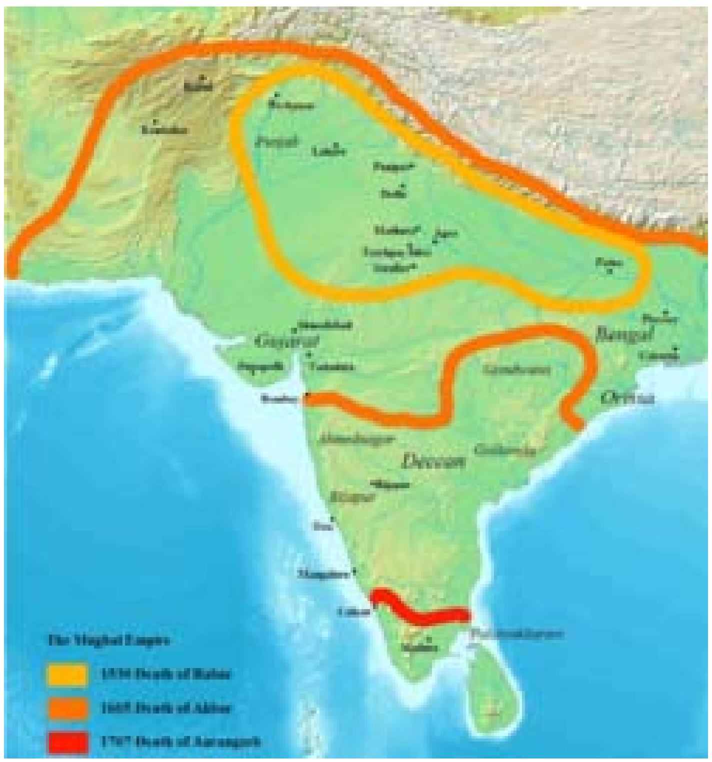
Lectures on the beginnings of calculus in Kerala and in Europe

Math 191, April 5, 7, 12, 14, 19

The School of Mādhava in Kerala, c.1350-1600 CE

- Exodus of Brahmins from North following Muslim invasions (Delhi Sultanate, Mughal Empire of Babur, Akbar, etc.)



- Tightly knit *guru-parampara* (“chain of teachers”): Mādhava (the founder) →Parameśvara →Dāmodara →Nīlakantha, Jyeṣthadeva → Narayana (+ non-Brahmins! Śankara Variyar, Acyuta Pisārati)
- All in a small area, at small village temples, under protection of the Mahārājah of Calicut.
- Jyeṣthadeva wrote a unique book, informal ‘lecture notes’, in Malayalam, the *Yukti-Bhāṣa* (vernacular <exposition> of rationales). A translation by K. V. Sarma was recently published as *Ganita-Yukti-Bhāṣa.*
- The basic ideas were attributed to Mādhava , and ideas such as a virtually heliocentric model of the solar system to Nīlakantha (1444-c.1540)
- Their work never spread and was forgotten until c.1820 when C.M.Whish learned Malayalam, collected palm-leaf manuscripts from Kerala and found, to his astonishment, a “*complete system of fluxions*”

- First big discovery: $$\int_0^R x^n dx = \frac{R^{n+1}}{n+1}$$
- In §6.4 of the *Yukti*: *"Summation of Series"*
- Goal is to approximate the Riemann sum

$$s\sum_{k=1}^{n}(k.s)^p,\ \ ns = R,\ \text{"radius"},\ s = \text{"segment"}$$

as *s* becomes *"as small as an atom ("aṇu")"* while *n* becomes as large as *"parārdha"* (1 trillion!)

- Now $$s\sum_{k=1}^{n}(k.s)^p = s^{p+1}\left(\sum_{k=1}^{n}k^p\right)$$

so the sums of integers are the key and he says:

*"Now suppose the radius to be the same number of units as the number of segments to which it has been divided, in order to facilitate remembering their number",*i.e. make *s* =1.

