HOMEWORK 7

- Section 9.2 in the book: Exercises 6, 12, 20, 26, 30, 34, 36, 40, 42, 46, 50, 54.
- Section 9.3 in the book: Exercises 10, 14.

Problem 1. Initially, tank A contains 50 gallons of water in which 25 pounds of salt are dissolved, while a second tank B contains 50 gallons of pure water. Pure water is pumped into tank A at the rate of 3 gallons per minute. There are two pipes connecting tank A and tank B. The first pumps salt solution from tank A into tank B at the rate of 4 gallons per minute, while the second pumps salt solution from tank B into tank A at the rate of 1 gallon per minute. Finally, tank B is drained at the rate of 3 gallons per minute. We assume perfect mixing for both tanks.
(a) Write down the two-dimensional system that models the salt content in each tank over time.
(b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (a). Use this to write down the general solution to the two-dimensional system in part (a).
(c) Find the particular solution that satisfies the initial conditions given in the problem.

Problem 2. (Cayley–Hamilton Theorem) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a matrix with constant real entries. By direct computation show that

$$A^2 - TA + DI = 0,$$

where $T = a + d$ denotes the trace of the matrix A, $D = \det(A) = ad - bc$ denotes the determinant of A, and I denotes the identity matrix.