HOMEWORK 2

- Section 2.4 in the book: Exercises 4, 16, 22, 28, 40.
- Section 2.5 in the book: Exercises 8, 12.

Problem 1. (a) Let $a \in \mathbb{R}$ be a constant and let $b_1 : \mathbb{R} \to \mathbb{R}$ and $b_2 : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that $b_1(t) \leq b_2(t)$ for all $t \geq 0$. Consider the linear differential equations

$$\frac{dx}{dt} = ax + b_1(t) \quad \text{and} \quad \frac{dy}{dt} = ay + b_2(t).$$

Assuming that $x(0) = y(0)$, show that $x(t) \leq y(t)$ for all $t \geq 0$.
(b) Show that if z is a differentiable function that satisfies $\frac{dz}{dt} \leq az$, then

$$z(t) \leq z(0)e^{at} \quad \text{for all} \quad t \geq 0.$$

Problem 2. Beer containing 6% alcohol per gallon is pumped into a vat that initially contains 400 gallons of beer at 3% alcohol. The rate at which the beer is pumped in is 3 gallons per minute, whereas the mixed liquid is pumped out at a rate of 4 gallons per minute. Find the number of gallons of alcohol $A(t)$ in the tank at any time. What is the percentage of alcohol in the tank after 60 minutes? When is the tank empty?

Problem 3. When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A,$$

where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount memorized at time t, M is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized. Solve for $A(t)$ and graph the solution, assuming that $A(0) = 0$. Find the limiting value of A as $t \to \infty$.