ON THE RING OF INVARIANT POLYNOMIALS ON A SEMISIMPLE LIE ALGEBRA.[1]

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1. We shall show in this note how a formula due to Harish-Chandra [1] may be used to obtain simple and elementary proofs of some results of Kostant [1] [2] concerning the algebra of invariant polynomials on a complex semisimple Lie algebra. Since we use Harish-Chandra's formula only in a very special case, we have included a simple proof of it in that special case, so as to make the present note self-contained.

2. Let $\mathfrak{g}$ be a Lie algebra over the field $\boldsymbol{C}$ of complex numbers. Let G denote the complex analytic adjoint group of $\mathfrak{g}$ and let $x \rightarrow \mathrm{Ad}(x)$ denote the complex adjoint representation of G in $\mathfrak{g}$. For $X \in \mathfrak{g}$ and $x \in G$ we write X^x for $\mathrm{Ad}(x) \cdot X$. A subset $\Omega \subseteq \mathfrak{g}$ is said to be *invariant* if $X \in \Omega$ implies $X^x \in \Omega$ for all $x \in G$. A function f, defined on an invariant set Ω, is said to be *invariant* if $f(X^x) = f(X)$ for all $X \in \Omega$ and $x \in G$. We write $\mathfrak{J}$ for the algebra of all invariant polynomials on $\mathfrak{g}$. It is a well known theorem of Chevalley that if $\mathfrak{g}$ is semisimple and $l =$ rank of $\mathfrak{g}$, then $\mathfrak{J}$ is generated by l homogeneous algebraically independent elements. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, the restriction of $\mathfrak{J}$ to $\mathfrak{h}$ is an algebra isomorphism of $\mathfrak{J}$ onto the algebra of all polynomials on $\mathfrak{h}$ which are invariant under the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let $p_1, \cdots, p_l$ denote l such homogeneous generators of $\mathfrak{J}$ and let $\nu_1, \cdots, \nu_l$ be their respective degrees. Since the Weyl group is generated by reflexions, it follows from the work of Chevalley [1] and Shepherd and Todd [1] that the integers ν_j are canonically determined by $\mathfrak{J}$; in particular[2]

$$\nu_1 + \nu_2 + \cdots + \nu_l = \tfrac{1}{2}(l + \dim \mathfrak{g}). \tag{1}$$

Kostant, in his paper [1] gave a description of the ν_j's in terms of the root structure of $(\mathfrak{g}, \mathfrak{h})$. To describe his result and also to set up notation for our subsequent discussion, we proceed as follows. An element $X \in \mathfrak{g}$ is said to be *nilpotent* if $\mathrm{ad}\, X$ $(Y \rightarrow [X, Y])$ is a nilpotent endomorphism of $\mathfrak{g}$.

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[2] In fact, it is shown in the paper of Shepherd and Todd [1] that $(\nu_1 - 1) + \cdots + (\nu_l - 1)$ is the number of reflexions in the Weyl group which is one half of the number of roots of $(\mathfrak{g}, \mathfrak{h})$. The relation (1) follows at once from this.

Let $X_0 \neq 0$ be any nilpotent element of $\mathfrak{g}$. Then, a well known theorem of Jacobson and Morozov asserts the existence of nonzero elements H_0 and Y_0 in $\mathfrak{g}$ such that

$$(2) \qquad [H_0, X_0] = 2X_0, \quad [H_0, Y_0] = -2Y_0, \quad [X_0, Y_0] = H_0.$$