<u>To give the flavor, here's the case *p*=3.</u>

*"Now, to the method or deriving the summation of cubes: Summation of cubes, it is clear, is the summation where the square of each number (*bhuja) *in the summation of squares is multiplied by the number. Now, by how much will the sum of cubes increase if all the numbers squared were to be multiplied by the radius. By the principle enunciated earlier, the square number next-to-last will increase by itself being multiplied by 1. The square numbers below will increase by multiples of 2,3, etc. in order. That sum will be equal to the summation of summation of squares. It has already been shown that the summation of squares is equal to ⅓ the cube of the radius. Hence ⅓ the cube of each number will be equal to the summation of all square numbers ending with that number. Hence it follows that the summation of summation of square numbers is equal to ⅓ the sum of cube numbers. Therefore the summation of squares multiplied by the radius will be equal to the summatiuon of cubes plus ⅓ of itself. Hence, when ¼ of it is subtracted, what remains will be the summation of cubes. Hence, it also follows that the summation of cubes is equal to ¼ the square of the square of the radius.*

$$\sum_{1}^{R} k^3 = \sum_{1}^{R} k^2.k = \sum_{1}^{R} k^2.(R-(R-k))$$

$$= R\sum_{1}^{R} k^2 - \sum_{\ell=1}^{R}\sum_{k=1}^{\ell} k^2$$

$$\approx R.\frac{R^3}{3} - \sum_{\ell=1}^{R}\frac{\ell^3}{3}$$

$$\therefore \quad \frac{4}{3}\sum k^3 = \frac{R^4}{3}, \;\; \sum k^3 = \frac{R^4}{4}$$

Same argument shows:

$$s.\sum_{k=1}^{n}(k.s)^p \approx s^{p+1}.\frac{n^{p+1}}{p+1} = \frac{R^{p+1}}{p+1}, \quad \text{equality in the limit}$$

Using double integrals and induction, this becomes:

$$\int_0^R x^p dx = \int_0^R x^{p-1}(R-(R-x))dx = R.\int_0^R x^{p-1}dx - \int_0^R x^{p-1}\left(\int_x^R dy\right)dx$$

