First Name: ____________________ ID#____________________

Last Name: ____________________

Section: ____________________ $= \begin{cases} 1a & \text{Tuesday with M. Bulkow} \\ 1b & \text{Thursday with M. Bulkow} \\ 1c & \text{Tuesday with L. Vera} \\ 1d & \text{Thursday with L. Vera} \\ 1e & \text{Tuesday with A. Mennen} \\ 1f & \text{Thursday with A. Mennen} \end{cases}$

Rules.

- There are **NINE** problems; ten points per problem.
- The problems are in no particular order.
- Use the backs of the pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	5
6	7	8	9	$\sum$

(1) Indicate TRUE/FALSE or fill in the missing number, as appropriate:

T / F : If $A = QR$ in the sense of the QR-decomposition, then $\det(A^T A) = \big[\det(R)\big]^2$.

T / F : If $\mathcal{B}$ is a basis for a subspace of $\mathbb{R}^7$, then $\vec{0} \in \operatorname{span}(\mathcal{B})$.

T / F : If $\dim(\ker(A)) = 2$, then $A\vec{x} = \vec{b}$ has at most two solutions $\vec{x}$.

T / F : A square matrix with distinct eigenvalues is orthogonally diagonalizable.

T / F : The nullity of a square matrix coincides with the algebraic multiplicity of zero as an eigenvalue.

T / F : If A is a square matrix, then the eigenvalues of A and A^T are the same.

$\dim\left(\operatorname{span}\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\right)\right) =$ ______

For the next **THREE** problems, consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

The volume of the parallelepiped generated by the columns of B is ______

T / F : If $q(\vec{x}) = \vec{x}^T B \vec{x}$ then $q(\vec{x}) \geq 0$ for all $\vec{x}$.

T / F : The matrix B is invertible.

(2) (a) Diagonalize the following matrix

$$A = \begin{bmatrix} -5 & 4 & -7 \\ -18 & 13 & -21 \\ -6 & 4 & -6 \end{bmatrix}$$

(b) Using this (or otherwise) find a 3×3 matrix X so that $X^2 = A$.

(3) Define any FIVE of the following terms:

geometric multiplicity of an eigenvalue	*singular value* of a matrix A
linear dependent collection of vectors	*positive definite* quadratic form $Q(\vec{x})$
a *subspace* of $\mathbb{R}^m$	a *linear transformation* T from $\mathbb{R}^m$ to $\mathbb{R}^n$

Be **precise** and write in **complete sentences**.

(4) (a) Find the solution to the following linear system:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 12 \\ 3 \end{bmatrix}$$

(b) Demonstrate the use of Cramer's rule by using it to check the bottom entry of $\vec{x}$.

(5) Consider the following matrix given in terms of its singular value decomposition:

$$A = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

(a) What is the rank of A?

(b) Find an orthonormal basis for the image of A.

(c) Find an orthonormal basis for the kernel of A.

(d) What is the largest value of $\|A\vec{x}\|$ among all vectors $\vec{x}$ with $\|\vec{x}\| = 1$.

(e) Describe *all* vectors that achieve this maximal value in part (d).

(6) (a) Determine the following partial derivative (of the determinant of a general $n \times n$ matrix):

$$\frac{\partial}{\partial a_{1n}} \det\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) =$$

(b) Let $\mathcal{P}$ denote the plane (in $\mathbb{R}^3$) through the origin that is perpendicular to $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Determine the matrix R representing reflection in this plane.

(7) Find coefficients A and B so that the following model

$$y = A\cos(t) + B\sin(t)$$

best fits the data:

t	0	$\pi/2$	π	2π
y	3	7	-3	0

(8) Consider two bases for $\mathbb{R}^3$, namely, $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ and $\mathcal{G} = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{w}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

(a) If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, find $[\vec{x}]_{\mathcal{G}}$.

(b) If the linear transformation T has $[Tx]_{\mathcal{G}} = [x]_{\mathcal{B}}$ find $[T]_{\mathcal{G}}$.

(9) Consider the quadratic form

$$Q(x, y, z) = 25x^2 + 13y^2 + 144z^2 - 120xz$$

(a) Find all points $(x, y, z) \in \mathbb{R}^3$ for which $Q(x, y, z) = 0$.

(b) Find the point (or points) closest to the origin for which $Q(x, y, z) = 169$.

Answer key

- Problem 1: T, T, F, F, F, T, 1, 2, F, T.
- Problem 2: With

$$S = \begin{bmatrix} 2 & -7 & 1 \\ 3 & 0 & 3 \\ 0 & 6 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have $A = SDS^{-1}$. We can take $X = SDS^{-1}$.

- Problem 4: $\vec{x} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$.
- Problem 5: (a) 2; (b) $\left\{ \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$; (c) $\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$; (d) 1;

(e) $\left\{ c_1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} : c_1^2 + c_2^2 = 1 \right\}$

- Problem 6: (a) $(-1)^{n+1} \det(B)$, where the matrix B is obtained from A by deleting the first row and the last column; (b)

$$R = \frac{1}{9} \begin{bmatrix} 1 & -4 & -8 \\ -4 & 7 & -4 \\ -8 & -4 & 1 \end{bmatrix}$$

- Problem 7: $A = 2$, $B = 7$.
- Problem 8: (a) $\begin{bmatrix} 2/5 \\ 0 \\ 0 \end{bmatrix}$; (b) $\begin{bmatrix} 5/2 & 1/2 & -1 \\ 0 & 0 & 2 \\ 5/2 & -1/2 & 1 \end{bmatrix}$
- Problem 9: (a) $\operatorname{span}\left\{ \begin{bmatrix} 12 \\ 0 \\ 5 \end{bmatrix} \right\}$; (b) $\pm \begin{bmatrix} -5/13 \\ 0 \\ 12/13 \end{bmatrix}$.