The representation $Z \to \operatorname{ad} Z$ of the three dimensional Lie algebra $\boldsymbol{C} \cdot X_0 + \boldsymbol{C} \cdot H_0 + \boldsymbol{C} \cdot Y_0$ acting in $\mathfrak{g}$ may be decomposed into its irreducible constituents. Let $\lambda_1, \cdots, \lambda_r$ be integers $\geqq 0$ such that the dimensions of these irreducible constituents are $\lambda_1 + 1, \cdots, \lambda_r + 1$. Then $r = \dim \mathfrak{z}_{X_0}$ where $\mathfrak{z}_{X_0}$ is the centralizer of X_0 i. e.,

$$(3) \qquad \mathfrak{z}_{X_0} = \{X : X \in \mathfrak{g}, [X_0, X] = 0\}.$$

In general, $r \geqq l$. If $r = l$, X_0 is said to be a *principal* nilpotent element. Of course, H_0 and Y_0 are not uniquely determined by X_0. However, given X_0, the Lie subalgebra $\boldsymbol{C} \cdot X_0 + \boldsymbol{C} \cdot H_0 + \boldsymbol{C} \cdot Y_0$ is determined up to conjugacy by an element of G which fixes X_0. This shows that the numbers $\lambda_1, \cdots, \lambda_r$ are uniquely determined by X_0. It follows from the theory of representations of the three dimensional simple Lie algebra that the centralizer $\mathfrak{z}_{Y_0}$ of Y_0 and the range $\mathfrak{g}_{X_0}$ of $\operatorname{ad} X_0$ are linearly independent and have $\mathfrak{g}$ as their direct sum. It also follows from the same theory that $\operatorname{ad} H_0$ leaves both $\mathfrak{g}_{X_0}$ and $\mathfrak{z}_{Y_0}$ invariant, that $\dim \mathfrak{z}_{Y_0} = r$, and that there exists a basis $\{Y_1', \cdots, Y_r'\}$ of $\mathfrak{z}_{Y_0}$ such that $[H_0, Y_j'] = -\lambda_j Y_j'$, $j = 1, 2, \cdots, r$. Clearly, if U is any linear manifold complementary to $\mathfrak{g}_{X_0}$ and invariant under $\operatorname{ad} H_0$, one can find a basis $\{Y_1, \cdots, Y_r\}$ for U such that

$$(4) \qquad [H_0, Y_j] = -\lambda_j Y_j, \quad j = 1, 2, \cdots, r.$$

With these definitions we can describe Kostant's results. One of his main results in [1] is that if X_0 is a principal nilpotent element, then the numbers $1 + \frac{1}{2}\lambda_1, \cdots, 1 + \frac{1}{2}\lambda_l$ associated with X_0 coincide with the degrees of any l homogeneous algebraically independent generators of $\mathfrak{J}$. The expressions for these degrees in terms of the root structure of $(\mathfrak{g}, \mathfrak{h})$ follow from this without much difficulty (cf. [1]). We are also concerned with another result of his (cf. [2]) which asserts that if we choose U to be a linear manifold complementary to $\mathfrak{g}_{X_0}$ and invariant under $\operatorname{ad} H_0$ (X_0 being a principal nilpotent), then the restriction of $\mathfrak{J}$ to the affine l-plane $X_0 + U$ is an algebra isomorphism of $\mathfrak{J}$ onto the algebra of *all* polynomials on $X_0 + U$. The present note sketches an elementary method of obtaining these results from the theory of differential operators on $\mathfrak{g}$. It is also shown how the same ideas lead to the well-known determination of the set of points of $\mathfrak{g}$ at which the differentials $dg\,(g \in \mathfrak{J})$ span an l-dimensional space.

3. Let $\mathfrak{g}$ be an *arbitrary* Lie algebra over $\boldsymbol{C}$ and $X_0 \in \mathfrak{g}$. We define $\mathfrak{g}_{X_0}$ as the range of $\operatorname{ad} X_0$ and write U for a linear manifold complementary to $\mathfrak{g}_{X_0}$. Let $r = \dim U$ and let $\{Y_1, \cdots, Y_r\}$ be a basis for U. For any function f defined on an open set containing X_0, let $f^{\sim}$ be the function defined on an open neighborhood of the origin in $\boldsymbol{C}^r$ given by

$$f^{\sim}(u_1, \cdots, u_r) = f(X_0 + u_1Y_1 + \cdots + u_rY_r). \tag{5}$$

LEMMA 1. *There exists an open set N containing the origin in $\boldsymbol{C}^r$ such that the invariant set*

$$\Omega_N = \{(X_0 + u_1Y_1 + \cdots + u_rY_r)^x : (u_1, \cdots, u_r) \in N, x \in G\} \tag{6}$$

is open in $\mathfrak{g}$. The mapping $f \to f^{\sim}$ is an algebra isomorphism of the algebra of invariant holomorphic functions on Ω_N into the algebra of holomorphic functions on N.