$$= R.\frac{R^p}{p} - \iint\limits_{0\le x\le y\le 1} x^{p-1}dxdy$$

$$=R.\frac{R^p}{p} - \int_0^R\left(\int_0^y x^{p-1}dx\right)dy$$

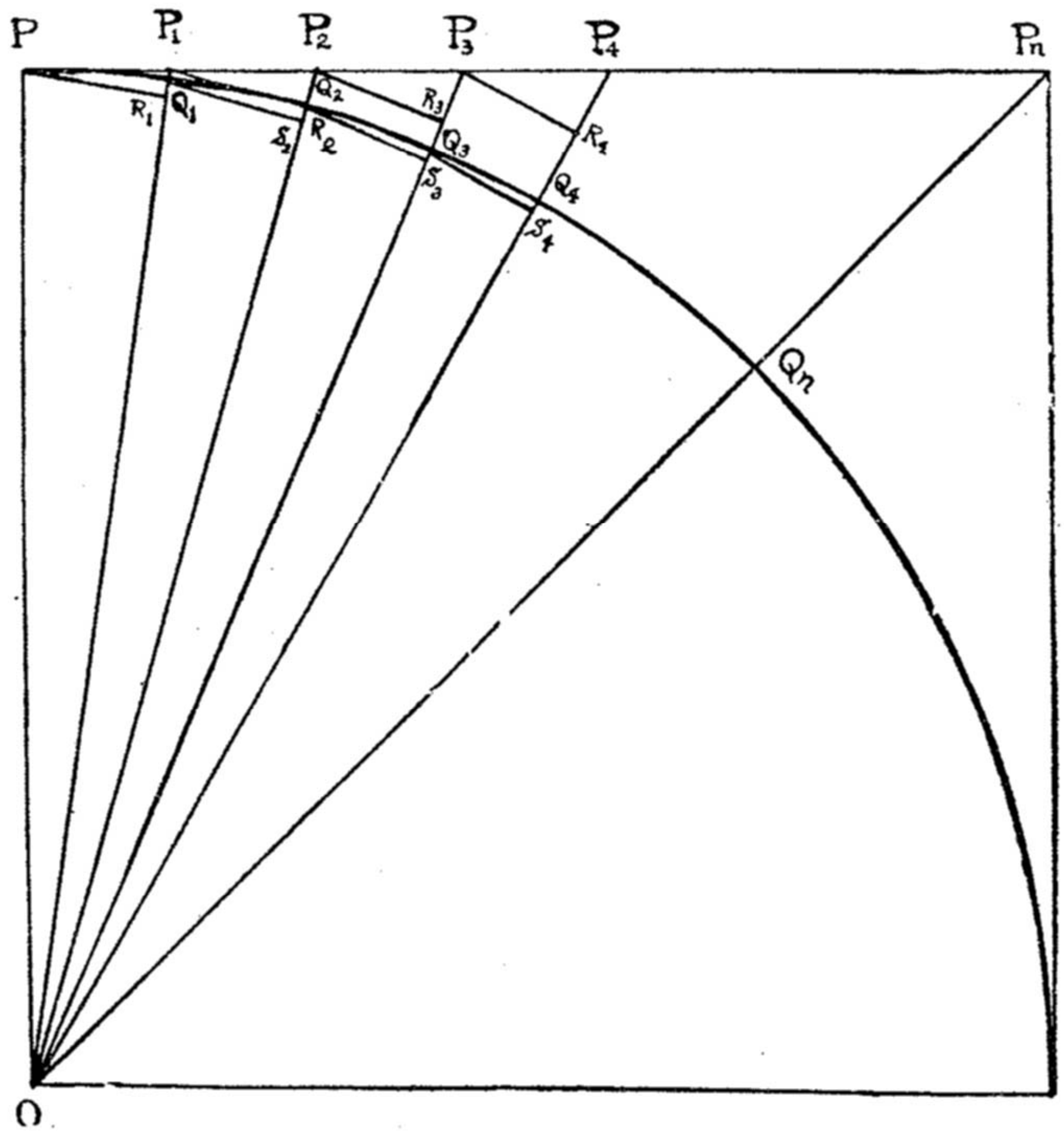
$$=R.\frac{R^p}{p} - \int_0^R \frac{y^p}{p}dy$$

so $\left(1+\frac{1}{p}\right)\int_0^R x^p dx = \frac{R^{p+1}}{p}$, hence $\int_0^R x^p dx = \frac{R^{p+1}}{p+1}$

He applies this summation to compute first

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

in §6.3 and later to get the power series for arctan(x), any x. Here is his basic diagram, a quadrant of a circle of radius R, P the “East” point, the line PP_n divided in a very large number n of segments



"Now is described the procedure for arriving at the circumference of a circle of desired diameter without involving calculation of square roots. Construct a square with four sides equal to the diameter of the proposed circle. Inscribe the circle inside the square in such a manner that the circumference of that circle touches the centers of the four sides of the square. Them through the center of the circle, draw the east-west line and the north-south line with their tips being located at the points of contact of the circumference and the sides. Then the interstice between the east-point and the south-east corner of the square will be equal to the radius of the circle. Divide this line into a number of equal parts by marking a large number of points closely at equal distances. The more the divisions, the more accurate would be the calculated circumference."

Outline of proof:

Let $R = s.n, \;\; s = \overline{P_{i-1}P_i},\; 1 \le i \le n,\; k_i = \overline{OP_i},$

$\theta_i =$ angle $P_{i-1}OP_i$, so that $\pi/8 = \sum_{i=1}^{n} \theta_i$

Sections 6.3.1, 6.3.2 are devoted to showing:

$$\theta_i \approx \sin(\theta_i) = \frac{s \cdot R}{k_{i-1}.k_i} \approx \frac{s \cdot R}{k_i^2}$$

or, since $\overline{P_0P_i} \mapsto \theta_i$ is $\arctan(x/R)$, $x =$ coord along P_0P_n,

