \( \mathcal{Q} \).

We now assume that \( V \) is a real Hilbert space. Then \( V \) becomes a complex Hilbert space in a natural fashion. The norm and inner product in \( V \) are denoted by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) respectively. Any \( z \in V \) can be written as \( z = x + (-1)^{1/2} y \), \( x, y \in V \); we write \( x = \text{Re} z \), \( y = \text{Im} z \), and note that \( \| z \|^2 = \).
\[ |x|^2 + |y|^2. \] \( S(V_e) \) is the symmetric algebra over \( V_e \). Each element of \( V_e \) acts as a holomorphic differential operator on holomorphic functions defined on open subsets of \( V_e \); for \( u \in S(V_e) \), the corresponding operator is written \( \partial(u) \). If \( V_i \) is an \( \mathbb{R} \)-linear subspace of \( V_e \) with \( \dim_{\mathbb{R}} V_i \leq \dim_{\mathbb{C}} V_e \), then for any \( u \in S(V_i) \), \( \partial(u) \) may be regarded as a differential operator acting on \( C^\infty \) functions defined on open subsets of \( V_i \).

If \( U \) is any subset of \( V \), the set \( T(U) = U + (-1)^{i/2} V \) is called the tube based on \( U \). Let \( U \) be an open subset of \( V \). We then define \( \mathcal{S}_U \) to be the space of all functions \( F \) defined and holomorphic on \( T(U) \) such that for any open set \( U_i \subset U \) and any integer \( l \geq 0 \),

\[ \sup_{z \in T(U)} (1 + |z|^2)^l |F(z)| < \infty; \]

here \( U_i \subset U \) means that \( \text{Cl}(U_i) \) is compact and contained in \( U \). It follows from Cauchy’s formula that \( \partial(u)F \in \mathcal{S}_U \) for all \( u \in S(V_e) \) and \( F \in \mathcal{S}_U \). The space \( \mathcal{S}^\prime \) is defined as the set of all functions \( F \) defined and holomorphic on \( T(U) \) such that for any open set \( U_i \subset U \), there is an integer \( l \geq 0 \) such that

\[ \sup_{z \in T(U)} (1 + |z|^2)^{-l} |F(z)| < \infty; \]

in this case, by Cauchy’s formula we can find an integer \( l_i \geq 0 \) such that

\[ \sup_{z \in T(U)} (1 + |z|^2)^{-l_i} |F(z; \partial(u))| < \infty \]

for all \( u \in S(V_e) \). \( \mathcal{S}_U \) and \( \mathcal{S}^\prime \) are algebras, and \( \mathcal{S}^\prime \mathcal{S}_U \subseteq \mathcal{S}_U \).

**Lemma 3.** Let \( U \) be an open subset of \( V \), \( F \in \mathcal{S}_U \). Then

\[ g(x) = \int_{\nu} F(x + (-1)^{i/2} y) dy \]

is an absolutely convergent integral for all \( x \in U \), and \( g \) is constant on each connected component of \( U \) (here \( dy \) is a Lebesgue measure on \( V \)).

**Proof.** Obvious.

**Lemma 4.** Let \( U \) be an open subset of \( V \); \( F \), a function defined and holomorphic on \( T(U) \). Let \( \Lambda_j^i \) \((1 \leq j \leq l)\) be complex linear functions on \( V \) which take real values on \( V \). Let \( \Lambda \) be defined by \( \Lambda_j(z) = \Lambda_j^i(z) + a_j \) \((z \in V_e)\), \( a_j \) being constants \((1 \leq j \leq l)\). Assume that \( \Lambda \neq 0 \) for all \( j \) and let \( \Lambda = \Lambda_1 \cdots \Lambda_l \). If \( \Lambda F \in \mathcal{S}_U \) (resp. \( \mathcal{S}^\prime \)), then \( F \in \mathcal{S}_U \) (resp. \( \mathcal{S}^\prime \)).

**Proof.** It is enough to consider the case \( l = 1 \); the general case follows by induction on \( l \). A simple argument shows that we may assume \( V = \mathbb{R}^s \) (with the usual scalar product) and \( \Lambda(z_1, \ldots, z_s) = z_i \) for all \((z_1, \ldots, z_s) \in \mathbb{C}^s \). We write \( H(z_1, \ldots, z_s) = z_i F(z_1, \ldots, z_s) \) for \((z_1, \ldots, z_s) \in T(U) \). We shall assume that \( H \in \mathcal{S}_U \) (resp. \( \mathcal{S}^\prime \)) and prove that \( F \in \mathcal{S}_U \) (resp. \( \mathcal{S}^\prime \)). It is enough to prove that for each \((\xi_1, \ldots, \xi_s) \in U \) there is an open set \( U_i \) with \((\xi_1, \ldots, \xi_s) \in U_i \)
$U_1 \subset U$ such that

$$\sup_{z \in \mathcal{T}(U_1)} (1 + ||z||^2)^l |F(z)| < \infty$$

for all integers $l \geq 0$ (resp.

$$\sup_{z \in \mathcal{T}(U_1)} (1 + ||z||^2)^{-l} |F(z)| < \infty$$

for some integer $l \geq 0$). Fix $(\xi_1, \ldots, \xi_s) \in U$. Two cases arise. Case 1: $\xi_i \neq 0$. Choose $\varepsilon$ such that $0 < \varepsilon < |\xi_i|$ and $U_1 = \{(x_1, \ldots, x_s): x_i \in \mathbb{R}, |x_i - \xi_i| < \varepsilon, 1 \leq i \leq s\} \subset U$. Then, for $(z_1, \ldots, z_s) \in T(U_1)$,

$$|F(z_1, \ldots, z_s)| \leq (|\xi_i| - \varepsilon)^{-1} |H(z_1, \ldots, z_s)|.$$

What we want is obvious in this case. Case 2: $\xi_i = 0$. Let $\varepsilon > 0$ be chosen such that $U_1$, defined as above, is $\subset U$. Then for $(z_1, \ldots, z_s) \in T(U_1)$, $(\tau z_1, z_2, \ldots, z_s) \in T(U_1)$ if $0 \leq \tau \leq 1$. Assume first that $H \in \mathcal{L}_U$. Let $Q$ be any polynomial in $z_1, \ldots, z_s$. Write $F^0 = QF$, $H^0 = QH$. If $g(\tau: z_1, \ldots, z_s) = H^0(\tau z_1, z_2, \ldots, z_s)$, then for $(z_1, \ldots, z_s) \in T(U_1)$ we have

$$H^0(z_1, \ldots, z_s) = z_1 \int_0^1 H^0_1(\tau z_1, z_2, \ldots, z_s)d\tau,$$

where $H^0_1 = (\partial/\partial z_1)H^0$, so that

$$F^0(z_1, \ldots, z_s) = \int_0^1 H^0_1(\tau z_1, z_2, \ldots, z_s)d\tau.$$

As $H^0 \in \mathcal{L}_U$, we can find a constant $C > 0$ such that $|H^0_1(\xi_1, \ldots, \xi_s)| \leq C$ for all $(\xi_1, \ldots, \xi_s) \in T(U_1)$. This implies that $F^0 = QF$ is bounded on $T(U_1)$. As $Q$ was arbitrary, $\sup_{z \in \mathcal{T}(U_1)} (1 + ||z||^2)^l |F(z)| < \infty$ for all $l \geq 0$. The case of $\mathcal{L}^U$ is handled similarly.

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