Proof. We write $\boldsymbol{u} = (u_1, \cdots, u_r)$ to denote a typical point of $\boldsymbol{C}^r$. Let us consider the map

$$\psi : x, \boldsymbol{u} \to (X_0 + u_1Y_1 + \cdots + u_rY_r)^x$$

of $G \times \boldsymbol{C}^r$ into $\mathfrak{g}$. ψ is evidently holomorphic. A simple calculation shows that the differential $d\psi$ is given by

$$d\psi_{x,\boldsymbol{u}}(Z, \boldsymbol{v}) = [Z, X_0 + u_1Y_1 + \cdots + u_rY_r]^x + (v_1Y_1 + \cdots + v_rY_r)^x; \tag{7}$$

here we identify the tangent space of the complex manifold $G \times \boldsymbol{C}^r$ at x, $\boldsymbol{u}$ canonically with $\mathfrak{g} \times \boldsymbol{C}^r$, and identify canonically the tangent space of $\mathfrak{g}$ at any point of it with $\mathfrak{g}$ itself. From (7) we have

$$d\psi_{1,\boldsymbol{0}}(Z, \boldsymbol{v}) = [Z, X_0] + v_1Y_1 + \cdots + v_rY_r. \tag{8}$$

(8) Shows that $d\psi_{1,\boldsymbol{0}}$ is surjective since $\mathfrak{g} = \mathfrak{g}_{X_0} + U$, and hence has full rank at $(1, \boldsymbol{0})$. By continuity, there exists an open set N containing $\boldsymbol{0}$ in $\boldsymbol{C}^r$ such that $d\psi_{1,\boldsymbol{u}}$ has full rank for all $\boldsymbol{u} \in N$. Now, it follows from (7) that

$$d\psi_{x,\boldsymbol{u}} = \operatorname{Ad}(x) \cdot d\psi_{1,\boldsymbol{u}}$$

so that $d\psi_{x,\boldsymbol{u}}$ has full rank whenever $(x, \boldsymbol{u}) \in G \times N$. Consequently the image of $G \times N$ under ψ is open in $\mathfrak{g}$. This image is obviously Ω_N. This proves the first statement of the lemma. If f is invariant and holomorphic on Ω_N and $f^{\sim} = 0$ on N, then $f = 0$ on Ω_N by invariance. This proves the second statement.

COROLLARY. *Let* $\mathfrak{g}$ *be semisimple and let* $p_1, \cdots, p_l$ *be* l *homogeneous algebraically independent generators of* $\mathfrak{J}$. *Then the polynomials* $p_1^{\sim}, \cdots, p_l^{\sim}$ *(on* $\boldsymbol{C}^r$*) are algebraically independent.*

Proof. If Q is a complex polynomial in l variables and $Q(p_1^{\sim}, \cdots, p_l^{\sim}) = 0$, then $Q(p_1, \cdots, p_l) = 0$ on Ω_N and is hence identically zero. This implies that $Q = 0$.