he shows $\dfrac{d\theta}{dx} = \dfrac{R}{k^2} = \dfrac{R}{R^2 + x^2}$

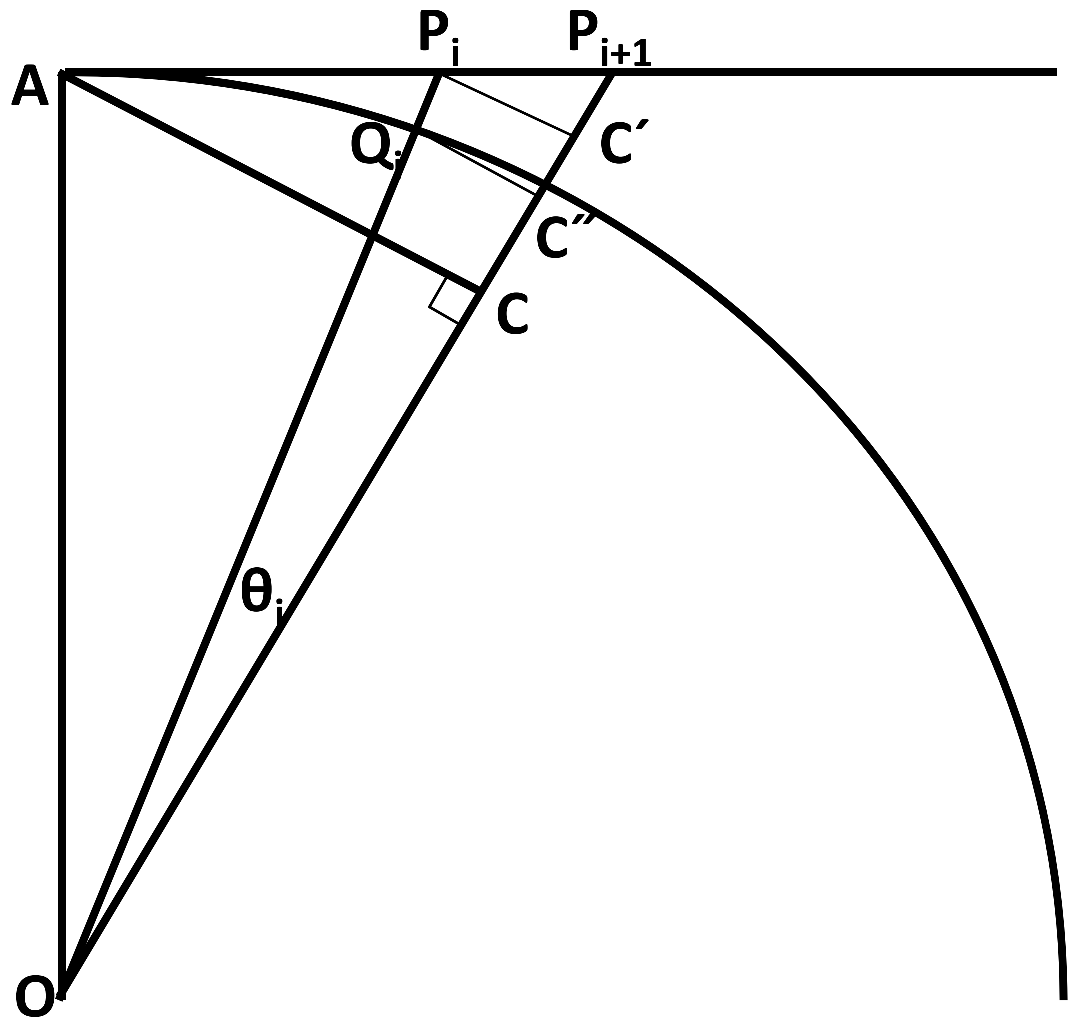
“6.3.1: Dividing the circum. into arc-bits, approx. the arc-bits by sines”

$\Delta(AOP_{i+1})$ congruent to $\Delta(ACP_{i+1})$ and to $\Delta(P_iC'P_{i+1})$ and $\Delta(P_iC'O)$ congruent to $\Delta(Q_iC''O)$.