It follows from the work of Harish-Chandra [1] that there exists an open set N containing $\mathbf{0}$ in $\boldsymbol{C}^r$ with the properties: (i) the set Ω_N, defined in (6), is open (ii) if D is a holomorphic differential operator defined on Ω_N, there is a holomorphic differential operator $D^{\sim}$ defined on N such that for any *invariant* holomorphic function f on Ω_N,

$$(Df)(X_0 + u_1Y_1 + \cdots + u_rY_r) = (D^{\sim}f^{\sim})(u_1, \cdots, u_r) \tag{9}$$

for $(u_1, \cdots, u_r) \in N$, $f^{\sim}$ being defined by (5). Actually, we do not need the construction of $D^{\sim}$ except in a very simple case for which a special argument can be given. Suppose that $\mathfrak{g}$ is semisimple, $X_0 \neq 0$ any nilpotent element and that H_0, Y_0 are chosen to satisfy (2). Let U be a linear manifold complementary to $\mathfrak{g}_{X_0}$, the range of $\operatorname{ad} X_0$, and invariant under $\operatorname{ad} H_0$. Let $Y_1, \cdots, Y_r$ be a basis for U such that the relations (4) are satisfied. Let N_0 be any open set containing $\mathbf{0}$ in $\boldsymbol{C}^r$. For D we take the vector field E which assigns to each point $X \in \mathfrak{g}$ the tangent vector X.

LEMMA 2. *With the notation as above,*

$$E^{\sim} = \sum_{j=1}^{r} (1 + \tfrac{1}{2}\lambda_j) u_j \frac{\partial}{\partial u_j}, \tag{10}$$

i.e., $E^{\sim}$ *is a differential operator on* N_0 *satisfying* (9).

Proof. This follows from Harish-Chandra's theory [1] (see Lemma 30). We include however a simple proof, for the sake of completeness. We use the notation of Lemma 1 and its proof. For any holomorphic function f on Ω_{N_0}, let f^{ψ} be the function on $G \times N_0$ given by

$$f^{\psi}(x, \boldsymbol{u}) = f((X_0 + u_1Y_1 + \cdots + u_rY_r)^x).$$

Now, putting $x = 1$ in (7) we get

$$d\psi_{1,\boldsymbol{u}}(Z, \boldsymbol{v}) = [Z, X_0 + u_1Y_1 + \cdots + u_rY_r] + (v_1Y_1 + \cdots + x_rY_r).$$

We now choose $Z = \frac{1}{2}H_0, v_j = (1 + \frac{1}{2}\lambda_j)u_j$, $1 \leq j \leq r$. For this choice of Z and $\boldsymbol{v}$ we find, using (4), that

$$d\psi_{1,\boldsymbol{u}}(Z,\boldsymbol{v}) = X_0 + u_1Y_1 + \cdots + u_rY_r. \tag{11}$$

If we now notice that

$$(d\psi_{1,\boldsymbol{u}}(Z,\boldsymbol{v})f)(X_0 + u_1Y_1 + \cdots + u_rY_r) = ((Z,\boldsymbol{v})f^{\psi})(1,\boldsymbol{u}),$$

we obtain, using (11),

$$((\tfrac{1}{2}H_0, \sum_{j=1}^{r}(1+\tfrac{1}{2}\lambda_j)u_j\frac{\partial}{\partial u_j})f^{\psi})(1,\boldsymbol{u}) = (Ef)(X_0 + u_1Y_1 + \cdots + u_rY_r).$$

Assume now that f is *invariant* on Ω_N. Then $f^{\psi}(x,\boldsymbol{u}) = f^{\sim}(\boldsymbol{u})$ is independent of x, and hence we conclude from the last equation that for all $\boldsymbol{u} \in N$,

$$(\sum_{j=1}^{r}(1+\tfrac{1}{2}\lambda_j)u_j\frac{\partial}{\partial u_j}f^{\sim})(\boldsymbol{u}) = (Ef)(X_0 + u_1Y_1 + \cdots + u_rY_r).$$

In view of (9), this proves (10). This completes the proof of Lemma 2.