Thus:

$$\frac{R}{k_{i+1}} = \frac{\overline{OA}}{\overline{OP_{i+1}}} = \frac{\overline{C'P_i}}{\overline{P_iP_{i+1}}} = \frac{\overline{C'P_i}}{s}$$

and $$\frac{\overline{C'P_i}}{k_i} = \frac{\overline{C'P_i}}{\overline{OP_i}} = \frac{\overline{C''Q_i}}{\overline{OP_i}} = \sin(\theta_i), \quad \text{hence } \sin(\theta_i) = \frac{s \cdot R}{k_i \cdot k_{i+1}}$$



Last step:

Start with the identity:

$$\frac{a}{b+c}=\frac{a}{b}-\frac{ac}{b(b+c)}$$

and iterate, giving the "sequence of subtractive corrections":

$$\frac{a}{b+c}=\frac{a}{b}-\left(\frac{ac}{b^2}-\frac{ac^2}{b^2(b+c)}\right)=\frac{a}{b}-\left(\frac{ac}{b^2}-\left(\frac{ac^2}{b^3}-\left(\frac{ac^3}{b^4}-\cdots\right)\right)\right)$$

Applying this to $a=s.R,\ b=R^2,\ c=(is)^2,\ b+c=k_i^2,$

$$\frac{\pi}{4}=\sum_i\theta_i\approx\sum_i\frac{s.R}{k_i^2}=\sum_i\frac{s}{R}-\left(\frac{s^3i^2}{R^3}-\left(\frac{s^5i^4}{R^5}-\cdots\right)\right)$$

$$\approx\frac{sn}{R}-\frac{s^3n^3}{3R^3}+\frac{s^5n^5}{R^5}-\cdots=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

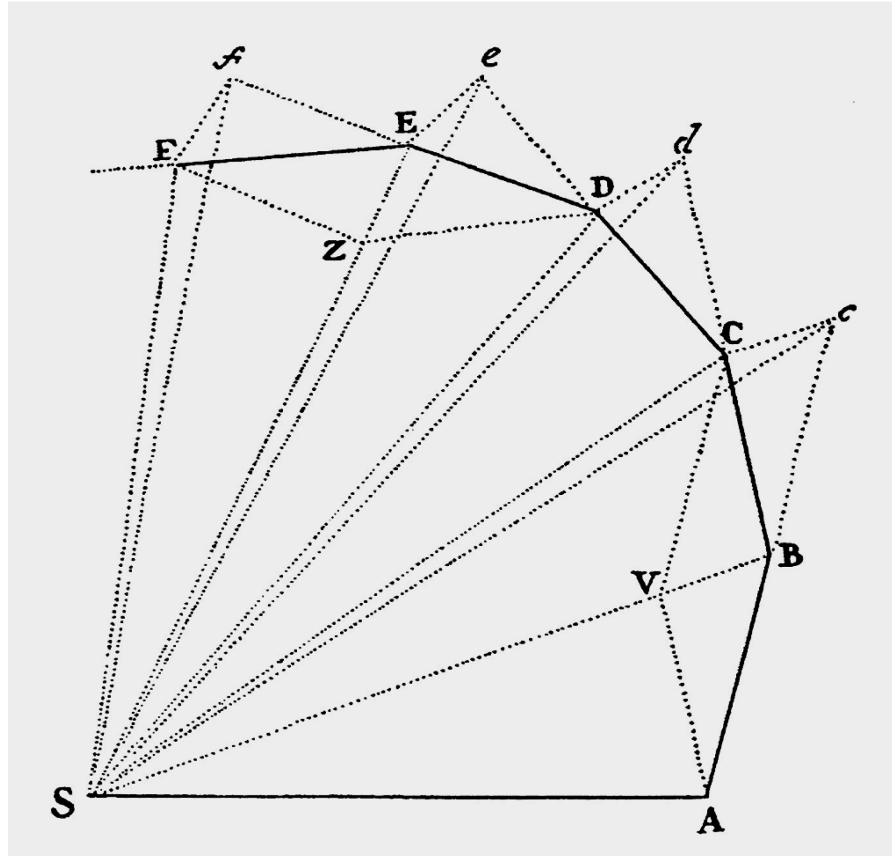
How rigorous is this?

- Bounds comparing $\sum k^p, \int x^p$ are easy and strong
- Bounds on $\theta - \sin(\theta)$ are also easy
- A tricky part is that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ converges only conditionally, not absolutely
- If $x < 1$, then we get absolute convergence for

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

This is called "*Abel summation*": that the limit of this as $x \to 1$ can be taken term by term.

It is interesting to compare this argument with Newton's derivation of Kepler's Second Law: that planets move so that the area swept out the line connecting them to the sun increases at a constant speed. **Both use geometry in discrete approximations, then a loose passage to the limit**.



Proposition 1[a] **Theorem 1** *The areas which bodies* [b]*made to move in orbits*[b] *describe by radii drawn to an unmoving center of forces lie in unmoving planes and are proportional to the times.*

Isaac Newton, *Philosophiæ Naturalis Principia Mathematica,* 1687

Let the time be divided into equal parts, and in the first part of the time let a body by its inherent force describe the straight line AB. In the second part of the time, if nothing hindered it, this body would (by law 1) go straight on to *c*, describing line B*c* equal to AB, so that—when radii AS, BS, and *c*S were drawn to the center—the equal areas ASB and BS*c* would be described. But when the body comes to B, let a centripetal force act with a single but great impulse and make the body deviate from the straight line B*c* and proceed in the straight line BC. Let *c*C be drawn parallel to BS and meet BC at C; then, when the second part of the time has been completed, the body (by corol. 1 of the laws) will be found at C in the same plane as triangle ASB. Join SC; and because SB and C*c* are parallel, triangle SBC will be equal to triangle SB*c* and thus also to triangle SAB. By a similar argument, if the centripetal force acts successively at C, D, E, . . . , making the body in each of the individual particles of time describe the individual straight lines CD, DE, EF, . . . , all these lines will lie in the same plane; and triangle SCD will be equal to triangle SBC, SDE to SCD, and SEF to SDE. Therefore, in equal times equal areas are described in an unmoving plane; and by composition [or componendo], any sums SADS and SAFS of the areas are to each other as the times of description. Now let the number of triangles be increased and their width decreased indefinitely, and their ultimate perimeter ADF will (by lem. 3, corol. 4) be a curved line; and thus the centripetal force by which the body is continually drawn back from the tangent of this curve will act uninterruptedly, while any areas described, SADS and SAFS, which are always proportional to the times of description, will be proportional to those times in this case. Q.E.D.

How did Newton justify his methods (my bold face)

“In any case, I have presented these lemmas before the propositions in order to avoid the tedium of working out lengthy proofs by reductio ad absurdum in the manner of the ancient geometers. Indeed, proofs are rendered more concise by the method of indivisibles. But since the hypothesis of indivisibles is problematical and this method is accounted less geometrical, I have preferred to make the proofs of what follows depend on the **ultimate** sums and ratios of vanishing quantities and on the first sums and ratios of nascent quantities, that is, on the **limits** of such sums and ratios, and therefore to present proofs of those limits beforehand as briefly as I could. For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground using principles that have been proved. ….

It may be objected that there is no such thing as an ultimate proportion of vanishing quantities, inasmuch as before vanishing the proportion is not ultimate, and after vanishing it does not exist at all. But the answer is easy: … the ultimate ratio of vanishing quantities is to be understood not as the ratio of quantities before they vanish or after they have vanished but **the ratio with which they vanish**.”