The formula (10) is central in our analysis. We shall always assume that the Y_j's and λ_j's are so numbered that

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_r.$$

THEOREM 1. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\boldsymbol{C}$ and let X_0 be a principal nilpotent element, so that $r = l$. Let $p_1, \cdots, p_l$ be l homogeneous algebraically independent generators of $\mathfrak{I}$, of degrees $\nu_1, \cdots, \nu_l$ respectively, and let these be so numbered that $\nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_l$. Then*

$$\nu_j = 1 + \tfrac{1}{2}\lambda_j \quad (j = 1, 2, \cdots, l). \tag{12}$$

Proof. To start with, we work with an arbitrary nilpotent element $X_0 \neq 0$, not necessarily principal. Since $Ep_j = \nu_j p_j$, we have, by (9) and (10), $E^{\sim}p_j^{\sim} = \nu_j p_j^{\sim}$. For any vector $\boldsymbol{m} = (m_1, \cdots, m_r)$ with non-negative integral components, let $M(\boldsymbol{m})$ be the monomial $u_1^{m_1} \cdots u_r^{m_r}$. Then

$$E^{\sim}(M(\boldsymbol{m})) = \{\sum_{j=1}^{r}(1+\tfrac{1}{2}\lambda_j)m_j\} \cdot M(\boldsymbol{m}).$$

This shows that for any j, $p_j^{\sim}$ is a linear combination of those $M(\boldsymbol{m})$ for which

$$\sum_{k=1}^{r}(1+\tfrac{1}{2}\lambda_k)m_k = \nu_j. \tag{13}$$

In general, $r \geqq l$. We claim that

$$\nu_j \geqq 1 + \tfrac{1}{2}\lambda_j \quad j = 1, 2, \cdots, l. \tag{14}$$

Suppose that for some $j \leqq l$, $\nu_j < 1 + \frac{1}{2}\lambda_j$. Then

$$\nu_s < 1 + \tfrac{1}{2}\lambda_t \tag{15}$$

for $s \leqq j \leqq t$. Comparing (15) with (13) we find that for any $s \leqq j$, $p_s{}^{\sim}$ is a polynomial only of the variables $u_1, \cdots, u_{j-1}$. Since this is true for $s = 1, 2, \cdots, j$, we conclude that $p_1{}^{\sim}, \cdots, p_j{}^{\sim}$ are algebraically dependent, thus contradicting the corollary to Lemma 1. We thus have (14) for all $j = 1, 2, \cdots, l$.

We now assume that X_0 is principal. Then, $r = l$ and $\sum_{j=1}^{l} (\lambda_j + 1) = \dim \mathfrak{g}$, as the numbers $\lambda_1 + 1, \cdots, \lambda_l + 1$ are the dimensions of the subspaces into which $\mathfrak{g}$ splits under the action of $\boldsymbol{C} \cdot X_0 + \boldsymbol{C} \cdot H_0 + \boldsymbol{C} \cdot Y_0$. Thus, using (1), we find

$$\sum_{j=1}^{l} (1 + \tfrac{1}{2}\lambda_j) = \sum_{j=1}^{l} \nu_j.$$

Theorem 1 now follows at once from (14).

We shall now take up the second result of Kostant mentioned in § 2.

THEOREM 2. *Let $\mathfrak{g}$ be a semisimple Lie algebra, X_0 a principal nilpotent element and let U be a linear manifold complementary to $\mathfrak{g}_{X_0}$ and invariant under* $\operatorname{ad} H_0$. *Then, the restriction of $\mathfrak{J}$ to the affine l-plane $X_0 + U$ is an algebra isomorphism of $\mathfrak{J}$ onto the algebra of all polynomials on $X_0 + U$.*