<u>Nicole Oresme</u>, 1323-1382, *Tractatus de Configurationibus Qulaitatum*

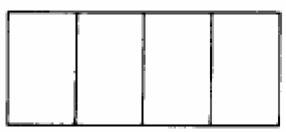
Every measurable thing except numbers is imagined in the manner of continuous quantity. Therefore, for the mensuration of such a thing, it is necessary that points, lines and surfaces, or their properties be imagined. For in them, as the Philosopher has it, measure or ratio is initially found, while in other things it is recognized by similarity as they are being referred to by the intellect to the geometrical entities. Although indivisible points, or lines, are non-existent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. Therefore, every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible thing, e.g. a quality. For whatever ratio is found to exist between intensity and intensity of the same kind, a similar ratio is found to exist between line and line, and vice versa. … Therefore, the measure of intensities can be fittingly imagined as the measure of lines.	1. Roughly, he's saying that all measurable things in the world are either discrete things like whole numbers, or vary continuously. 2. Given 2 such measurements, they always have a ratio, one to the other; and the most basic case of this sort of measurement is the length of line segments or the area of surfaces, because 2 lengths or 2 areas have a definite ratio, one to the other. "The Philosopher" is Aristotle. 3. Points are infinitely small and lines infinitely thin, so they are idealizations. 4. 'Successive' means a quantity that varies in time *f(t)*. 5. The 'subject' is the set of points on which the function *f* is defined, its domain. 6. His graph is given by imagining perpendicular lines erected on the domain, like a bar graph.

One thing he was very clear about is that the key thing about a graph is that its shape should depict accurately the *ratios of the quality being* measured against the true distances in the *subject,* an interval of space or time.

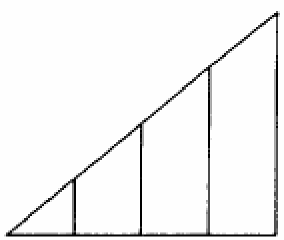
This is the point in problem #1, HW9.

The quantity of any linear quality is to be imagined by a surface whose length or base is a line protracted in a subject of this kind and whose breadth or altitude is designated by a line erected perpendicularly on the aforesaid base. And I understand by "linear quality" the quality of some line in the subject informed with a quality.

That the quantity of such a linear quality can be imagined by a surface of this sort is obvious, since one can give a surface equal to the quality in length or extension and which would have an altitude similar to the intensity of the quality. But it is apparent that we ought to imagine a quality in this way in order to recognize its disposition more easily, for its uniformity and its difformity are examined more quickly, more easily and more clearly when something similar to it is described in a sensible figure. … Thus it seems quite difficult for some people to understand the nature of a quality which is uniformly difform. But what is easier to understand than that the altitude of a right triangle is is uniformly difform.



(a)



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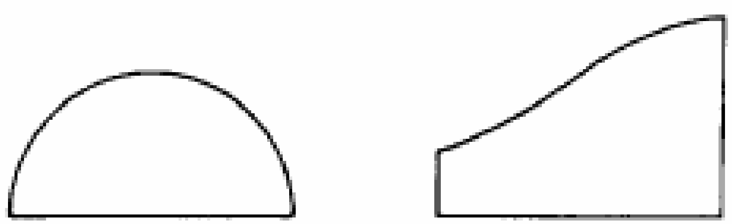
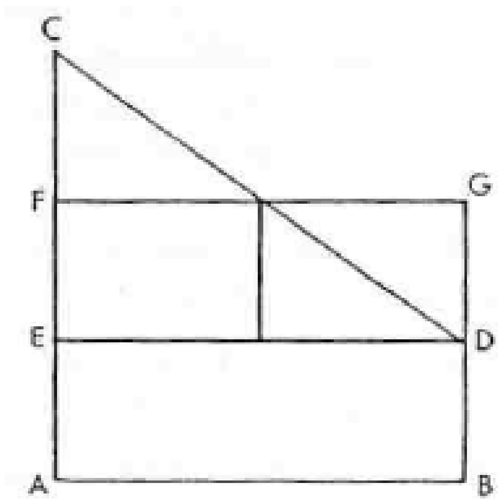


(c)

1. In modern terminology, 'linear quality' means a dependent variable y depending on 1 independent variable, i.e. $y=f(x)$.
2. 'Subject' means the domain of x, in this case a line segment I.
3. The surface referred to is the plane figure $0\leq y\leq f(x)$, x in I. The "quantity" of the quality means the area of this surface or the integral of f over I. Note that the x and y axes are required to be perpendicular.
4. Next, he says that any such quality can be graphed like this. *Note that his qualities are always positive.*

5. Some such qualities are "uniform", meaning f is constant, and others "difform", meaning f is non-constant and he notes that one sees such things much better by making a graph, because it is then "sensible", i.e. visible to the eye.