Proof. Choose a basis $\{Y_1, \cdots, Y_l\}$ for U satisfying (4). Let us use the notation of Theorem 1. Let $k_1 < k_2 < \cdots < k_m$ $(m \leqq l)$ be the distinct elements of the set $\{\nu_1, \cdots, \nu_l\}$ and let

$$B_s = \{j : \nu_j = k_s\} \quad (s = 1, 2, \cdots, m). \tag{16}$$

Since $E^{\sim} p_j{}^{\sim} = \nu_j p_j{}^{\sim}$, we use (13) and (12) to conclude that $p_j{}^{\sim}$ is a linear combination of those monomials $M(\boldsymbol{m})$ for which $m_1\nu_1 + \cdots + m_l\nu_l = \nu_j$. If $\nu_j = k_s$, then this shows that $p_j{}^{\sim}$ is a polynomial only of the u_i's for which $i \in \bigcup_{t \leqq s} B_t$, and furthermore, that the u_i's with $i \in B_s$ enter *linearly* in the expression for $p_j{}^{\sim}$. In other words,

$$p_j{}^{\sim} = \sum_{j' \in B_s} a_{jj'} u_{j'} + Q_j \quad (j \in B_s) \tag{17}$$

where the $a_{jj'}$ are constants and Q_j is a polynomial only of those u_i's with $i \in \bigcup_{t \leqq s-1} B_t$. Let A_s denote the matrix $(a_{jj'})_{j, j' \in B_s}$. Then a trivial computation yields the formula

$$J = \frac{\partial(p_1^{\sim}, \cdots, p_l^{\sim})}{\partial(u_1, \cdots, u_l)} = \det \begin{bmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & A_m \end{bmatrix}. \tag{18}$$

so that

$$J = \det(A_1) \cdots \det(A_m).$$

J is thus a *constant*. On the other hand, the corollary to Lemma 1 tells us that J cannot vanish identically. Hence $J \neq 0$, showing that the matrices $A_1, A_2, \cdots, A_m$ are invertible. For all $i \in B_1$, (17) shows that the $p_i^{\sim}$ are linear functions of the u_j $(j \in B_1)$, and, coupled with the invertibility of A_1, enables us to conclude that the algebra generated by $p_1^{\sim}, \cdots, p_l^{\sim}$ contains all the u_j, $j \in B_1$. An easy induction, based on the 'triangular' form of (17) and the invertibility of the matrices A_k, now shows that the algebra generated by $p_1^{\sim}, \cdots, p_l^{\sim}$ contains all the u_j's. This completes the proof of Theorem 2.

We point out a simple corollary. Here $\mathfrak{g}$ is *simple.*

Corollary. *Let dp_l be the differential of p_l and $X_0 \in \mathfrak{g}$ a nilpotent element. Then, dp_l vanishes at X_0 unless X_0 is principal.*

Proof. Let us write dp_l^0 for the linear function on $\mathfrak{g}$ which corresponds to the covector defined by dp_l at X_0. Suppose that X_0 is principal. From (17) we get, as $l \in B_m$,

$$\left(\frac{\partial p_l^{\sim}}{\partial u_j}\right)(0) = a_{lj} \quad (j \in B_m).$$

As $\det(A_m) \neq 0$, $a_{lj_0} \neq 0$ for some $j_0 \in B_m$, and hence $dp_l^0(Y_{j_0}) \neq 0$. Suppose on the other hand that X_0 is not principal. If $X_0 = 0$, $dp_l^0 = 0$ as p_l is homogeneous and is of degree $\geqq 2$. We may assume therefore that $X_0 \neq 0$. Let the linear manifold U be chosen as in Theorem 1. Since p_l is constant on the orbit X_0^G, it follows that for any $Z \in \mathfrak{g}$,

$$dp_l^0([Z, X_0]) = \{\frac{d}{dt} p_l(X_0^{\exp tZ})\}_{t=0} = 0.$$

This shows that dp_l^0 vanishes on $\mathfrak{g}_{X_0}$. To prove that $dp_l^0 = 0$, it is sufficient now to prove that dp_l^0 vanishes on U i.e., $(\partial/\partial u_j)(p_l^{\sim})(\mathbf{0}) = 0$ for $j = 1, 2, \cdots, r$. Suppose for some j_0, $(\partial/\partial u_{j_0})(p_l^{\sim})(\mathbf{0}) \neq 0$. Since $E^{\sim} p_l^{\sim} = \nu_l p_l^{\sim}$, $p_l^{\sim}$ is a linear combination of the monomials $u_1^{m_1} \cdots u_r^{m_r}$ with $m_1(1 + (\lambda_1/2)) + \cdots + m_r(1 + (\lambda_r/2)) = \nu_l$, and the monomial u_{j_0} must appear in this expression with a nonzero coefficient. But then $\nu_l = (1 + (\lambda_{j_0}/2))$ from which we conclude easily that $(\operatorname{ad} X_0)^{2\nu_l - 2} = (\operatorname{ad} X_0)^{\lambda_{j_0}} \neq 0$. This shows that X_0 must be principal, a contradiction (cf. Kostant [1] Corollary 5.4, Lemma 9.1).

Our final theorem also due to Kostant [2] determines the set of all points of $\mathfrak{g}$ at which the differentials $dp_1, \cdots, dp_l$ are linearly independent. We recall that for any $A \in \mathfrak{g}$, the dimension of the centralizer $\mathfrak{z}_A$ of A is $\geqq l$. Let $\mathfrak{M}$ be the set of all $A \in \mathfrak{g}$ with $\dim \mathfrak{z}_A = l$. $\mathfrak{M}$ is a dense open set of $\mathfrak{g}$. $\mathfrak{M}$ is clearly invariant.

THEOREM 3. *If $A \in \mathfrak{g}$, the differentials $dp_1, \cdots, dp_l$ are linearly independent at A if and only if $A \in \mathfrak{M}$.*

Proof. We begin by introducing the well known decomposition $A = H + X$ where H is semisimple,[3] X is nilpotent and $[H, X] = 0$. H and X are uniquely determined by A, $\mathfrak{z}_A = \mathfrak{z}_H \cap \mathfrak{z}_X$ and the Lie algebra $\mathfrak{z}_H$ is reductive. The derived algebra $\mathfrak{z}_H' = [\mathfrak{z}_H, \mathfrak{z}_H]$ is semisimple and $X \in \mathfrak{z}_H'$. It is known (Kostant [2], Proposition 13) that $A \in \mathfrak{M}$ if and only if either $X = 0$ and H is regular, or H is non regular and X is a principal nilpotent of $\mathfrak{z}_H'$. This said, we come to the proof of Theorem 3. We shall assume that $A \notin \mathfrak{M}$ and prove that $dp_1, \cdots, dp_l$ are linearly dependent at A. If A is nilpotent, the corollary to Theorem 2 already proves this; in this case, the space spanned by the differentials $dg\, (g \in \mathfrak{J})$ at A is of dimension $< l$. Suppose next that $A \notin \mathfrak{M}$ but $H \neq 0$. Then H is non regular and X is not principal in $\mathfrak{z}_H'$. Let dp_i^0 be the linear functional on $\mathfrak{g}$ corresponding to the covector defined by dp_i at A, and write

$$\mathfrak{N} = \{Z : Z \in \mathfrak{g}, dp_i^0(Z) = 0 \text{ for } i = 1, 2, \cdots, l\}.$$

Since the p_i are constant on the orbit A^G we conclude as before that $\mathfrak{g}_A \subseteq \mathfrak{N}$. On the other hand, the polynomials $g_i(Z \to p_i(H + Z), Z \in \mathfrak{z}_H')$ on $\mathfrak{z}_H'$ are invariant under the adjoint group of $\mathfrak{z}_H'$, so that, the fact that X is not principal in $\mathfrak{z}_H'$ implies that $dg_1, \cdots, dg_l$ span a subspace of dimension $<$ rank $\mathfrak{z}_H'$ at X. $\mathfrak{z}_H'$ being the derived algebra of $\mathfrak{z}_H$, $\mathfrak{z}_H$ is known to be the direct sum of its center and $\mathfrak{z}_H'$; in particular, $\dim \mathfrak{z}_H' - \operatorname{rank} \mathfrak{z}_H' = \dim \mathfrak{z}_H - l$. In other words,