6. Finally "uniformly difform" means the rate of change of y is constant, or equivalently the graph is a straight line and so it is part of the hypotenuse of a right triangle erected in the x axis.

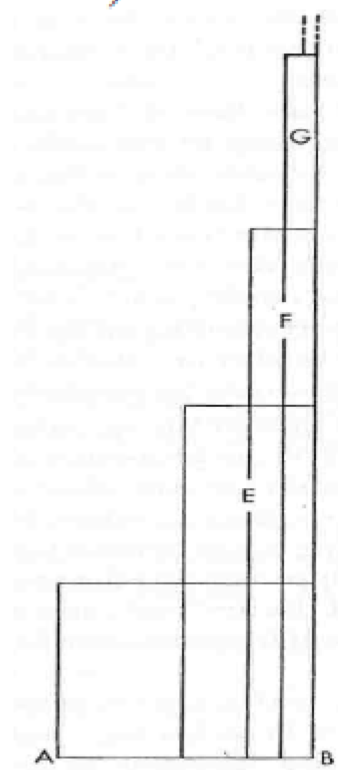


Oresme's assertion:
area(ABDC) =
area(ABGF)

"Every quality, if it is uniformly difform, is of the same quantity as would be the quality of the same or equal subject that is uniform according to the degree of the middle point of the same subject"

In modern terminology:

$$\int_a^b (Cx+D)dx = \left(C\left(\frac{a+b}{2}\right)+D\right).(b-a)$$



He considers a 'quality' that has value n between points 2^{-n} and $2^{-(n+1)}$, so that it 'blows up' when x=0. The graph from his book appears on the left. By rearranging the blocks as shown in the figure to make one rectangle, he showed that the area is just twice the length AB times the height of the graph over A. Again, in modern terminology, he is approximating the evaluation of an improper integral:

$$\int_0^1 \log(1/x)dx = 1$$

Gottfried Wilhelm von Leibniz, 1646-1716

- Polymath: Philosopher, Mathematician, Scientist, he aspired to understand *everything* and reduce to a logical system
- Discovered calculus seemingly independently of Newton, drawing on many ideas 'in the air' in the work of Fermat, Descartes, Cavalieri, Huygens, Pascal, Barrow
- Introduced $dx, \int x$ $(\text{also } ddx)$ and the formalism for working with them., being strongly motivated by finite differences of discrete sequences, as were the Indian mathematicians.

Leibniz and Newton, as proud old men, fell into a bitter priority fight. Leibniz wrote his story in *Historia et Origo Calculi Differentialis*, in the 3rd person. Here's where he describes the germ of the fundamental theorem:

If *A, B, C, D, E* are supposed to be quantities that continually increase in magnitude, and the differences between successive terms are denoted by *L, M, N, P*, it will follow that

$$L+M+N+P = E - A$$

that is, sums of the differences, no matter how great their number, will be equal to the difference between terms at the beginning and end of the series. For example, let us take the squares 0 1 4 9 16 25 with differences 1 3 5 7 9. It is evident that

$$1+3+5+7+9 = 25 - 0 = 25$$

and the same will hold good whatever the number of terms or the differences may be. Delighted by this easy elegant theorem, our young friend considered a large number of numerical series, and also proceeded to to the second differences or differences of the differences, ….

- Leibniz wrote many unpublished notes and many letters.
- In an unpublished manuscript addressed to the the "*Journal des Savans*" (from the 1670's), he announces

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

  by a method which starts very differently from Madhavan but ends the same way [next slide]
- He publishes *New Method for Maxima, Minima and Tangents* in 1784 using differentials dx and showing how to calculate *dy/dx* for all functions *y=f(x)* obtained by rational expressions and powers.
- His notes and letters show much more detail including integrals, the fundamental theorem and even higher order differentials ddx, dddx, … [describe]

Leibniz's derivation of the formula for π