[3] i.e., $\operatorname{ad} H$ is a semisimple endomorphism of $\mathfrak{g}$.

$$(19) \qquad \dim(\mathfrak{N} \cap \mathfrak{z}_H') > \dim \mathfrak{z}_H - l.$$

Hence, as
$\dim(\mathfrak{N}) \geqq \dim(\mathfrak{g}_A + (\mathfrak{N} \cap \mathfrak{z}_H')) \geqq \dim \mathfrak{g}_A + \dim(\mathfrak{N} \cap \mathfrak{z}_H') - \dim(\mathfrak{g}_A \cap \mathfrak{z}_H)$,
we have,

$$(20) \qquad \dim \mathfrak{N} > \dim \mathfrak{g}_A + \dim \mathfrak{z}_H - l - \dim(\mathfrak{g}_A \cap \mathfrak{z}_H).$$

We now notice that $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_H$ and $\mathfrak{z}_H$ as $\operatorname{ad} H$ is semisimple, and that $\operatorname{ad} A$ leaves both of them invariant as $[A, H] = 0$. Thus $\mathfrak{g}_A$ is the direct sum of $\mathfrak{g}_A \cap \mathfrak{z}_H$ and $\mathfrak{g}_A \cap \mathfrak{g}_H$, and in particular, $\mathfrak{g}_A \cap \mathfrak{z}_H = (\operatorname{ad} A)[\mathfrak{z}_H]$. This shows that

$$\dim(\mathfrak{g}_A \cap \mathfrak{z}_H) = \dim \mathfrak{z}_H - \dim(\mathfrak{z}_A \cap \mathfrak{z}_H) = \dim \mathfrak{z}_H - \dim \mathfrak{z}_A.$$

Substituting in (20) and noting that $\dim \mathfrak{g}_A + \dim \mathfrak{z}_A = \dim \mathfrak{g}$, we obtain the inequality

$$(21) \qquad \dim \mathfrak{N} > \dim \mathfrak{g} - l.$$

(21) proves that $dp_1, \cdots, dp_l$ are dependent at points of $\mathfrak{g} - \mathfrak{M}$.

We now use Theorem 2 and its notation. Since the restriction of $\mathfrak{J}$ on $X_0 + U$ is surjective, it follows that $dp_1, \cdots, dp_l$ are linearly independent at the points of $X_0 + U$ and thus at the points of $(X_0 + U)^G$. By the first half of the theorem, $(X_0 + U)^G \subseteq \mathfrak{M}$. Moreover Theorem 2 implies that the map, which assigns to each orbit in $((X_0 + U)^G$ the point of $\boldsymbol{C}^l$ composed of the values of $p_1, \cdots, p_l$ on that orbit, has for range the *whole of* $\boldsymbol{C}^l$, while it is well-known that this map is bijective on the set of all orbits of $\mathfrak{M}$ itself (Kostant [2], Theorem 2). Hence we infer that $(X_0 + U)^G = \mathfrak{M}$. This proves that $dp_1, \cdots, dp_l$ are linearly independent precisely at the points of $\mathfrak{M}$. Theorem 3 is completely proved.

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REFERENCES.

C. Chevalley [1] "Invariants of finite groups generated by reflexions," *American Journal of Mathematics*, vol. 77 (1955), pp. 778-782.

Harish-Chandra [1] "Invariant distributions on Lie algebras," *American Journal of Mathematics*, vol. 86 (1964), pp. 271-309.

B. Kostant [1] "The principal three dimensional subgroup and the Betti numbers of a complex simple Lie group," *American Journal of Mathematics*, vol. 81 (1959), pp. 973-1032.

[2] "Lie group representations on polynomial rings," *American Journal of Mathematics*, vol. 85 (1963), pp. 327-404.

G. C. Shepherd and J. A. Todd [1] "Finite unitary reflexion groups," *Canadian Journal of Mathematics*, vol. 6 (1954), pp. 274-304.